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	3: Expected utility	
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Last week, we used preference relations to describe all instances of decision making. Preferences are cumbersome, which led to representation functions and the vNM representation. We saw that if we fix a finite set of outcomes $\Omega = \{\omega_1, \ldots, \omega_d\}$, a convex set $\mathcal{X}$ of lotteries over these outcomes, and a preference relation $>$ on $\mathcal{X}$, then a representation U for $>$ on $\mathcal{X}$ has a Von Neumann-Morgenstern Representation if there exists a function $u : \Omega \to \mathbb{R}$ such that

$$U(p) = \sum_{i=1}^{d} p_i u(\omega_i) \quad \text{for all } p \in \mathcal{X}. \tag{1}$$

We saw cases where a Von Neumann-Morgenstern Representation doesn't exist, and when one does exist. This week, we first look at conditions for Von Neumann-Morgenstern Representations to exist. We then define risk aversion using preferences and Von Neumann-Morgenstern Representations.

Venn diagram: preference relations can be partitioned into those defined over outcomes that are numbers, and those over abstract outcomes. They be further partitioned into those with representation U and those without: with representation U, we can quantify how much one alternative is preferred over another, and talk about minimizing or maximizing an objective. Those preferences with U can be further partitioned into those with vNM and those without.

1 Existence of Von Neumann-Morgenstern Representation

Remark 1 (Affine transformation). For $a \in [0, 1]$, and a pair of lotteries p, q, the convex mixture $ap + (1 - a)q$ is a compound lottery. A Von Neumann-Morgenstern Representation U satisfies the affinity property, i.e., for every $p, q \in \mathcal{X}$ and every $a \in [0, 1]$, we have

$$U(ap + (1 - a)q) = aU(p) + (1 - a)U(q). \tag{2}$$

Preference relations that have a Von Neumann-Morgenstern Representation have special properties. Moreover, if a preference relation has these properties, then it has a vNM representation.

Definition 1.1 (Independence)**.** A preference relation $>$ on $\mathcal{X}$ satisfies the independence axiom (or substitution axiom) if for all $p, q, r \in \mathcal{X}$ and all $a \in (0, 1]$,

$$p > q \implies ap + (1 - a)r > aq + (1 - a)r. \tag{3}$$

Remark 2. Think of the lottery $ap + (1-a)r$ as a coin flip for p or r, followed by the selected lottery.

The independence axiom is reasonable?

Example 1.1. Consider three pizza with different ingredient compositions (cheese, peppers, fish): Mexican Pizza, American Pizza, Napoli Pizza.

Definition 1.2 (Continuity)**.** A preference relation $>$ on $\mathcal{X}$ satisfies the Archimedean axiom (or continuity axiom) if for $p > r > q$, there exist $a, b \in (0,1)$ such that

$$ap + (1-a)q > r > bp + (1-b)q. \tag{4}$$

Remark 3. Observe that $ap + (1-a)q$ converges to p as $a \to 1$, and converges to q as $a \to 0$.

Example 1.2. Consider three pizza with different ingredient compositions: Mexican Pizza, American Pizza, Napoli Pizza. Three armies with different compositions. Doing activities with different amounts of interactions with three friends.

Example 1.3. Consider the following three deterministic distributions: p yields 1000 CAD, r yields 10 CAD, and q is the lottery where one dies for sure. Even for small a it is not clear that someone would prefer the gamble $ap + (1-a)q$, which involves the probability of dying, over the conservative r. However, that most people would not hesitate to drive a car for a distance of 50 km in order to receive 1000 CAD, even though this might involve the risk of a deadly accident.

Theorem 1.1 (Corollary 2.23 of Textbook)**.** *Suppose that $\mathcal{X}$ is the set of all probability distributions on a finite set Ω and that $>$ is a preference relation on $\mathcal{X}$ that satisfies both the Archimedean and the independence axiom. Then there exists a von Neumann–Morgenstern representation U for $>$, and it is unique up to positive affine transformations. In other words, there exists a function u such that for every pair of lotteries $p, q \in \mathcal{X}$,*

$$p > q \iff \mathbb{E}_{Z\sim p} u(Z) > \mathbb{E}_{Z\sim q} u(Z). \tag{5}$$

Remark 4. Conversely, any decision maker who maximizes expected value $\mathbb{E}_{Z\sim p} u(Z)$ over $p \in \mathcal{X}$ has a preference relation that satisfies the Archimedean and independence axioms.

2 Risk Aversion for von Neumann–Morgenstern preference relations

For von Neumann–Morgenstern preference relations, there's a simple way to check monotonicity and risk aversion.

Proposition 2.1 (Risk aversion)**.** *Suppose that $>$ has von Neumann–Morgenstern representation $U(p) = \mathbb{E}_{Z\sim p} u(Z)$. $>$ is monotone if and only if u is strictly increasing. $>$ is risk averse if and only if u is strictly concave.*

Remark 5. With this proposition, it's much easier to check which preference relation is risk averse by taking the second derivative of u.

Definition 2.1 (Utility function)**.** A function $u : \Omega \to \mathbb{R}$ is a utility function if it is strictly increasing, strictly concave, and continuous on Ω.

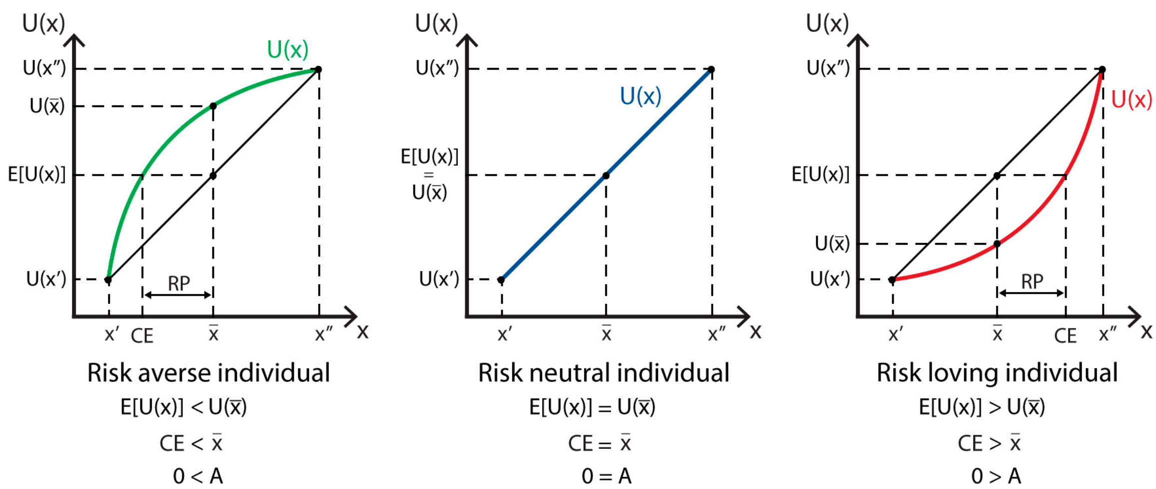

Figure 1: From `https://policonomics.com/risk-aversion/`

Homework: Show that for a risk averse U: we have $\mathbb{E}_p u(Z) < U(\mathbb{E}_p Z)$.

Example 2.1 (Financial assets)**.** In an environment subject to uncertain, every decision is a lottery p. In finance, each decision is an asset, and p describes the distribution of payoff for purchasing this asset now and selling it a fixed period of time in the future. The expected value $m(p)$ is called the fair price of the asset. In the case where the asset is an insurance contract, $m(p)$ is also called a fair premium. The problem in practice is that p is unknown and must be estimated.

2.1 Quantifying risk aversion via $m(p) - c(p)$

Previously, we only talked about risk-averse or not risk-averse. Let's try to quantify how risk-averse is a decision maker or a preference relation.

Example 2.2. Suppose that you hold a lottery ticket. The outcomes of the lottery are 100 CAD, 10 CAD, and 0 CAD. The expected value of the outcome is 50 CAD. Before finding out the outcome, would you be willing to sell the ticket for 49 CAD? 48 CAD? The lowest value you are willing to accept is called the certainty equivalent $c(p)$.

Consider $\Omega \subset \mathbb{R}$. Consider a decision maker with a preference $>$ that admits an utility function u (von Neumann–Morgenstern representation with an utility function).

For a given lottery p, we have

$$U(p) = \sum_i p_i u(\omega_i). \tag{6}$$

Since u is continuous, clearly there exists a real number $c(p)$ called certainty equivalent such that[1]

$$u(c(p)) = U(p) = U(\delta_{c(p)}). \tag{7}$$

Therefore, the decision maker is indifferent between the lottery p and a sure outcome with payoff $c(p)$. The difference $m(p) - c(p)$ is called the risk premium of p.

If u is a risk-averse utility function, then it is concave, we can apply Jansen's inequality to conclude that

$$c(p) \leq m(p). \tag{8}$$

Moreover, by the definition of risk averse preference, the decision maker prefers a certain payoff $m(p)$ to a lottery p, hence

$$c(p) < m(p) \quad \text{if } p \neq \delta_{m(p)}. \tag{9}$$

In the context of financial assets, one interpretation of the risk premium is the amount that the decision maker is willing to pay to replace the payoff distribution p by its expected value $m(p)$ (fair price). It is the monetary cost of the uncertainty in p. This is why bonds have such low yields, and some stocks have high yield.

Example 2.3. Numerical example with $u(z) = \sqrt{z}$, p uniform distribution on $[0, 1]$. What is the fair price $m(p)$ and certainty equivalent $c(p)$?

3 References

- Chapter 2.3 of textbook.

[1] Compare $\delta_{m(p)}$ and $\delta_{c(p)}$.