INSE6320 Jia Yuan Yu

5: Measures of Risk

Concordia November 15, 2021

1 Representation of risk aversion is not perfect

Example 2.75 of Textbook (Ellsberg paradox). Ask each student in class to write down their choices on a piece of paper. You are faced with a choice between two urns, each containing 100 balls which are either red or black. In the first urn, the proportion p of red balls is know; assume, e.g., $p = 0.49$. In the second urn, the proportion q unknown. Suppose that you get 1000 CAD if you draw a red ball and 0 otherwise. Let X_1 denote the payoff for choosing urn 1, X_2 the payoff for choosing urn 2. Which urn do you choose? What if you get 1000 CAD for drawing a black ball and 0 for a red one? Is this behavior compatible with the paradigm of expected utility? Ask each student to write down their vNM representation function u, then calculate $U((0.49, 0.51))$ and $U((q, 1-q))$. For any unknown probability q, the first choice implies $p > q$, the second choice implies $1 - p > 1 - q$.

2 Quantifying risk, Part II: risk measures

Preference relations are binary operators on lotteries, they can be risk-averse, risk-seeking. Risk measure are functions (unary operators) of one lottery, a lottery can have positive risk, negative risk. The set of functions that can be used to measure risk is a subset of the set of functions:

$$\{\text{risk measures}\} \subset \{\text{all functions}\}.$$

We will further subdivide the set of risk measures later in this section.

So far, we have looked at a decision maker, and how risk sensitive he or she is, when he or she is risk averse, etc. We have quantified risk with the notion of risk premium $c(p) - m(p)$. Just as the best decision depends on the decision maker, the proper way of quantifying risk depends on the decision maker. Let's look at different ways of quantifying risk. We want to look at choices available to a decision maker, and quantify the risk associated with each choice.

So far, we first fix a decision maker, then obtain a preference relation.

Another approach is to first measure the risk associated with a random variable or lottery, then ask the decision maker to choose.

Example: worst-case risk measure:

$$\rho(X) = -\inf_{\omega \in \Omega} X(\omega). \tag{1}$$

This is the least potential loss (e.g., shortfall of face masks) over all scenarios.

Example: Value at risk corresponding to a quantile λ:

$$VAR_\lambda(X) = \inf\{m \in \mathbb{R} : \mathbb{P}(m + X < 0) \le \lambda\} \tag{2}$$

Let X be the payoff for building a train track along path 1, and Y along path 2. Consider a uniform X with support $[-3, 5]$, what is the minimum subsidy m that ensures that $\mathbb{P}(m + X < 0) \le 0.1$?

3 Quantifying risk of a decision

Each decision is associated with a random payoff X, which is described as a probability distribution (lottery). Comparing distributions is cumbersome. We so far compared two payoffs between each other. Let's try to quantify the risk with a single number so that we can describe how much one payoff is prefered to another.

One classical approach to measure risk is using the variance, but it does not capture the asymmetry: the downside risk is more important than the upside risk.

3.1 Risk measures

Let Ω be a set of scenarios (e.g., outbreak scenarios of coronavirus). A decision is described by a mapping $X : \Omega \to \mathbb{R}$, where $X(\omega)$ is the payoff if the scenario ω is realized (e.g., number of face masks available at a hospital). We assume that X belongs to a set $\mathcal{X}$ of decisions.

A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is a measure of risk if it satisfies two conditions for all $X, Y \in \mathcal{X}$:

1. Monotonicity: if $X \le Y$ (for all ω), then $\rho(X) \ge \rho(Y)$,
2. Translation invariance (Cash invariance): if $m \in \mathbb{R}$, then $\rho(X) - \rho(X + m) = m$.

Example 3.1. Monotonicity: you can build a road from A to B, or a road A to C to B. The cost is the same, but the number of users $X \le Y$, but $\rho(X) \ge \rho(Y)$.

Example 3.2. Monotonicity: You have two job offers from car dealerships, first offer you make 100 dollars per car you sell, second offer you make 110 per car sold.

Example 3.3. Monotonicity: You can choose between driving a fast car and taking a bus to go from A to B. Let X and Y denote the time saved compared to walking.

Example 3.4. Translation invariance: You can by an electric vehicle with subsidy or without. Let X be the cost of owning the EV over one year, which is random due to fluctuation in price of electricity.

Example 3.5. Translation invariance: X is reading from a broken speedometer, m is amount of deceleration, the risk measure is the maximum fine you can receive if stopped by police (assumed equal to your actual speed).

Example 3.6. You are the decision maker at a hospital. Let ω denote a policy for hospital staff to change face mask every ω hours. Then $X(\omega)$ is the number of face masks remaining after one day. Think of $\rho(X)$ as the number of additional masks needed at the start of the day to avoid a shortfall of masks with probability 0.99. In other words, $\rho(X)$ is the number of donation face masks that can be added to decision X in order for the supervising agency (WHO) to accept the decision ω.

In this example, let $Y(\omega)$ denote the number of masks remaining in another smaller hospital with fewer workers and patients. Monotonicity means that you need more masks to avoid a shortfall in the bigger hospital. Translation invariance means that if you receive m masks at the start of the day from another source as a gift, then the number of additional masks you need is m less than if you didn't receive the gift.

Observe that

$$\rho(X + \rho(X)) = 0, \tag{3}$$

$$\rho(m) = \rho(0) - m. \tag{4}$$

For purpose of normalization, we can assume that $\rho(0) = 0$.

A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called a convex measure of risk if for all $\lambda \in [0, 1]$ and $X, Y \in \mathcal{X}$, we have

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y) \tag{5}$$

Example 3.7. In the hospital example, suppose that you simultaneously increase the number of employees and patients in the two hospitals by factor of $1/\lambda$ and $1/(1 - \lambda)$ and they share their supply of masks, then convexity means that the total number of additional masks needed to avoid shortfall is less than $1/\lambda$ and $1/(1-\lambda)$ times the daily additional masks needed by each hospital when they don't share their mask supplies.

Exercise: What is risk pooling? How is it related to convexity of risk measures?

The mixture decision $\lambda X + (1 - \lambda)Y$ corresponds to putting some resources in one decision and the remaining in the alternative. The convexity axiom captures the idea that diversification does not increase risk. If ρ is convex and normalized, let $Y = 0$, then

$$\rho(\lambda X) \leq \lambda\rho(X) \quad \text{for } \lambda \in [0, 1], \tag{6}$$

$$\rho(\lambda X) \geq \lambda\rho(X) \quad \text{for } \lambda \geq 1. \tag{7}$$

A convex measure of risk ρ is called a coherent risk measure if it satisfies

$$\text{(Positive Homogeneity)} \quad \lambda \geq 0 \implies \rho(\lambda X) = \lambda\rho(X). \tag{8}$$

Example 3.8. Positive homogeneity means that multiplying the number of workers and patients in the hospital by λ, the number of masks required to avoid shortfall with the same probability is also multiplied by λ.

Note that positive homogeneity implies normalization. Moreover, positive homogeneity and convexity imply subadditivity:

$$\rho(X + Y) \leq \rho(X) + \rho(Y). \tag{9}$$

Subadditivity is useful to decompose risk management into smaller tasks.

3.2 Comparison with vNM representation

Recall that a vNM preference relation has a representation U that is affine:

$$U(ap + (1-a)q) = aU(p) + (1-a)U(q).$$

Compare this with the properties of a risk measure ρ.

3.3 Acceptance set

Given a measure of risk ρ, we define the acceptance set of ρ:

$$A_\rho \triangleq \{X \in \mathcal{X} : \rho(X) \leq 0\}, \tag{10}$$

which contains decisions that are acceptable because they do not require donations.

Proposition 3.1. *Suppose that ρ is a measure of risk with acceptance set A_ρ. Then we have the following properties:*

1. *A_ρ is non-empty and satisfies:*

$$\inf m \in R : m \in A_\rho > -\infty, \tag{11}$$

$$X \in A_\rho, Y \in \mathcal{X}, Y \geq X \implies Y \in A_\rho. \tag{12}$$

2. *The measure of risk ρ can be recovered from A_ρ:*

$$\rho(X) = \inf\{m \in \mathbb{R} : m + X \in A_\rho\}. \tag{13}$$

3. *ρ is a convex risk measure if and only if A_ρ is convex.*

4. *ρ is positively homogeneous if and only if A_ρ is a cone. In particular, ρ is coherent if and only if A_ρ is a convex cone.*

Conversely, given a set A of acceptable decisions, we can define the risk measure of X as the minimal amount m that makes $m + X$ acceptable:

$$\rho_A(X) \triangleq \inf\{m \in \mathbb{R} : m + X \in A\} \tag{14}$$

Example: worst-case risk measure:

$$\rho(X) = -\inf_{\omega \in \Omega} X(\omega). \tag{15}$$

This is the least potential loss (e.g., shortfall of face masks) over all scenarios.

Example: Value at risk corresponding to a quantile λ:

$$VAR_\lambda(X) = \inf\{m \in \mathbb{R} : \mathbb{P}(m + X < 0) \leq \lambda\} \tag{16}$$

4 References

- Chapter 2 of textbook.