ELEC 372 LECTURE NOTES, WEEK 11

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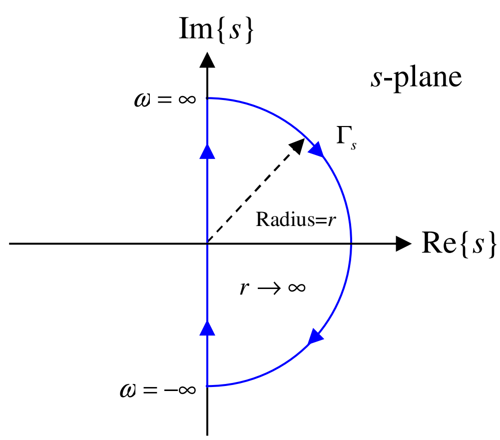
Parts of these notes are adapted from the materials in the following references:

- Modern Control Systems by Richard C. Dorf and Robert H. Bishop, Prentice Hall.
- Feedback Control of Dynamic Systems by Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, Prentice Hall.
- Automatic Control Systems by Farid Golnaraghi and Benjamin C. Kuo, John Wiley & Sons, Inc., 2010.

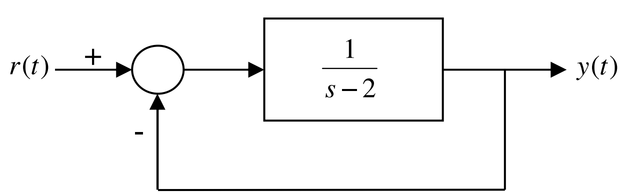
Stability analysis in frequency domain (cont'd)

- For stability analysis it is very important to note that the closed-loop poles are the zeros of $1+G(s)$ and the open-loop poles are the poles of $G(s)$ (or $1+G(s)$).
- To use the principle of the argument, we need to define a closed path that contains all points in the RHP. We define the following closed path as the Nyquist path or Nyquist contour:

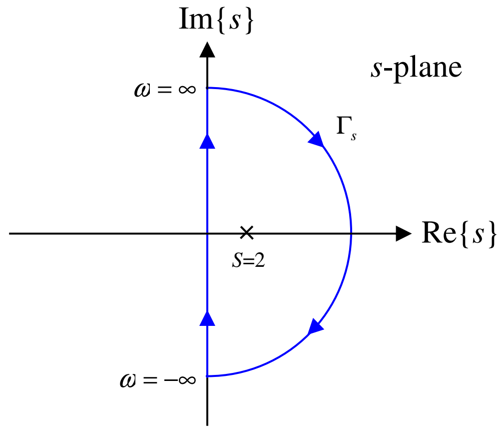


- Now, suppose that Z is the number of zeros of $1+G(s)$ inside Γ_s (unstable closed-loop poles) and P is the number of poles of $1+G(s)$ inside Γ_s (since the poles of $1+G(s)$ and $G(s)$ are equal, this is in fact number of unstable open-loop poles). In stability analysis P is known (from the open-loop transfer function) and it is desired to find Z. This can be accomplished by finding N and using the principle of the argument.

- We define $N(0,\Gamma_s,1+G(s))$ to be the number of encirclements of the origin by the image of Γ_s under $1+G(s)$.
- Using a simple shift in the $1+G(s)$-plane, it can be easily seen that $N(0,\Gamma_s,1+G(s)) = N(-1,\Gamma_s,G(s))$. In other words, the number of encirclements of the origin in the $(1+G(s))$-plane is equal to the number of encirclements of the point (-1,0) in the $G(s)$-plane. The point (-1,0) is usually referred to as the critical point.
- For the stability of the closed-loop system we must have $Z = 0$ or $N = -P$.
- If the open-loop transfer function is stable, we have $P = 0$ and then for the stability of the closed-loop system we must have $N = 0$.
- It is to be noted that the image of the imaginary axis in the Nyquist path under $G(s)$ is the Nyquist diagram of $G(j\omega)$.
- The Nyquist stability criterion states that a closed-loop system with negative feedback is stable if and only if the number of counterclockwise encirclements of the point (-1,0) in the $G(s)$-plane is equal to the number of poles of $G(s)$ in the RHP, where $G(s)$ represents the open-loop transfer function.
- **Example 11.1:** Use the Nyquist criterion to ascertain the stability of the following closed-loop system:



- **Solution:** We have:



From the above figure we know that $P=1$ (one open-loop pole in the RHP). Now, we have to find the map of the Nyquist path under $G(s)$. For the portion of the Nyquist path which is on the imaginary axis (from $s=-j\infty$ to $s=j\infty$), the map is in fact the Nyquist diagram of $G(s)$. We have:

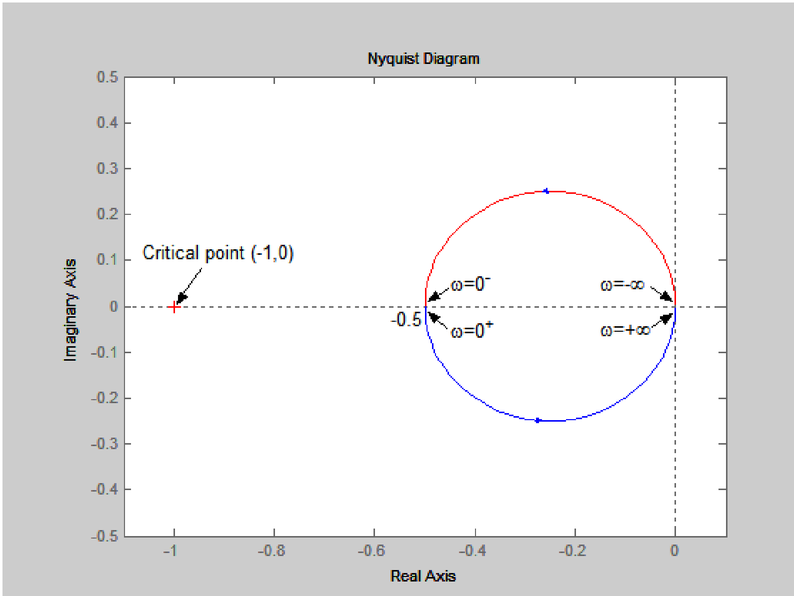
$$G(s)=\frac{1}{s-2}\Rightarrow G(j\omega)=\frac{1}{j\omega-2}$$

$$\omega=0\Rightarrow G(j\omega)=-0.5$$

$$\omega\rightarrow\infty\Rightarrow G(j\omega)\rightarrow\frac{1}{j\omega}\rightarrow 0\angle-90^{\circ}$$

$$G(j\omega)=\frac{-j\omega-2}{4+\omega^2}\Rightarrow\begin{cases}\mathrm{Re}\{G(j\omega)\}<0, & \omega>0\\ \mathrm{Im}\{G(j\omega)\}<0, & \omega>0\end{cases}$$

The portion of the Nyquist path that is a circle $s=re^{j\phi}$ ($r\rightarrow\infty$, $-90^{\circ}\le\phi\le 90^{\circ}$), will be mapped into the origin of the $G(s)$-plane (this is always the case for strictly proper transfer functions). Therefore, the Nyquist map will be as follows:



As it can be seen from the diagram, the number of encirclements of the critical point (-1,0) by the image of Γ_s under $G(s)$ is equal to zero or $N(-1,\Gamma_s,G(s))=0$. So, from the principle of the argument we will have:

$$Z = N + P = 1.$$

This implies that the closed-loop system has one pole in the RHP and so, it is unstable.

For this simple example, without using the Nyquist criterion, we knew that the characteristic equation of the closed-loop system is $1+G(s)=1+\dfrac{1}{s-2}=\dfrac{s-1}{s-2}$. So, the characteristic equation has a root at $s=1$ and so the closed-loop system is unstable. Nyquist criterion can be very useful for more complex systems in general.

- If there is a constant gain K in the open-loop transfer function, the characteristic equation will be $1+KG(s)$. In this case, for Nyquist stability analysis, one can find the map of the Nyquist path under $G(s)$ (similar to the previous case) but the critical point will be $-\dfrac{1}{K}$ instead of -1. This can be seen from the following equation:

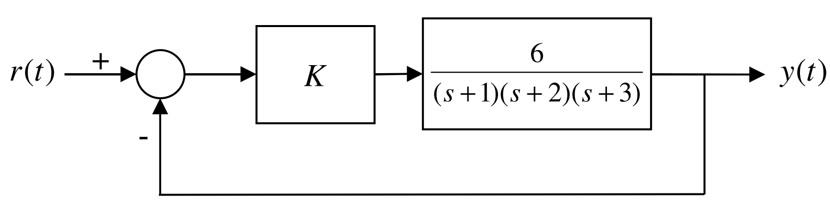
$$1+KG(s)=0 \Rightarrow \frac{1}{K}+G(s)=0.$$

This implies that the number of encirclements of the origin by the image of Γ_s under $1+KG(s)$ is equal to the number of encirclements of the point $(-\frac{1}{K},0)$ by the image of Γ_s under $G(s)$. In other words:

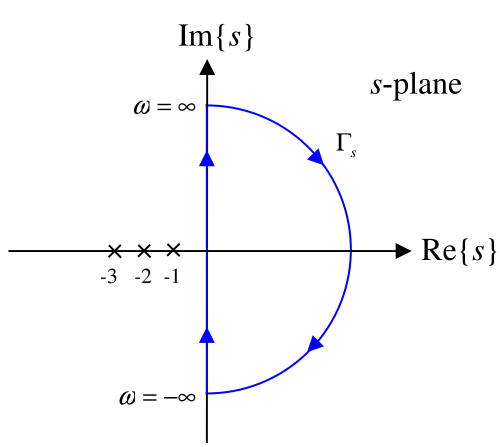
$$N(0,\Gamma_s,1+KG(s)) = N(-1,\Gamma_s,KG(s)) = N(-\frac{1}{K},\Gamma_s,G(s)).$$

So, in general we will use $(-\frac{1}{K},0)$ as the critical point.

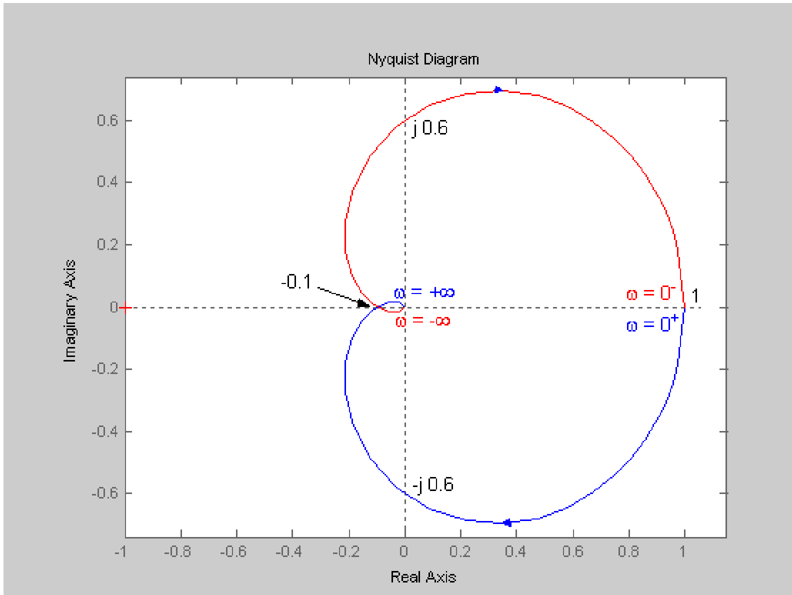
- It is to be noted that the stability analysis using the Nyquist criterion can be generalized to the non-unity feedback systems. If the forward path transfer function is denoted by $KG(s)$ and the transfer function in the feedback path is denoted by $H(s)$, the characteristic equation will be $1+KG(s)H(s)$. Therefore, the only difference is that for the stability analysis we must find the image of Γ_s under $G(s)H(s)$ instead of $G(s)$.
- **Example 11.2:** Use the Nyquist criterion to ascertain the stability of the following closed-loop system:



- **Solution:** We have:



Since there is no open-loop poles inside the Nyquist path, we have $P=0$. From the result of Example 9.5, we have the following Nyquist diagram (map of the imaginary axis under $G(s)$):



The number of encirclements of the critical point $(-\frac{1}{K},0)$ in this example depends on the value of K. We will have the following cases (note that $P=0$):

1. For $-\infty<-\frac{1}{K}<-\frac{1}{10}$ or equivalently $0<K<10$, we will have:

$$N(-\frac{1}{K},\Gamma_s,G(s))=0 \Rightarrow Z=N+P=0 \Rightarrow \text{The closed-loop system is stable}$$

2. For $-\frac{1}{10}<-\frac{1}{K}<0$ or equivalently $K>10$, we will have:

$$N(-\frac{1}{K},\Gamma_s,G(s))=2 \Rightarrow Z=N+P=2 \Rightarrow \text{The closed-loop system has 2 unstable poles}$$

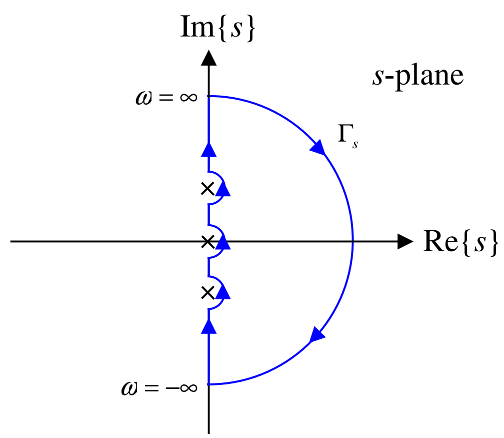
3. For $0<-\frac{1}{K}<1$ or equivalently $K<-1$, we will have:

$$N(-\frac{1}{K},\Gamma_s,G(s))=1 \Rightarrow Z=N+P=1 \Rightarrow \text{The closed-loop system has 1 unstable pole}$$

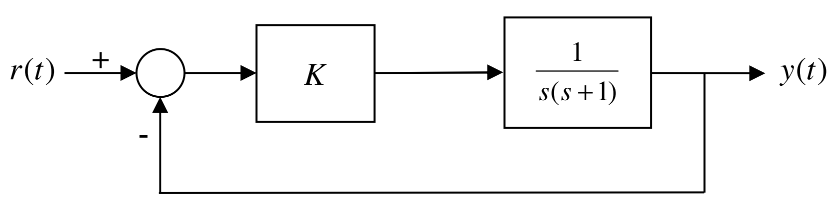
4. For $1 < -\dfrac{1}{K} < +\infty$ or equivalently $-1 < K < 0$, we will have:

$$N(-\frac{1}{K}, \Gamma_s, G(s)) = 0 \Rightarrow Z = N + P = 0 \Rightarrow \text{The closed-loop system is stable}$$

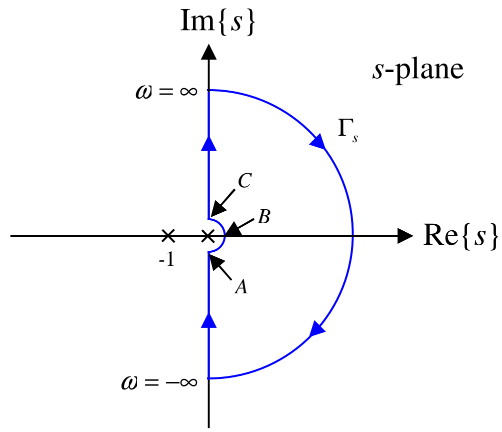
- Open-loop poles on the imaginary axis: Since the Nyquist path must not pass through any poles and zeros of the characteristic equation, if the open-loop transfer function has poles on the imaginary axis, we will have to use small semicircles as shown in the following figure to go around these poles.



- **Example 11.3:** Use the Nyquist criterion to ascertain the stability of the following closed-loop system:



- **Solution:** We will use the following Nyquist path:



The Nyquist diagram can be obtained as follows:

$$G(s)=\frac{1}{s(s+1)}\Rightarrow G(j\omega)=\frac{1}{j\omega(j\omega+1)}$$

$$\omega<<1\Rightarrow G(j\omega)\cong\frac{1}{j\omega}\rightarrow\infty\angle-90^{\circ}$$

$$\omega>>1\Rightarrow G(j\omega)\cong\frac{1}{(j\omega)^{2}}\rightarrow 0\angle-180^{\circ}$$

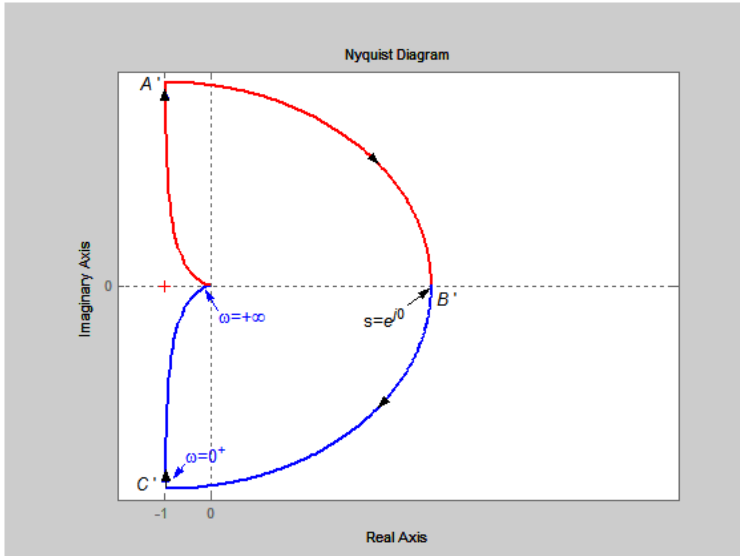
$$G(j\omega)=\frac{1}{j\omega-\omega^{2}}=\frac{-j\omega-\omega^{2}}{\omega^{2}+\omega^{4}}\Rightarrow\begin{cases}\mathrm{Re}\{G(j\omega)\}<0, & \omega>0\\ \mathrm{Im}\{G(j\omega)\}<0, & \omega>0\end{cases}$$

Since the semicircle is very close to the pole, it will be mapped into infinity but we need to find the shape of the map. For this purpose, we can choose three points A, B and C as shown in the figure and find the angles of the images of these points. Assume that the radius of the small semicircle is equal to $\varepsilon\rightarrow 0$. Define A', B' and C' as the images of A, B and C, respectively. We will have:

$$A:\quad s=\varepsilon e^{-j\frac{\pi}{2}}\Rightarrow G(s)\cong\frac{1}{s}=\frac{1}{\varepsilon e^{-j\frac{\pi}{2}}}\Rightarrow A'=Me^{j\frac{\pi}{2}}\rightarrow\infty\angle 90^{\circ}$$

$$B:\quad s=\varepsilon e^{j0}\Rightarrow G(s)\cong\frac{1}{s}=\frac{1}{\varepsilon e^{j0}}\Rightarrow B'=Me^{j0}\rightarrow\infty\angle 0^{\circ}$$

$$C:\quad s=\varepsilon e^{j\frac{\pi}{2}}\Rightarrow G(s)\cong\frac{1}{s}=\frac{1}{\varepsilon e^{j\frac{\pi}{2}}}\Rightarrow C'=Me^{-j\frac{\pi}{2}}\rightarrow\infty\angle-90^{\circ}$$



We know that the image of a curve in the complex plane under a rational function $G(s)$ is a conformal map. This means that the angles of the s-plane contour will be retained in the $G(s)$-plane. One can use this result to simplify the process of finding the image of the corners of the Nyquist path. For example, assume that using the techniques given for drawing the Nyquist diagram, assume that the image of the imaginary axis under the function $G(s)$ is obtained. If we move from $\omega = +\infty$ towards $\omega = 0^+$ on the imaginary axis, we will have to turn 90 degrees to the left at the C point. We have exactly the same thing in the image of the Nyquist path under $G(s)$ at the C' point.

We will now use the Nyquist criterion for the stability analysis of Example 11.3. Since there are no poles of the open-loop transfer function inside the Nyquist path, we have $P = 0$. We will have the following two cases:

1. For $-\infty < -\dfrac{1}{K} < 0$ or equivalently $K > 0$, we will have:

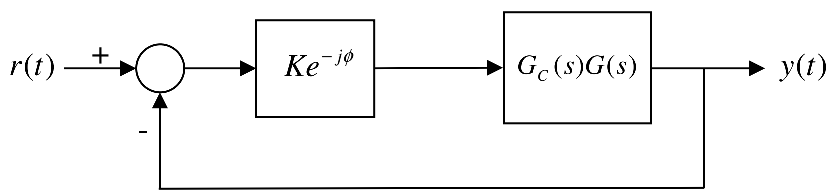
$$N(-\frac{1}{K}, \Gamma_s, G(s)) = 0 \Rightarrow Z = N + P = 0 \Rightarrow \text{The closed-loop system is stable}$$

2. For $0 < -\dfrac{1}{K} < +\infty$ or equivalently $K < 0$, we will have:

$$N(-\frac{1}{K},\Gamma_s,G(s))=1 \Rightarrow Z=N+P=1 \Rightarrow \text{The closed-loop system has 1 unstable pole}$$

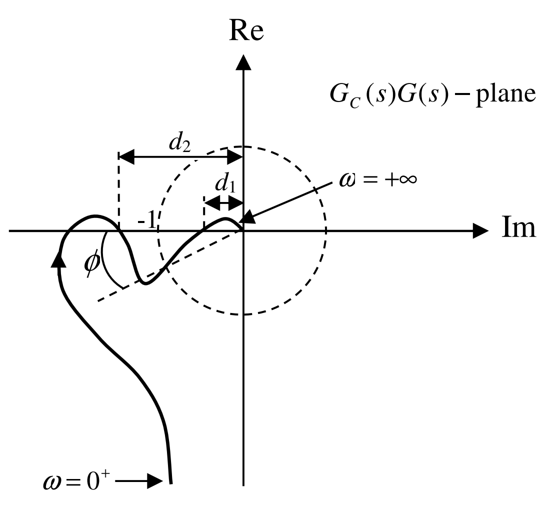
Conditional stability

- Consider the following closed-loop system:



where $G_C(s)G(s)$ represents the nominal open-loop transfer function (for $K=1$ and $\phi=0$).

- The term $Ke^{-j\phi}$ can represent the modeling error of the process $G(s)$. Assume that the nominal closed-loop system (corresponding to $K=1$ and $\phi=0$) is stable.
- For $\phi=0$, the largest and smallest values of K which result in a stable closed-loop system are called *upward gain margin* and *downward gain margin*, respectively.
- For $K=1$, the largest value of ϕ which results in a stable closed-loop system is called *phase margin*.
- The phase margin and gain margin are indicated in the following Nyquist diagram:



Assume that the closed-loop system is stable. From this figure we see that rotating the Nyquist plot by any angle less than $-\phi$ will not change the number of encirclements of the critical point $s=-1$. This implies that the phase margin is equal to ϕ. On the other hand, if the open-loop transfer function $G_C(s)G(s)$ is multiplied by any value greater than $\frac{1}{d_2}$ and less than $\frac{1}{d_1}$, the number of encirclements of the critical point will not change. This means that the upward gain margin is equal to $\frac{1}{d_1}$ (or $-20\log(d_1)$ in the dB scale) and the downward gain margin is equal to $\frac{1}{d_2}$ (or $-20\log(d_2)$ in the dB scale). Note that $d_1 < 1 < d_2$ and so, $\frac{1}{d_2} < 1 < \frac{1}{d_1}$.

- As a designer, we want the phase margin and upward gain margin to be as large as possible and downward gain margin to be as small as possible (to make sure that the closed-loop system will remain stable in the presence of uncertainty in the plant model).
- A system with a nonzero downward gain margin and finite upward gain margin is called a conditionally stable system. Usually for many practical systems the downward gain margin is equal to 0 and the term gain margin is referred to the upward gain margin.
- Phase margin and upward gain margin are denoted by PM and GM, respectively.
- PM and GM represent the relative position of the Nyquist plot with respect to the critical point and are used as quantitative measures for relative stability. In other words, PM represents the angle by which the Nyquist diagram should rotate, such that the Nyquist plot passes through the critical point. GM represents the gain by which the Nyquist diagram should be multiplied such that the Nyquist plot passes through the critical point.