ELEC 372 LECTURE NOTES, WEEK 1

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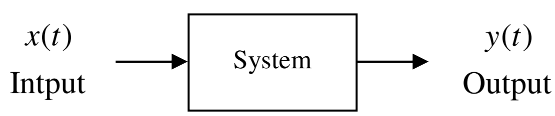
Parts of these notes are adapted from the materials in the following references:

- Modern Control Systems by Richard C. Dorf and Robert H. Bishop, Prentice Hall.
- Feedback Control of Dynamic Systems by Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, Prentice Hall.
- Automatic Control Systems by Farid Golnaraghi and Benjamin C. Kuo, John Wiley & Sons, Inc., 2010.

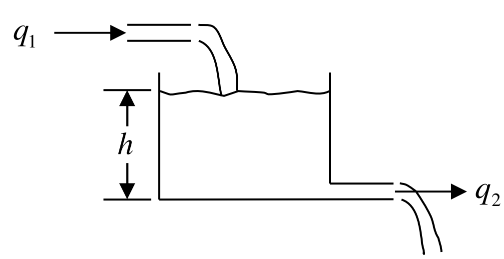
Introduction

- **System:** "An integrated whole even though composed of diverse, interacting, specialized structures and sub-junctions. *Notes:* Any system has a number of objectives and the weights placed on them may differ widely from system to system. (2) A system performs a function not possible with any of the individual parts. Complexity of the combination is implied." (*IEEE Standard Dictionary of Electrical and Electronics Terms*).
- A system can be an industrial process, automobile, airplane, etc. In a wider sense, the concept of system can be extended to economical, biological and social processes as well.
- A system (process or plant) is mathematically characterized by an operation between input(s) and output(s) and is denoted as follows:



$$y(t) = f(x(t))$$
$$x(t) \rightarrow y(t)$$

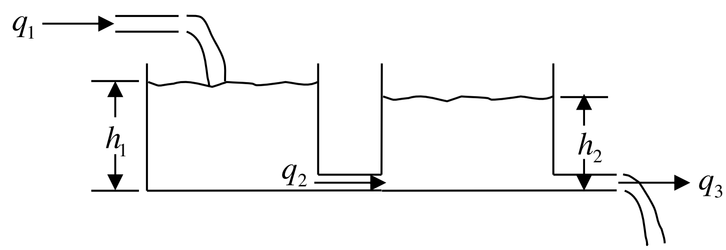
- **Control system:** "A system in which a desired effect is achieved by operating on the various inputs to the system until the output, which is a measure of the desired effect, falls within an acceptable range of values." (*IEEE Standard Dictionary of Electrical and Electronics Terms*).
- In one sentence, "control is the process of making a system variable adhere to a particular value, called the reference value." (G. F. Franklin, J. D. Powell and A. E. Naeini, *Feedback Control of Dynamic Systems*).
- A control system is a system that will provide a desired system response.
- Control is found in all sectors of industry and sometimes is referred to as a *hidden technology* "because of its essential importance to so many devices and systems while being mainly out of sight." (G. F. Franklin, J. D. Powell and A. E. Naeini, *Feedback Control of Dynamic Systems*, 6th Edition, Prentice Hall, 2010).
- Examples of control systems include quality control of manufactured products, automatic assembly line, machine-tool control, space-vehicle systems, missile guidance systems, robotics systems, disk drive, transportation systems, power systems, MicroElectroMechanical Systems (MEMS), nanotechnology.
- More recent applications of modern control theory include such non-engineering systems as biological, biomedical, and economic systems.
- As a simple example, consider the following water-tank system.



- In the water tank height control system it is desired to control the height of the water in the tank (h), which is also called controlled variable, by changing the flow rate of the input water (q_1) and/or the flow rate of the output water (q_2), which are also called manipulated variables.
- Different classes of control systems include:

- Multi-input multi-output (MIMO) control systems versus single-input single-output (SISO) control systems.
- Discrete-time control systems versus continuous-time control systems.
- Open-loop control systems versus closed-loop control systems.

- An example of MIMO control system is given below.



- Robots are examples of discrete-time control (computer-controlled) systems.
- An example of a closed-loop control system is the automobile, which is controlled by the driver. The driver makes changes in steering and velocity by measuring the relative position of the car through his/her eyes and comparing it with the desired position.
- Figures 1(a), (b) show the open-loop and closed-loop control of the height of the water.

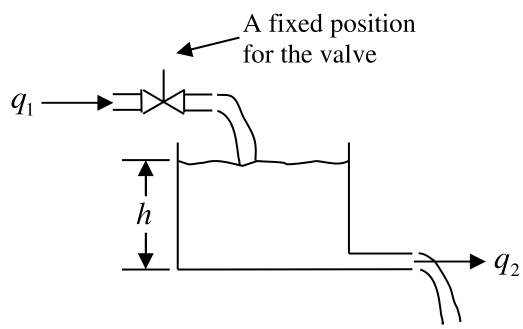


Figure 1(a): Open-loop control of the height of the water

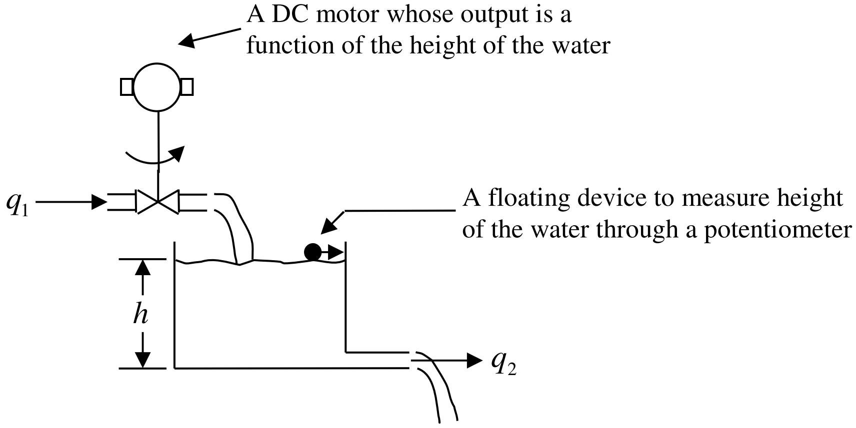
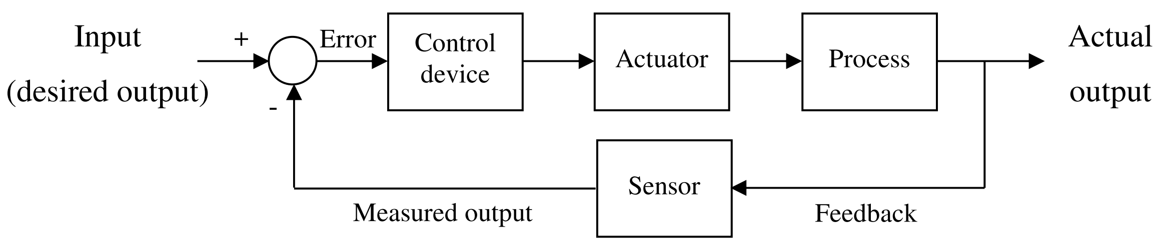


Figure 1(b): Closed-loop control of the height of the water

- Block diagram of a basic closed-loop control system is as follows.



- **Regulator:** "A system designed to maintain an output fixed against unknown disturbances is called a *regulating control* or a *regulator*. For example the cruise control of an automobile is a regulator." (G. F. Franklin, J. D. Powell and A. E. Naeini, *Feedback Control of Dynamic Systems*).
- **Servo:** "A system designed to follow a changing reference signal is called *tracking control* or a *servo*." (G. F. Franklin, J. D. Powell and A. E. Naeini, *Feedback Control of Dynamic Systems*). For example an anti-aircraft missile control system is a servo.
- Typical control objectives:
    - Good transient response
    - Good steady-state response
    - Disturbance rejection
    - Robustness

Mathematical Foundation

- **Linearity:** A linear system is a system that possesses the property of superposition, which is for any constant values a and b, the following equation is satisfied:

$$x_1(t) \to y_1(t), \quad x_2(t) \to y_2(t) \quad \Rightarrow \quad ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

- It can be easily verified that for linear systems, an input which is zero at all times, results in an output which is zero at all times.
- An example of a linear system: $y(t) = t\,x(t)$.
- **Time invariance:** A system is time invariant if the behaviour and characteristics of the system are fixed over time. Time invariant systems have the following property:

$$x(t) \to y(t) \Rightarrow x(t - t_0) \to y(t - t_0)$$

- An example of a time invariant system: $y(t) = x^2(t)$.
- **Causality:** A system is causal (or nonanticipative) if the output at any time depends only on values of the input at the present time and in the past.
- A capacitor is an example of a causal system whose input and output are related through the equation: $v(t) = \dfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$.
- **Bounded-Input Bounded-Output (BIBO) Stability:** A BIBO stable system is one in which a bounded input results in a bounded output.
- A resistor is an example of a BIBO stable system whose input and output are related through the following equation: $v(t) = Ri(t)$.
- A constant amplifier $y(t) = K\,x(t)$ is also an example of a BIBO stable system.
- **Convolution:** For linear time invariant (LTI) systems, the input $x(t)$, output $y(t)$ and the impulse response (the output when the input is a unit impulse function) $h(t)$ are related through the following equation which is called convolution.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau$$

- This means that an LTI system can be completely characterized by its impulse response.
- For LTI systems with impulse response $h(t)$:
  - Causality $\Leftrightarrow h(t) = 0, \quad t < 0$.
  - BIBO stability $\Leftrightarrow \int_{-\infty}^{+\infty} |h(t)| dt < \infty$.
- **Laplace transform:** Given a real-valued time function $x(t) \in R$, the unilateral Laplace transform is defined as follows:

$$X(s) = L(x(t)) = \int_{0^-}^{+\infty} x(t) e^{-st} dt, \quad s \in C, \quad X(s) \in C.$$

- The Laplace transform pair is also denoted by $x(t) \xleftrightarrow{L} X(s)$.
- Not every signal has a Laplace transform. For example, signals that grow faster than $e^{at}, \forall a$, such as e^{t^2} do not have a Laplace transform.
- **Properties of Laplace transform:** For a causal signal (a signal which is equal to zero for all $t < 0$), we have:
  - Linearity: $a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{L} a_1 X_1(s) + a_2 X_2(s), \quad \forall a_1, a_2 \in C$.
  - Time shifting: $x(t - t_0) \xleftrightarrow{L} e^{-st_0} X(s), \quad \forall t_0 > 0$.
  - Time scaling: $x(at) \xleftrightarrow{L} \frac{1}{a} X(\frac{s}{a}), \quad \forall a > 0$.
  - Convolution property: $x_1(t) * x_2(t) \xleftrightarrow{L} X_1(s).X_2(s)$.
  - Differentiation in the time domain:

    $$\frac{d^n x(t)}{dt^n} \xleftrightarrow{L} s^n X(s) - s^{n-1} x(0^-) - \cdots - s \frac{d^{n-2} x(0^-)}{dt^{n-2}} - \frac{d^{n-1} x(0^-)}{dt^{n-1}}.$$

    In particular:

    $$\frac{dx(t)}{dt} \xleftrightarrow{L} sX(s) - x(0^-), \text{ and } \frac{d^2 x(t)}{dt^2} \xleftrightarrow{L} s^2 X(s) - sx(0^-) - \frac{dx(0^-)}{dt}.$$

  - Differentiation in the *s*-domain: $-tx(t) \xleftrightarrow{L} \frac{d}{ds} X(s)$.
  - Integration in the time domain: $\int_{0^-}^{t} x(\tau) d\tau \xleftrightarrow{L} \frac{1}{s} X(s)$.

- Initial value theorem: For a signal $x(t)$ which contains no singularity at $t=0$, we have $x(0^+) = \lim_{s\to\infty} sX(s)$.
- Final value theorem: A signal for which $sX(s)$ is analytic for all s on the imaginary axis and the right half plane, has the following property: $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$.

- The Laplace transforms of some important functions are given below ($\delta(t)$ and $u(t)$ represent unit impulse function and unit step function, respectively).
  - $\delta(t) \xleftrightarrow{L} 1$.
  - $\dfrac{t^{n-1}}{(n-1)!}u(t) \xleftrightarrow{L} \dfrac{1}{s^n}$.
  - $e^{-at}u(t) \xleftrightarrow{L} \dfrac{1}{s+a}$.
  - $(\cos \omega_n t)u(t) \xleftrightarrow{L} \dfrac{s}{s^2+\omega_n^2}$.
  - $(\sin \omega_n t)u(t) \xleftrightarrow{L} \dfrac{\omega_n}{s^2+\omega_n^2}$.

- **Example 1.1:** Find the solution of the following differential equation:

$$\frac{dy(t)}{dt} + y(t) = \delta(t), \quad y(0^-) = 0.$$

Solution: By taking the Laplace transform of both sides of the above differential equation, we will get the following algebraic equation:

$$sY(s) - y(0^-) + Y(s) = 1.$$

This results in:

$$Y(s) = \frac{1}{s+1} \Rightarrow y(t) = e^{-t}u(t).$$

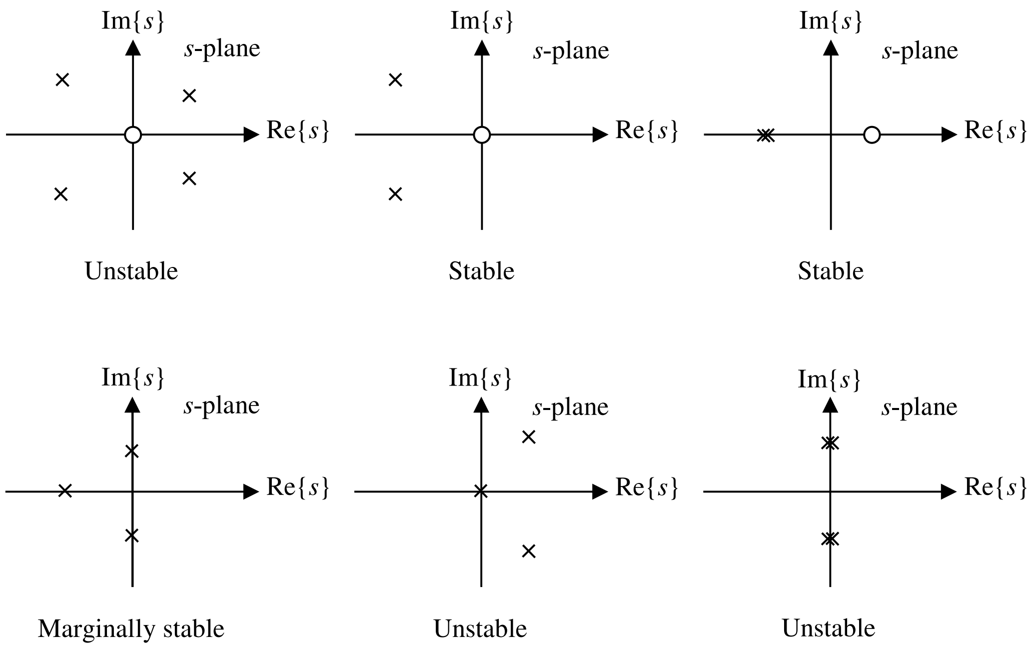
- In this course, we are mainly concerned with the systems whose inputs and outputs are related through a linear constant coefficient differential equation of the following form:

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}, \quad a_N = 1, \quad N \geq M \ . \tag{1.1}$$

- If $N = M$, then the system is called "proper".
- If $N > M$, then the system is called "strictly proper". All physical systems are strictly proper.
- Transfer function: The transfer function of a LTI system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero.
- The transfer function of a system with the differential equation given by (1.1) is a rational function of s as follows:

$$H(s) := \frac{Y(s)}{X(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0} := \frac{b(s)}{a(s)}$$

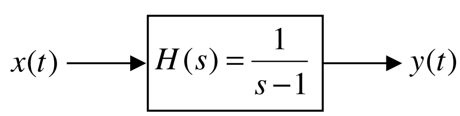
- The roots of $b(s)$ are called the zeros and the roots of $a(s)$ are called the poles of the transfer function.
- The behaviour of a system depends highly on the location of its poles and zeros.
- A causal system with a rational transfer function and $N \geq M$ is BIBO stable if and only if all of its poles lie in the open left-half plane (LHP). For practical reasons, a system with simple poles on the imaginary axis and no poles in the right-half plane (RHP) is referred to as a *marginally stable* system because only a bounded input with exactly the same frequency as the imaginary poles will result in an unbounded output. For example if a system has a pair of poles $s = \pm j\omega_0$ and all other poles are in the LHP, only the input signal $A\cos(\omega_0 t + \theta)$ (which is a bounded signal) for any value of A and θ will cause an unbounded output. Sometimes an integrator (which has a single pole in the origin) is considered stable.
- For example, consider systems with the following pole-zero configurations:



- *Step response* of an LTI system is the output of the system when the input is a unit step and all initial conditions are zero.
- Step input is one of the commonly used test signals in the control systems. For example changing the set-point in the automobile cruise control means applying a step input to the system.
- *Impulse response* $h(t)$ is equal to the derivative of the step response $s(t)$ as follows:

$$h(t) = \frac{ds(t)}{dt}.$$

- **Example 1.2:** Find the step response of the following system:



- **Solution:** $\frac{Y(s)}{X(s)} = \frac{1}{s-1}, \quad X(s) = \frac{1}{s},$

$$Y(s) = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s} \text{ (partial fraction expansion) } \Rightarrow y(t) = (e^t - 1)u(t).$$

This system is unstable (because of the RHP pole, step response goes to infinity as time goes to infinity. In other words the output is not finite while the input is finite).

It is to be noted that the final value theorem cannot be applied to $y(t)$ as $sY(s) = \frac{1}{(s-1)}$ is not analytic in the RHP (it has a pole in the RHP). If one applies the final value theorem to $y(t)$ by mistake, it will result in $y(\infty) = -1$ which is not correct.