6: Engineering Applications

As engineers, you will build tomorrow's hospitals. What dimensions should you make them? The design will influence how many beds can fit in and how many respirators can fit in the storage room.

Some examples come from the paper "Engineering Decisions under Risk-averseness" by R. T. Rockafellar and J. O. Royset.

1 When risk neutral suffices

If the same decision is repeated many times, or by many people, and the total cost is the random variable to interest, then we can be risk neutral. Indeed, the central limit theorem and law of large numbers guarantee that the total cost is represented accurately by the expected value.

Examples:

- Government making decision over a long time horizon,
- Retail sales, etc.

2 When risk measures are needed

In contrast, we need risk averseness when we make a single decision and live with the consequences for a long time. For example, building a rocket, or a bridge.

Example 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a vector describing design decisions such as dimensions, material choices, maintenance frequency, etc. The random variable $Y(x)$ denotes the random cost of design x. This cost depends on both the design x and a random vector V describing uncertain parameters: $Y(x) = g(x, V)$. For example, you design a bar to support a random load S using material with yield stress z. You decide the cross section A. In this case, we have $V = (S, z)$, $x = A$, and $Y(x) = g(x, V) = S - zA$. The random variable $Y(x)$ describes the random load exceedance for design $x = A$; positive realizations imply that the load exceeds the capacity and nonpositive realizations give satisfactory performance.

Example 2.2 (Safety Margin)**.** A natural choice motivated by statistical confidence intervals is to set $\rho(Y) = \mathbb{E}Y + \gamma\mathbb{SD}(Y)$, where $\mathbb{SD}(Y)$ is the standard deviation of Y and γ a given positive constant. Here the risk includes variance, but does so in a symmetrical manner. Large variability on the high side can remain undetected if it is compensated by small variability on the low side.

2.1 Value at risk example

Recall the definition of Value at Risk:

$$VAR_\lambda(X) = \inf\{m \in \mathbb{R} : \mathbb{P}(m + X < 0) \le \lambda\}. \tag{1}$$

The VAR is easy to compute for continuous distributions without any jumps. Observe that

$$\mathbb{P}(m + X < 0) \le \lambda \tag{2}$$
$$1 - \mathbb{P}(m + X \ge 0) \le \lambda \tag{3}$$
$$1 - \lambda \le \mathbb{P}(-X \le m) = F_{-X}(m). \tag{4}$$

Hence, it can be computed from the CDF F_{-X}. The following example is one where there are jumps.

Example 2.3 (Example 4.41 of textbook)**.** Diversification should reduce risk (e.g., risk pooling, mutual funds, etc.). How about the effect on Value at Risk? Two bonds X_1 and X_2, independent of each other, each with a cost of w, a yield of 6 percent ($\hat{r}$-r = 0.06), and a probability of default of p. Buying one unit of the first bond yields $X_1 = -w$ in the event of a default, and $X_1 = 0.06w$ otherwise. Similarly for the second bond. Observe that

$$\mathbb{P}(X_1 - 0.06w < 0) = \mathbb{P}(\text{first bond defaults}) = p. \tag{5}$$

Suppose that $p \le \lambda$, then we have

$$\mathbb{P}(X_1 - 0.06w < 0) \le \lambda. \tag{6}$$

Observe also that for all $\epsilon \ge 0$,

$$\mathbb{P}(X_1 - 0.06w + \epsilon < 0) = \mathbb{P}(X_1 - 0.06w < 0), \tag{7}$$
$$\mathbb{P}(X_1 - 0.06w - \epsilon < 0) = 1. \tag{8}$$

so that we have

$$-0.06w = \inf\{m \mid \mathbb{P}(X_1 + m < 0) \le \lambda\} = VAR_\lambda(X_1). \tag{9}$$

Since the value at risk is negative, the payoff X_1 is acceptable according to the value at risk measure.

Consider next, a diversified investment of $w/2$ in each of X_1 and X_2. The corresponding payoff is $Y = (X_1 + X_2)/2$. This payoff takes value $Y = -w$ when both bonds default, $Y = 0.06w/2 - w/2$ when one of the two bonds defaults, and $Y = 0.06w$ otherwise. Observe that

$$\mathbb{P}(Y + m < 0) \tag{10}$$

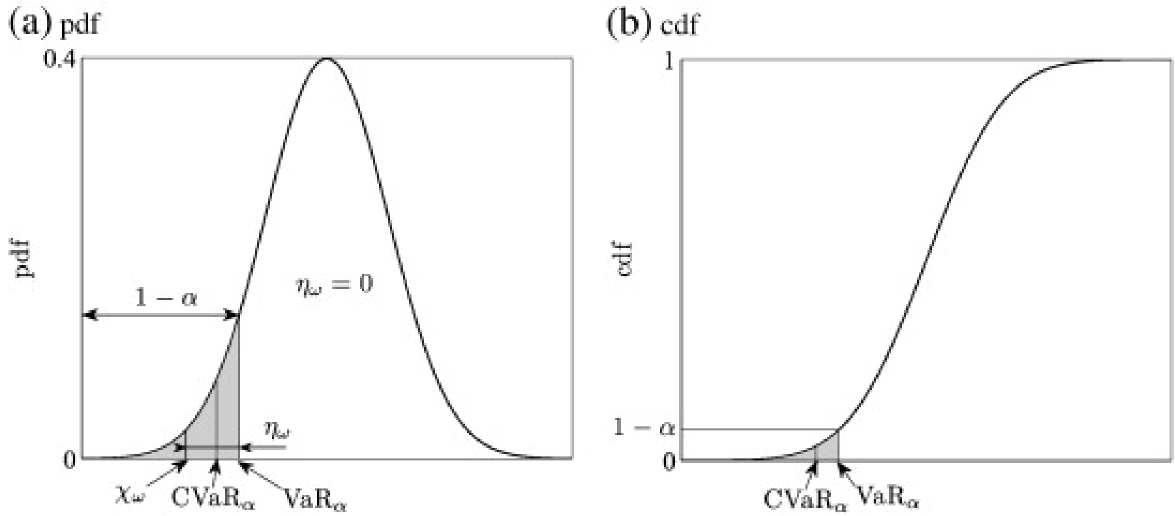

Figure 1: VAR and AVAR. From `https://www.sciencedirect.com/science/article/pii/S0140988312000606`

takes four possible values for different values of m:

$$0, p^2, p^2 + 2p(1-p) = 2p - p^2, 1. \tag{11}$$

Suppose that in addition to $p \leq \lambda$, we have $\lambda < 2p - p^2$. For example, we can set $p = 0.009$ and $\lambda = 0.01$. In this case ($p^2 < \lambda < 2p - p^2$), for every small $\varepsilon > 0$, we have

$$\mathbb{P}(Y + w/2 - 0.06w/2 + \varepsilon < 0) = \mathbb{P}(\text{both bonds default}) = p^2, \tag{12}$$
$$\mathbb{P}(Y + w/2 - 0.06w/2 - \varepsilon < 0) = \mathbb{P}(\text{at least one bond defaults}) \tag{13}$$
$$= 1 - (1-p)^2 = 2p - p^2. \tag{14}$$

Therefore,

$$w/2 - 0.06w/2 = 0.47w = \inf\{m \mid \mathbb{P}(X_1 + m < 0) \leq \lambda\} = VAR_\lambda(Y). \tag{15}$$

By comparison, investing in a single bond has $VAR_\lambda(X_1) = -0.06w$, but diversifying into two bonds has $VAR_\lambda(Y) = 0.47w$. This risk measure penalizes diversification!

This pushes us to find risk measures that takes into account the expected loss in the event of a loss. One example is Average Value at Risk:

$$AVAR_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda VAR_z(X)dz. \tag{16}$$

3 References

- Chapter 4 of textbook.

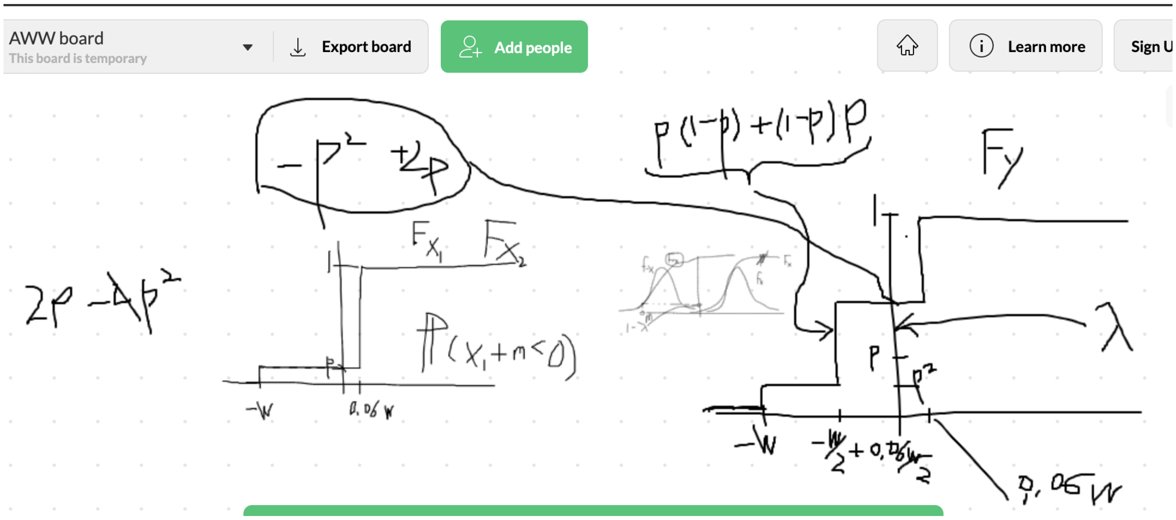

Figure 2: Example 2.3

- Examples: `https://statistik.econ.kit.edu/download/doc_secure1/7_StochModels.pdf`.
- Examples with different definitions: `http://www.cs.unh.edu/~mpetrik/pub/tutorials/risk/coherent_risk.pdf`