ELEC 372 LECTURE NOTES, WEEK 10

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Parts of these notes are adapted from the materials in the following references:

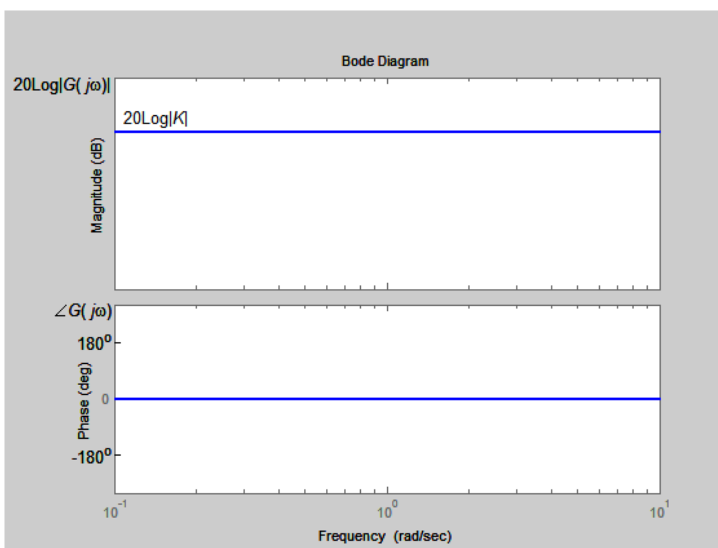
- Modern Control Systems by Richard C. Dorf and Robert H. Bishop, Prentice Hall.
- Feedback Control of Dynamic Systems by Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, Prentice Hall.
- Automatic Control Systems by Farid Golnaraghi and Benjamin C. Kuo, John Wiley & Sons, Inc., 2010.

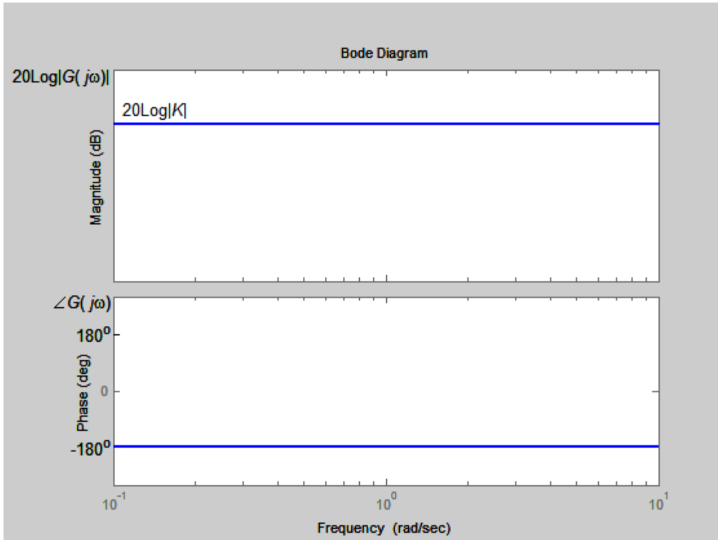
Bode plot (cont'd)

1. $G(s) = K$ (constant gain)

$$20\log\left|G(j\omega)\right| = 20\log\left|K\right|$$

$$\angle G(j\omega) = \begin{cases} 0 & K > 0 \\ -180^\circ & K < 0 \end{cases}$$





Note that since the horizontal axis in the Bode plot is logarithmic, only positive frequencies can be shown in the figure, with the zero frequency in $-\infty$ ($\omega \to 0 \Rightarrow \log\omega \to -\infty$).

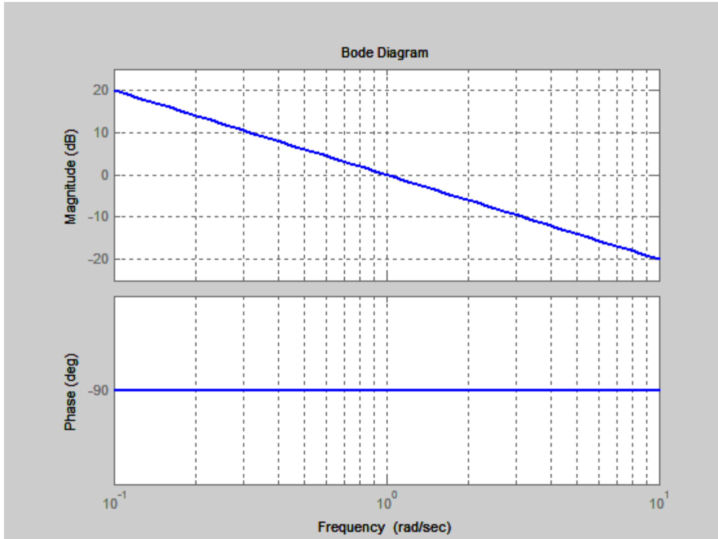
2.

a) $G(s) = \dfrac{1}{s}$ (integrator)

$$20\log\left|G(j\omega)\right| = 20\log\left|\frac{1}{j\omega}\right| = 20\log\frac{1}{\omega} = -20\log\omega$$

$$\angle G(j\omega) = \angle\frac{1}{j\omega} = -90^\circ$$

It is to be noted that since the horizontal axis is frequency in logarithmic scale or $\log\omega$, the equation $20\log\left|G(j\omega)\right| = -20\log\omega$ represents a straight line with the slope -20. In a logarithmic scale an interval between two frequencies with a ratio equal to 10 is called a decade. So, the range of frequencies from any arbitrary frequency ω_1 to $10\omega_1$ is called a decade. Therefore the slope of the line $-20\log\omega$ is -20 dB/dec. This is shown in the magnitude plot in the following figure.

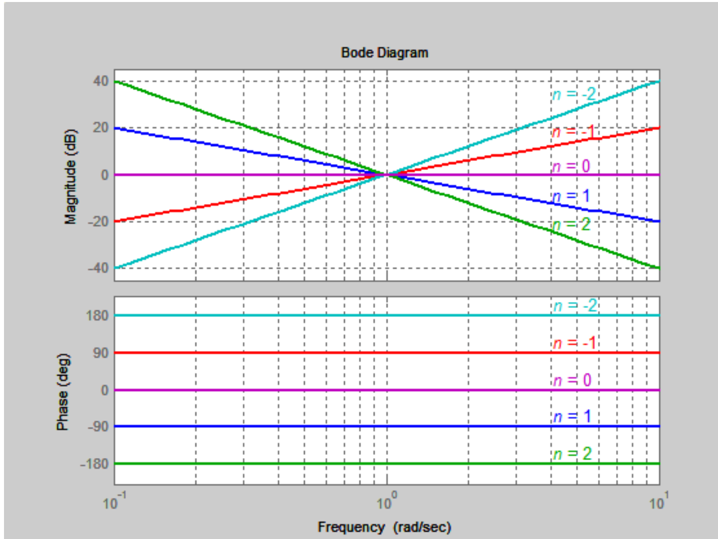


b) $G(s)=\dfrac{1}{s^n}$ (the cascade connection of n integrators)

$$20\log|G(j\omega)| = 20\log\left|\frac{1}{(j\omega)^n}\right| = 20\log\frac{1}{\omega^n} = -20n\log\omega$$

$$\angle G(j\omega) = \angle\frac{1}{(j\omega)^n} = -n\times 90^\circ$$

The magnitude plot of $G(j\omega)=\dfrac{1}{(j\omega)^n}$ is a straight line with the slope $-20n$ dB/dec. This can be seen from the magnitude plot in the following figure. The phase plot of $G(j\omega)=\dfrac{1}{(j\omega)^n}$ is constant and its value is equal to $-n\times 90^\circ$.



3. $G(s) = \dfrac{1}{\tau s + 1}$ (a first order pole with unity DC gain)

$$20\log|G(j\omega)| = 20\log\left|\frac{1}{j\omega\tau + 1}\right| = 20\log\frac{1}{\sqrt{(\omega\tau)^2 + 1}} = -20\log\sqrt{(\omega\tau)^2 + 1}$$

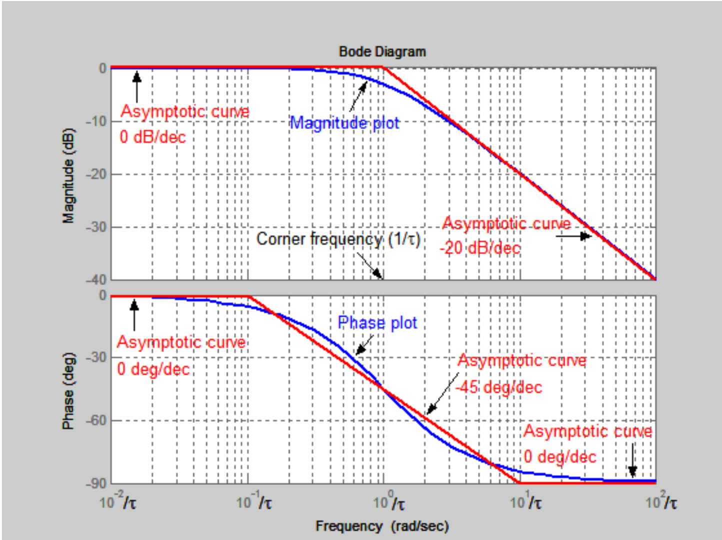
$$\angle G(j\omega) = \angle\frac{1}{j\omega\tau + 1} = -\tan^{-1}(\omega\tau)$$

$$\omega << \frac{1}{\tau} \Rightarrow \omega\tau << 1 \Rightarrow G(j\omega) \cong 1 \Rightarrow 20\log|G(j\omega)| \cong 0\,\text{dB},\ \ \angle G(j\omega) = 0^\circ$$

$$\omega >> \frac{1}{\tau} \Rightarrow \omega\tau >> 1 \Rightarrow G(j\omega) \cong \frac{1}{j\omega\tau} \Rightarrow 20\log|G(j\omega)| \cong -20\log(\omega\tau),\ \ \angle G(j\omega) = -90^\circ$$

$$\omega = \frac{1}{\tau} \Rightarrow 20\log|G(j\omega)| = -20\log\sqrt{2} \cong -3\,\text{dB},\ \ \angle G(j\omega) = -45^\circ$$

The Bode diagram (magnitude and phase plots) for $G(j\omega) = \dfrac{1}{j\omega\tau+1}$ is given in the following figure.



In this figure, both asymptotic curve and exact curve are given for magnitude and phase plot.

The asymptotic curve for the magnitude plot is obtained by considering the low frequency asymptote $20\log|G(j\omega)| \cong 0\,\text{dB}$ (which is a straight line with the slope 0 dB/dec) for the frequencies below $\omega = 1/\tau$ and considering the high frequency asymptote $20\log|G(j\omega)| \cong -20\log(\omega\tau) = -20\log\omega - 20\log\tau$ (which is a straight line with the slope -20 dB/dec) for the frequencies above $\omega = 1/\tau$.

The approximate curve for the phase plot is obtained by considering the low frequency asymptote $\angle G(j\omega) = 0^\circ$ (which is a straight line with the slope 0 deg/dec) for the frequencies below $\omega_1 = 0.1\times 1/\tau$ and considering the high frequency asymptote $\angle G(j\omega) = -90^\circ$ (which is a straight line with the slope 0 deg/dec) for the frequencies above $\omega_2 = 10\times 1/\tau$. For the frequencies between ω_1 and ω_2, a straight line with the slope -45 deg/dec is used (the approximate curve for the phase plot is not standard and some text-books may use different approximations for the frequencies around $1/\tau$).

For the first-order system $G(j\omega) = \dfrac{1}{j\omega\tau + 1}$, the maximum difference between the exact magnitude plot and asymptotic magnitude plot occurs at the frequency

$\omega = \dfrac{1}{\tau}$ and is equal to 3 dB (as it can be seen from the equations). This frequency is called the *corner frequency* or *break frequency*.

The asymptotic approximations (especially for the magnitude plot) are usually good approximations and are often used instead of the exact curves.

4. $G(s) = \tau s + 1$ (a first order zero with unity DC gain)

$$20\log|G(j\omega)| = 20\log|j\omega\tau + 1| = 20\log\sqrt{(\omega\tau)^2 + 1} = 20\log\sqrt{(\omega\tau)^2 + 1}$$

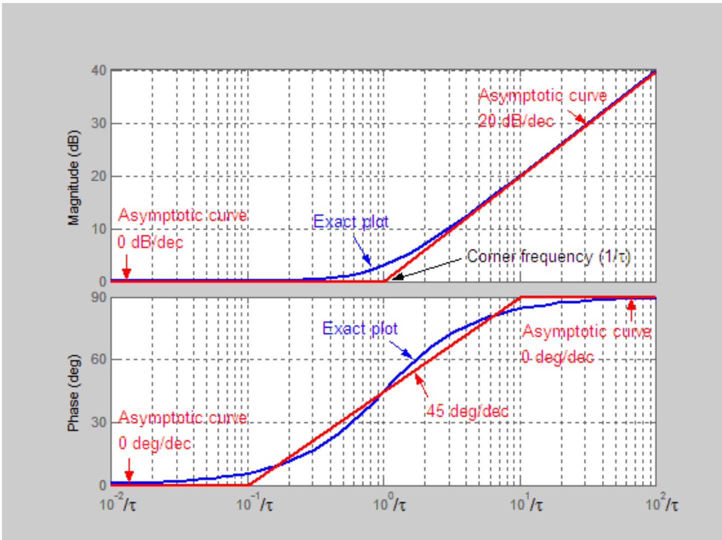
$$\angle G(j\omega) = \angle(j\omega\tau + 1) = \tan^{-1}(\omega\tau)$$

$$\omega << \frac{1}{\tau} \Rightarrow \omega\tau << 1 \Rightarrow 20\log|G(j\omega)| \cong 0\,\text{dB},\ \ \angle G(j\omega) = 0^\circ$$

$$\omega >> \frac{1}{\tau} \Rightarrow \omega\tau >> 1 \Rightarrow 20\log|G(j\omega)| \cong 20\log(\omega\tau),\ \ \angle G(j\omega) = 90^\circ$$

$$\omega = \frac{1}{\tau} \Rightarrow 20\log|G(j\omega)| = 20\log\sqrt{2} \cong 3\,\text{dB},\ \ \angle G(j\omega) = 45^\circ$$

The Bode diagram (magnitude and phase plots) for $G(j\omega) = (j\omega\tau + 1)$ is given in the following figure.



Note that the magnitude and phase plots of the system $G(s) = \tau s + 1$ are the mirror images of those of $\dfrac{1}{\tau s + 1}$ with respect to the frequency axis.

5. $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, $0 < \zeta \le 1$ (a standard second-order system with complex conjugate poles).

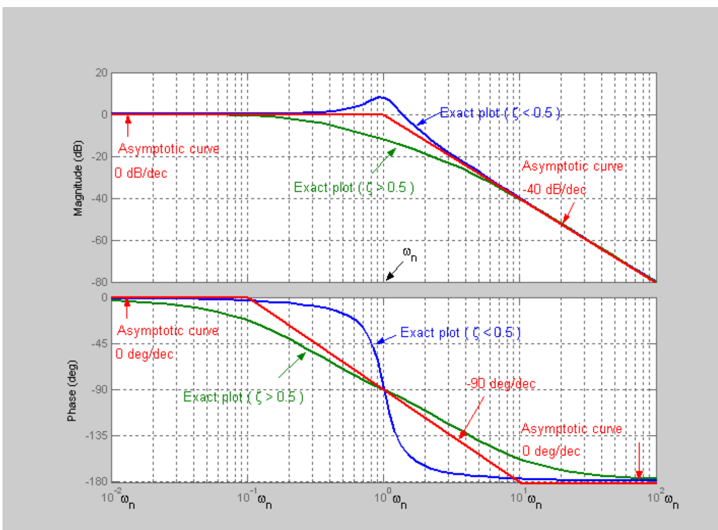
$$G(j\omega) = \frac{1}{(\frac{j\omega}{\omega_n})^2 + j\frac{2\zeta}{\omega_n}\omega + 1} = \frac{1}{\left[1 - (\frac{\omega}{\omega_n})^2\right] + j\frac{2\zeta}{\omega_n}\omega}$$

$$\omega << \omega_n \Rightarrow G(j\omega) \cong 1 \Rightarrow 20\log|G(j\omega)| \cong 0\,\text{dB},\ \angle G(j\omega) \cong 0^\circ$$

$$\omega >> \omega_n \Rightarrow G(j\omega) \cong -\frac{1}{(\frac{\omega}{\omega_n})^2} \Rightarrow 20\log|G(j\omega)| \cong -40\log\left|\frac{\omega}{\omega_n}\right|,\ \angle G(j\omega) \cong -180^\circ$$

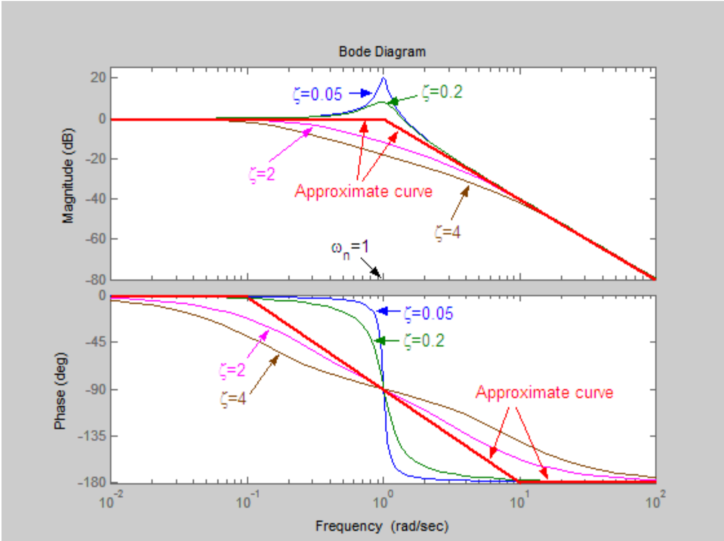
$$\omega = \omega_n \Rightarrow G(j\omega) = \frac{1}{j2\zeta} \Rightarrow 20\log|G(j\omega)| = -20\log(2\zeta),\ \angle G(j\omega) = -90^\circ$$

For the standard second-order systems, the frequency $\omega = \omega_n$ is the corner frequency and as it can be seen from the above equations, the value of the magnitude plot at this frequency depends on the value of the damping ratio ζ.



It is important to note that the above approximation is good only for underdamped or critically damped second-order system (i.e., a system with two complex conjugate poles). For the case when $\zeta > 1$, the second-order term must be written as a product of two first-order terms, and the Bode plot must be obtained by adding the Bode plots of the two first-order terms. Using this method, the resultant magnitude plot will have two distinct corner frequencies, as opposed to the standard second order term whose corner frequency is $\omega = \omega_n$.

The Bode plot for second-order systems with $\omega_n = 1$ and different values of ζ is given in the following figure. As it can be seen from this figure, the approximate curves for the magnitude and phase plot (red graphs) are good only for damping ratios around $\zeta = 0.5$.



The maximum value of the frequency response is called the resonant peak and is denoted by $M_{P\omega}$ or M_r. The frequency at which the resonant peak occurs is called the resonant frequency and is denoted by ω_r. The values of $M_{P\omega}$ and ω_r are given by:

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}, \quad \zeta < \frac{\sqrt{2}}{2}$$

$$M_{P\omega} = |G(\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \quad \zeta < \frac{\sqrt{2}}{2}$$

Note that the magnitude and phase plots of $G(s)=\dfrac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$ are the mirror images of those of $\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ with respect to the frequency axis.

The Bode plot of more complex systems can be obtained using the building blocks discussed above (constant gain, integrators, differentiators, first-order poles and zeros and second-order poles and zeros). Assume that the rational transfer function $G(s)$ is equal to the product of lower-order transfer functions $G_1(s),\ldots,\ G_n(s)$ as follows:

$$G(s)=G_1(s)\times\cdots\times G_n(s)$$

We will have:

$$20\log|G(j\omega)|=20\log|G_1(j\omega)|+\cdots+20\log|G_n(j\omega)|,$$

$$\angle G(j\omega)=\angle G_1(j\omega)+\cdots+\angle G_n(j\omega).$$

The approximate plot for the phase usually does not provide a good approximation if multiple poles and zeros exist close to each other.

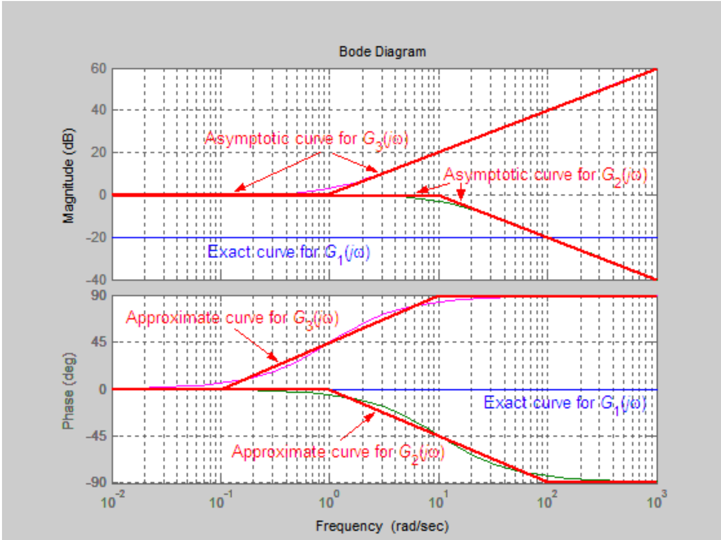
- One can also find the Bode plot for a delay system with the transfer function $G(s)=e^{-\tau s}$. For such a system, the magnitude of the frequency response $e^{-j\tau\omega}$ is equal to 1, and its phase is equal to $-\tau\omega$.

- **Example 10.1:** Find the Bode plot for the system $G(s)=\dfrac{s+1}{s+10}$.

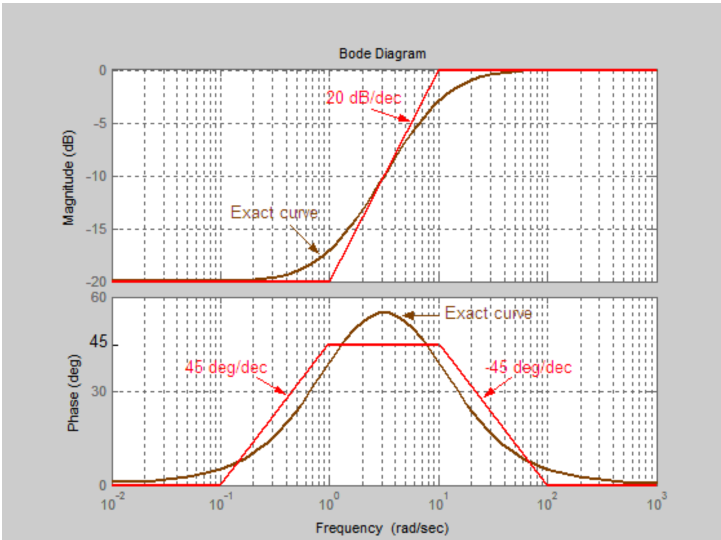
- **Solution:** The transfer function can be rewritten as $G(s)=\dfrac{1}{10}\times\dfrac{1}{0.1s+1}\times(s+1)$, which means that we can find the Bode plot of $G(s)$ by adding up the Bode plots of the systems $G_1(s)$, $G_2(s)$ and $G_3(s)$ given below:

$$G_1(s)=\frac{1}{10},\ G_2(s)=\frac{1}{0.1s+1},\ G_3(s)=(s+1)$$

The Bode plots for $G_1(s)$, $G_2(s)$ and $G_3(s)$ are given in the following figure (both exact curves and approximate curves are shown in the figure):



So, the Bode plot of the overall system will be:



Note that the red curves in this figure represent the approximate graphs for magnitude and phase in the Bode plot and are obtained by adding up the corresponding graphs for the individual terms $G_1(s)$, $G_2(s)$ and $G_3(s)$. Similarly, the brown curves represent the exact curves for the magnitude and phase in the Bode plot and are obtained by adding up the corresponding curves for the individual terms $G_1(s)$, $G_2(s)$ and $G_3(s)$.

The maximum phase of the exact phase plot is equal to 54.9° and occurs at the frequency $\omega = \sqrt{10}$ rad/sec (the point in the middle of $\omega = 1$ rad/sec and $\omega = 10$ rad/sec in the logarithmic scale).

- In order to draw the asymptotic approximation for the magnitude plot, use the following steps:
  1. Write the transfer function as a product of a constant term K, and the terms s^n, first-order terms $\tau s + 1$, and second-order terms $\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1$ ($\zeta < 1$) in the numerator and denominator.
  2. Find all corner frequencies ($|1/\tau|$ in the first-order terms and ω_n in the second-order terms).
  3. Start from low frequency with a slope equal to $-20n$ dB/dec, where n denotes the number of integrators in the transfer function (for example, if the transfer function has a term s^2 in the numerator only, then n is equal to -2 and the slope will be 40 dB/dec).
  4.
     a) At each corner frequency corresponding to a single pole, add -20 dB/dec to the slope.
     b) At each corner frequency corresponding to a single zero, add 20 dB/dec to the slope.
     c) At each corner frequency corresponding to a second-order pole, add -40 dB/dec to the slope.
     d) At each corner frequency corresponding to a second-order zero, add 40 dB/dec to the slope.
  5. The value of the asymptote at the smallest corner frequency is equal to $20\log\left|\frac{K}{\omega_1^n}\right|$, where K is the constant gain in the transfer function (see Step 1),

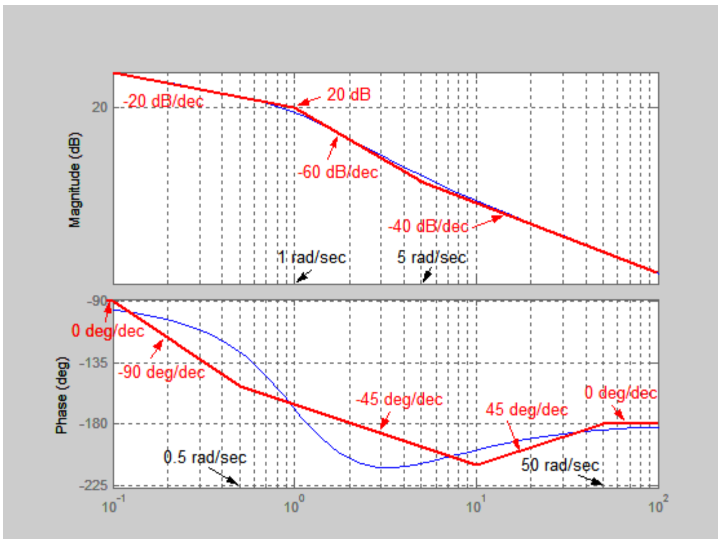
ω_1 is the smallest corner frequency (see Step 2), and n is the number of integrators in the transfer function (see Step 3).

- **Example 10.2:** Find the Bode plot for the system $G(s)=\dfrac{2s+10}{s(s^2+1.4s+1)}$.

- **Solution:** Using the above steps, we will have:

$$G(s)=10\frac{(0.2s+1)}{s(s^2+1.4s+1)}$$

The Bode plot is given in the following figure:



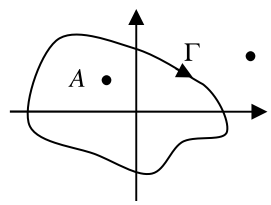
In this figure, the blue graphs represent the exact plots while the red curves represent the approximations.

The point $\omega=1$ rad/sec represents a corner frequency corresponding to a second-order pole and $\omega=5$ rad/sec represents a corner frequency corresponding to a first-order zero. That is why the slope of the asymptotes are changed by -40 dB/dec and +20 dB/dec at these two points, respectively. Also, the value of the low-frequency asymptote at the first corner frequency $\omega=1$ rad/sec is equal to $20\log 10=20$ dB.

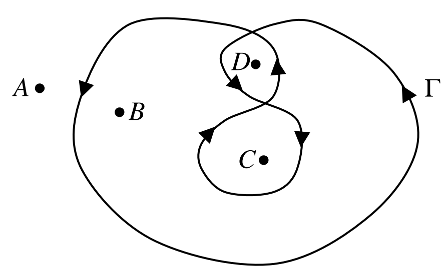
The phase plot is obtained by adding up the phase plots of $\dfrac{1}{s}$, $(0.2s+1)$, and $\dfrac{1}{s^2+1.4s+1}$.

Stability analysis in frequency domain

- The Nyquist diagram is very useful in stability analysis of LTI systems. For the stability analysis using the Nyquist method we need to know the principle of argument.
- Consider the following closed path:



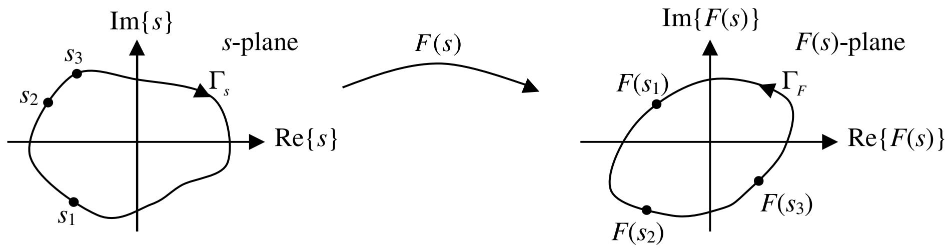
- A point or region is said to be encircled by the closed path (or the contour) if it is found inside the path. For example, point *A* in the above figure is encircled by the closed path Γ, since it is inside the closed path but point *B* is not encircled by the closed path Γ, since it is outside the path.
- The arrow on the closed path indicates the direction of the path.
- The positive direction for the encirclement of a point or a region by a closed path is the clockwise direction.
- To find the number of encirclements of a point X by a closed path Γ, consider a vector v from X to any point Y on the path. The net angle change of the vector v as Y traverses along Γ is a multiple of 360°, say $N \times 360^\circ$, where N is an integer. The number of encirclements of X by Γ is then equal to N. An encirclement in the clockwise direction is considered positive.
- **Example 10.3:** Find the number of encirclements of A, B, C, and D by the following closed path Γ.



- **Solution:** We have:

$$A: \quad N = 0$$
$$B: \quad N = -1$$
$$C: \quad N = 0$$
$$D: \quad N = -2$$

- For the stability analysis using the Nyquist method, it is important to know how a closed path in the s-plane is mapped by a function $F(s)$.



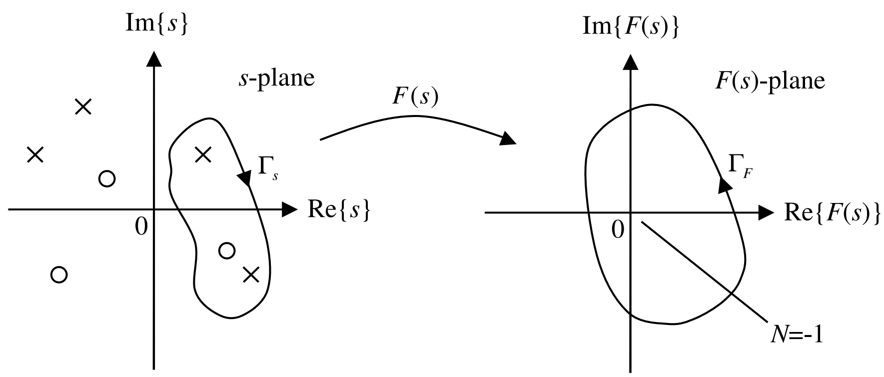
- **Principle of the argument:** Consider a rational function of s denoted by $F(s)$, and a closed path Γ_s in the clockwise direction such that:
  1. $F(s)$ is analytic on Γ_s and inside it except at some finite number of points inside Γ_s.
  2. $F(s)$ has no zero on Γ_s.

  Then we have:

$$N = Z - P$$

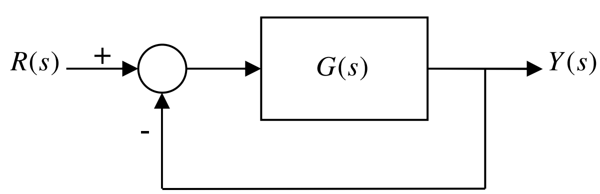
  where N is the number of encirclements of the origin of the $F(s)$-plane by the image of Γ_s under $F(s)$; Z is the number of zeros of $F(s)$ inside Γ_s; P is the number of poles of $F(s)$ inside Γ_s.
- In particular, the principle of the argument can be applied to the functions $F(s)$ of the form $F(s) = K \dfrac{\prod_{i=1}^{m}(s+z_i)}{s\prod_{j=1}^{n}(s+p_j)} e^{sT}$ (combination of a rational function and an exponential function of s).

- For example, consider the following map from the s-plane to the $F(s)$-plane where $F(s)$ is a rational function of s with a pole-zero configuration as shown in the s-plane.



As it can be seen from the $F(s)$-plane, the number of encirclements of the origin of this plane by the map $F(s)$ is equal to $N=-1$. On the other hand, the number of zeros and poles of $F(s)$ inside the closed path Γ_s are equal to $Z=1$ and $P=2$, respectively. This is confirmed by the principle of the argument which states $N=Z-P$.

- We now want to use the principle of the argument for the stability analysis of the closed-loop LTI systems.
- Consider the following closed-loop system:



- We have:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)}.$$

- The characteristic equation is:

$$1+G(s)=0$$