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9: Cooperative Game Theory in Supply Chains

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Supply chains are by definition made of multiple agents interacting and cooperating (suppliers, retailers, etc.). We can therefore use concepts from game theory to predict outcomes of these interactions. In INSE 6290, we cover competition or non-cooperative game theory. We will see that cooperative game theory can help find bargaining solutions.

1 Core

We consider a cooperative game with transferable utility described by

- A set of players (firms) N,
- A set of coalitions $\mathcal{S}$ of the form $S \subseteq N$,
- A characteristic function $v : \mathcal{S} \to \mathbb{R}$ that measures the amount of value $v(S)$ created by a coalition S. This is the amount of profit that must later be divided among the firms of the coalition.

Players are free to join and leave coalitions. For simplicity, we assume that the value $v(S)$ does not depend on the presence of other coalitions. This is realistic when the majority takes all (e.g., Bluray standard, majority-rule political systems), but less realistic in other situations (e.g., Android vs iOS, oil vs nuclear, etc.). We assume that there exists a common money (transferable utility) that is valued equally by all players, and that players can transfer this money.

Definition 1.1 (Core)**.** An allocation vector $x \in \mathbb{R}^N$ (of money among players) belongs the core if

$$\sum_{i \in N} x_i = v(N),$$

and for every coalition $S \subseteq N$:

$$\sum_{i \in S} x_i \geq v(S).$$

An alternative coalition S can block a feasible allocation if members of this coalition prefer a different allocation from the value $v(S)$. If the players are allocated $(x_1, \ldots, x_n)$, then no coalition S has an incentive to deviate from the coalition of n players. Think of the core as a contract for the distribution of wealth. If such a contract is in place, then no player is better off breaking the terms of the contract. Hence, the core is a stable contract for wealth redistribution, if it exists.

Figure 1: From passportchronicles.com

Example 1.1 (Miners, Bank Robbers (nonexistence))**.** Consider n miners who have found a large deposit of ores. Each piece of ore have value 1 and requires two miners to extract. The value of a coalition S is

$$v(S) = \begin{cases} |S|/2 & \text{if } |S| \text{ is even,} \\ (|S|-1)/2 & \text{otherwise.} \end{cases}$$

If $n = 2, 4, \ldots$, then the core is the allocation $(0.5, 0.5, \ldots, 0.5)$. If $n = 3$, the core is empty because there will be a miner with no allocation who would like to join a coalition if by himself or kick a third miner out of the coalition of three.

Example 1.2 (Engines (non-uniqueness))**.** Two knitters have each produced three engines. Each airplane requires a pair of engines and sells for 5 CAD. If they form a coalition, then they sell three airplanes and can make 15 CAD. The core contains all allocations of the form

$$(x, 15 - x), \quad x \in [5, 10].$$

Example 1.3 (Newboys and risk-pooling)**.** See INSE 6290 notes.

The core allows us to check if a coalition is station, but does not specify the exact distribution of wealth among coalition members. Next, we consider a solution concept that specifies not only the outcome in terms of coalitions, but also the outcome for each player.

2 Shapley value

Consider a cooperative game with transferable utility, players N and characteristic function v. The Shapley value (vector) for the game is an allocation vector $(\phi_1, \ldots, \phi_N) \in \mathbb{R}^N$, where for player $i \in N$:

$$\phi_i = \sum_{S \subseteq N \setminus i} \frac{|S|!(|N| - |S| - 1)!}{|N|!}(v(S \cup i) - v(S)).$$

Figure 2: From pintrest.com

The sum is over all subsets S that do not contain i. $|S|!(|N|-|S|-1)!$ is the number of ordering of players with exactly $|S|$ other players who joined the coalition before i. $|N|!$ is the number of orderings of all players. Think of a bucket with $|N|!$ identical blue balls (cf. Figure 2), each with a distinct ordering written on it. Suppose that you color in red all the balls where player i appears in the $(|S|+1)$-th position. The above ratio is the probability of taking a red ball from the bucket at random. Hence, the Shapley value is the expected marginal contribution of adding player i to a random coalition, where every coalition is equally likely.

The Shapley value has the following properties:

- Efficiency:

$$\sum_{i \in N} \phi_i = v(N),$$

- If $v(S \cup i) = v(S)$, then $\phi_i = 0$.
- If $v(S \cup i) = v(S \cup j)$, then $\phi_i = \phi_j$.

Example 2.1 (Refinery)**.** Consider a game with three players, player 1 and 2 have oil field each, player 3 has oil refinery plant. Assume that each player has the same operating cost. In order to produce petrol, we need crude from an oil field and a refinery plant. The characteristic function is therefore

$$v(S) = \begin{cases} 1 & \text{if } S = \{1,3\}, \{2,3\}, \{1,2,3\}, \\ 0 & \text{otherwise.} \end{cases}$$

For this small example, we can enumerate all orderings and subsets. We have:

$$\begin{aligned}
\phi_1 &= \sum_{S\subseteq N\setminus 1} \frac{|S|!(|N|-|S|-1)!}{|N|!}(v(S\cup 1)-v(S)) \\
&= \frac{1!\cdot 1!}{3!}(v(\{2\}\cup 1)-v(\{2\})) + \frac{1!\cdot 1!}{3!}(v(\{3\}\cup 1)-v(\{3\})) \\
&+ \frac{2!\cdot 0!}{3!}(v(\{2,3\}\cup 1)-v(\{2,3\})) + \frac{0!\cdot 2!}{3!}(v(\{1\})-v(\{\})) \\
&= 0 + 1/6 + 0 + 0.
\end{aligned}$$

By symmetry, $\phi_2 = 1/6$. By the efficiency property, $\phi_3 = 4/6$.

Example 2.2 (Fleet of trucks)**.** There is one owner of the fleet of trucks (player 0), and $N-1$ drivers (players 1 to $N-1$). Each driver contributes p to the total profit. However, without the trucks, no profit. Hence,

$$v(S) = \begin{cases} 0 & \text{if } 0 \notin S, \\ (|S|-1)p & \text{otherwise.} \end{cases}$$

The Shapley values are:

$$\begin{aligned}
\phi_0 &= \sum_{S\subseteq N\setminus i} \frac{|S|!(|N|-|S|-1)!}{|N|!}(|S|-1)p, \\
\phi_i &= \sum_{S\subseteq N\setminus i} \frac{|S|!(|N|-|S|-1)!}{|N|!} p 1_{[0\in S]}, \quad i \neq 0.
\end{aligned}$$

Half of the $|S|!(|N|-|S|-1)!$ contain player 0, hence the expected marginal gain from adding $i \neq 0$ is $p/2$, and $\phi_i = p/2$. By symmetry and efficiency, we have $\phi_0 = (N-1)p/2$.

Remark 1 (Rich get richer!). Observe that the profit going to the owner of the capital increases with the number of drivers! However, the profit of the driver is constant.

The following example is from Granot and Sosic: `http://pubsonline.informs.org/doi/pdf/10.1287/opre.51.5.771.16749`.

Example 2.3 (Inventory rebalancing)**.** N is a set of retailers of iPhones. The demand is random. Before observing the actual demand, each retailer must first place an order. Later, if there's unsatisfied demand at one retailer, excess inventory can be shipped from other retailers (this is called inventory rebalancing). The inventory rebalancing is done in such as way as to minimize total cost of shipping. The additional profit from inventory rebalancing (minus additional shipping costs) is shared among the retailers.

Example 2.4 (Ridesharing)**.** Two friends share a taxi. Two firms share a delivery truck. What is the characteristic function? Calculate the saving: total cost if each friend travels separately, then substract the cost of the taxi with two pickups and two dropoffs.

Remark 2. The Shapley value may not belong to the core: some players may form a coalition and improve their payoff.

3 References

- S. Froehlich, "Notes on Cooperative Game Theory and the Core" https://pdfs.semanticscholar.org/f4f4/dcc807da8f0d7d17adc6cce263e263fa6e34.pdf
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- Cachon and Netessine, "Game Theory in Supply Chain Analysis" http://opim.wharton.upenn.edu/~cachon/pdf/cachon_netessine_gt.pdf.