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4: Managing Risk	
Concordia	March 11, 2020

We see two examples of managing risk by mixing-in certainty.

1 Managing risk by adding certainty

Let $X : \Omega \to \mathbb{R}$ denote a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with a probability distribution F and a density function f. Suppose that X is bounded from below by a. What is the optimal mix $\lambda \in [0, 1]$ for a random payoff X and a fixed amount c? The payoff of a mix λ is

$$X_\lambda = (1 - \lambda)X + \lambda c. \tag{1}$$

Let f_λ denote the probability density function of X_λ.

Example 1.1. What decision are of the form of a mixture? What decisions are not? Immigration, unit-demand goods, indivisible.

Suppose that the preference relation admits a vNM representation, then the representation of the lottery f_λ is

$$U(f_\lambda) = \mathbb{E}u(X_\lambda) = \int u(z) f_\lambda(z) dz. \tag{2}$$

We can maximize the value of $U(f_\lambda)$ over λ if u is given. If u is a utility function, then the above integral is concave, and has an unique maximum.

Example 1.2. Consider a decision maker with $u(z) = \sqrt{z}$, $c = 0.45$, X uniform on $[0, 1]$. For $\lambda = 0$, we have $X_0 = X$, then

$$U(f_0) = \int_0^1 z^{1/2} dz = 1/(3/2) = 2/3. \tag{3}$$

For a general value of λ:

$$\begin{aligned}
F_\lambda(z) &= \mathbb{P}((1-\lambda)X + \lambda c \leq z) && (4)\\
&= \mathbb{P}(X \leq \frac{z-\lambda c}{1-\lambda}) && (5)\\
&= \frac{z-\lambda c}{1-\lambda} \text{ for } \frac{z-\lambda c}{1-\lambda} \in [0,1] && (6)\\
&= \frac{z-\lambda c}{1-\lambda} \text{ for } z \in [\lambda c, (1-\lambda)+\lambda c] && (7)\\
f_\lambda(z) &= \frac{1}{1-\lambda} 1_{[z\in[\lambda c,(1-\lambda)+\lambda c]]} && (8)\\
U(f_\lambda) &= \int z^{1/2} f_\lambda(z) dz && (9)\\
&= \int_{\lambda c}^{(1-\lambda)+\lambda c} z^{1/2} \frac{1}{1-\lambda} dz && (10)\\
&= \frac{2}{3(1-\lambda)} \left[(1-\lambda+\lambda c)^{3/2} - (\lambda c)^{3/2}\right], && (11)
\end{aligned}$$

which is plotted in Figure 1 for different values of c. What if $u(z) = \log(1+z)$[1]? What if X follows a normal distribution with mean 0.5?

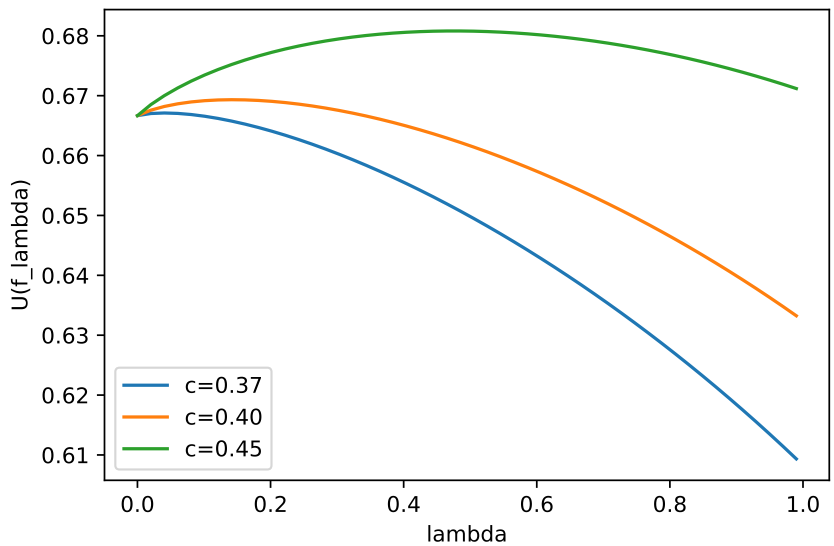

Figure 1: $U(f_\lambda)$

The following is a useful tool to analyze the mixture λ.

[1] Use integration by parts.

Proposition 1.1. *Let λ^* denote the value of λ that maximizes $U(f_\lambda)$. If u is differentiable utility function, then $\lambda^* = 1$ if and only if $\mathbb{E}X \leq c$. Moreover, $\lambda^* = 0$ if and only if*

$$c \leq \frac{\mathbb{E}Xu'(X)}{\mathbb{E}u'(X)}. \tag{12}$$

1.1 Investment

Example 1.3 (Investing in risky asset (Example 2.43 of Textbook))**.** Consider one risky asset with unit-price π and random payoff S. There is also a risk-free bond with interest rate r. You have w dollars, and your preference is represented by an utility function u. If you invest $(1-\lambda)w$ in the asset and λw in the bond, the payoff (profit) is:

$$\begin{aligned} X_\lambda &= \frac{(1-\lambda)w}{\pi}S - (1-\lambda)w + \lambda wr && (13) \\ &= \frac{(1-\lambda)w}{\pi}(S-\pi) + \lambda wr && (14) \\ & && (15) \end{aligned}$$

Under what condition, should you invest in the asset?

By Proposition 1.1, we have $\lambda^* = 1$ if and only if

$$\mathbb{E}\frac{w(S-\pi)}{\pi} \leq wr \quad \iff \mathbb{E}(S-\pi) \leq \pi r \quad \iff \mathbb{E}\frac{S}{1+r} \leq \pi. \tag{16}$$

In other words, to attract a risk averse investor, the price of the risky asset must be strictly less than the expected discounted payoff.

1.2 Insurance

Example 1.4 (Optimal amount of insurance (Example 2.44 of Textbook))**.** Suppose that you have utility function u. You start with w dollars. Let Y denote the random loss that you will incur in the next year. You are offered an insurance policy that costs $\lambda\pi$ dollars pays you λY dollars next year, where π is the maximum amount you can buy and you can choose the amount $\lambda \in [0,1]$. How much insurance λ should you buy?

The random payoff next year is

$$\begin{aligned} X_\lambda &= w - Y + \lambda Y - \lambda\pi && (17) \\ &= (1-\lambda)(w-Y) + \lambda(w - Y + Y - \pi) = (1-\lambda)(w-Y) + \lambda(w-\pi). && (18) \end{aligned}$$

By Proposition 1.1, we have $\lambda^* = 1$ if and only if

$$\mathbb{E}(w-Y) \leq w - \pi \quad \iff \mathbb{E}Y \geq \pi. \tag{19}$$

You should buy full insurance if and only if $\mathbb{E}Y \geq \pi$. However, if $\mathbb{E}Y < \pi$, then you should buy partial insurance. By Proposition 1.1 again, we have $\lambda^* \in (0, 1)$ when

$$\pi > \mathbb{E}Y, \tag{20}$$

$$\text{and } w - \pi > \frac{\mathbb{E}(w - Y)u'(w - Y)}{\mathbb{E}u'(w - Y)} \tag{21}$$

$$\iff w - \pi > w - \frac{\mathbb{E}Yu'(w - Y)}{\mathbb{E}u'(w - Y)} \tag{22}$$

$$\iff \pi < \frac{\mathbb{E}Yu'(w - Y)}{\mathbb{E}u'(w - Y)} \tag{23}$$

$$\tag{24}$$

This explains why people only buy partial insurance, even when the insurance premium is higher than the fair premium $\mathbb{E}Y$. When do we have $\lambda^* = 0$?

2 References

- Chapter 2.3 of textbook.