ELEC 372 LECTURE NOTES – WEEK 2

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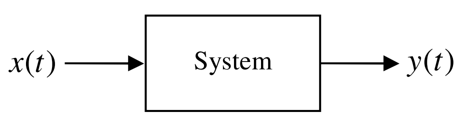
Parts of these notes are adapted from the materials in the following references:

- Modern Control Systems by Richard C. Dorf and Robert H. Bishop, Prentice Hall.
- Feedback Control of Dynamic Systems by Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, Prentice Hall.
- Automatic Control Systems by Farid Golnaraghi and Benjamin C. Kuo, John Wiley & Sons, Inc., 2010.

Linearization of nonlinear systems

- We linearize a nonlinear system around a nominal operating point.



1) **Static systems:** The general form of a static nonlinear system is $y = f(x)$. Let the system operate around the point (x_0, y_0), where $y_0 = f(x_0)$ (this point is referred to as the operating point). Then, using Taylor-series expansion, one can write:

$$f(x) = f(x_0) + \left.\frac{\delta f}{\delta x}\right|_{x=x_0} (x - x_0) + \cdots$$

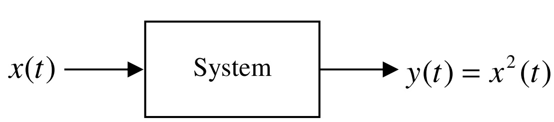
Neglecting the second order term and higher:

$$y \cong y_0 + \left.\frac{\delta f}{\delta x}\right|_{x=x_0} (x - x_0)$$

Define $\Delta y := y - y_0, \quad \Delta x := x - x_0 \quad \Rightarrow \quad \Delta y \cong \left.\frac{\delta f}{\delta x}\right|_{x=x_0} \Delta x$ (linear approximation)

This is called linearized or incremental model.

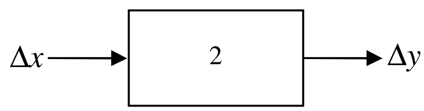
- **Example 2.1:** Consider the following static nonlinear system:



Linearize this system around the operating point $x_0 = 1$, $y_0 = 1$ (note that $y_0 = {x_0}^2$).

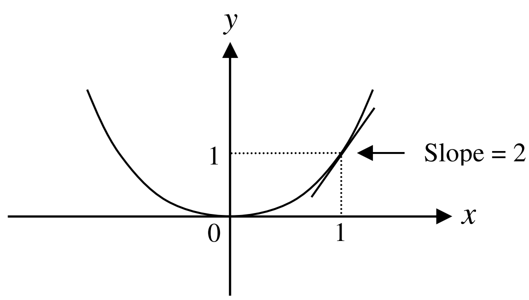
- **Solution:** $\left.\frac{\delta f}{\delta x}\right|_{x=x_0} = 2x_0 = 2 \quad \Rightarrow \quad \Delta y \cong 2\Delta x$ (linear approximation)

$y = y_0 + \Delta y \cong 1 + 2\Delta x$



Linear approximation

x	1	1.01	1.1	1.5	2	10
x^2	1	1.0201	1.21	2.25	4	100
$1+2\Delta x$	1	1.02	1.2	2	3	19



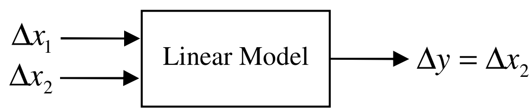
- The accuracy of the linear model decreases as we get farther from the operating point.
- We use linear approximation because we have very simple and powerful techniques for analyzing linear systems.
- **Example 2.2:** Consider a static nonlinear system with two inputs x_1 and x_2, and one output y, where $y = {x_1}^2 {x_2}^2 + x_2 + 2{x_1}^2$. Linearize this system around the operating point $y = x_1 = x_2 = 0$.
- **Solution:** Here, we have $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Therefore, $\frac{\delta f}{\delta x}\Delta x = \frac{\delta f}{\delta x_1}\Delta x_1 + \frac{\delta f}{\delta x_2}\Delta x_2$.

$$\left.\frac{\delta f}{\delta x_1}\right|_{\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}} = (2x_1x_2^{\ 2}+4x_1)\Big|_{x_1=x_2=0} = 0\,,$$

$$\left.\frac{\delta f}{\delta x_2}\right|_{\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}} = (2x_1^{\ 2}x_2+1)\Big|_{x_1=x_2=0} = 1\,,$$

$$\Delta y \cong \left.\frac{\delta f}{\delta x_1}\right|_{\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}} \Delta x_1 + \left.\frac{\delta f}{\delta x_2}\right|_{\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}} \Delta x_2 = \Delta x_2\,.$$

This gives the following linear approximation:



2) **General linearization (for dynamic systems):** Consider the following system

$$\dot{y}(t) = f(y(t), x(t), t)$$

(note that $\dot{y} := \dfrac{dy}{dt}$).

- We will linearize the system around a nominal operating trajectory $y_0(t)$, which corresponds to the nominal input $x_0(t)$, as follows:

$$\dot{y}_0(t) = f(y_0(t), x_0(t), t)$$

- If $\dot{y}_0(t) = 0$, the nominal operating point is called an equilibrium point.
- We have:

$$x(t) = x_0(t) + \Delta x(t), \quad y(t) = y_0(t) + \Delta y(t),$$

$$\frac{d}{dt}(y_0(t) + \Delta y(t)) = f(y_0(t) + \Delta y(t), x_0(t) + \Delta x(t), t),$$

$$\Rightarrow \underline{\underline{\dot{y}_0(t)}} + \Delta\dot{y}(t) = \underline{\underline{f(y_0(t), x_0(t), t)}} + \left.\frac{\delta f}{\delta y}\right|_{\substack{x=x_0\\y=y_0}} \Delta y(t) + \left.\frac{\delta f}{\delta x}\right|_{\substack{x=x_0\\y=y_0}} \Delta x(t) + \cdots$$

$$\Rightarrow \Delta\dot{y}(t) \cong a(t)\Delta y(t) + b(t)\Delta x(t) \quad \text{(linear differential equation)} \qquad (2.1)$$

- where:

$$a(t) := \left.\frac{\delta f}{\delta y}\right|_{\substack{x=x_0 \\ y=y_0}}, \quad b(t) := \left.\frac{\delta f}{\delta x}\right|_{\substack{x=x_0 \\ y=y_0}}.$$

- **Example 2.3:** Linearize the nonlinear system represented by the following differential equation:

$$z\frac{d^2 z}{dt^2} + x^2\frac{dz}{dt} + \sqrt{z} = x \tag{2.2}$$

around the equilibrium point $x_0 = \sqrt{z_0}$, where x_0 and z_0 are constants.

- **Solution:** We will solve this problem using two different approaches.

1. Systematic Approach: The given second order differential equation can be written as a set of two first order differential equations as follows:

$$y_1(t) := z(t), \quad y_2(t) := \dot{z}(t) \quad \Rightarrow \quad y_1\dot{y}_2 + x^2 y_2 + \sqrt{y_1} = x$$

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = \frac{-x^2}{y_1}y_2 - \frac{1}{\sqrt{y_1}} + \frac{x}{y_1} \quad \Rightarrow \quad \begin{bmatrix}\dot{y}_1 \\ \dot{y}_2\end{bmatrix} = \begin{bmatrix} y_2 \\ \dfrac{-x^2}{y_1}y_2 - \dfrac{1}{\sqrt{y_1}} + \dfrac{x}{y_1}\end{bmatrix}.$$

So, here we have:

$$y(t) := \begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}, \quad f(y(t), x(t), t) := \begin{bmatrix} y_2 \\ \dfrac{-x^2}{y_1}y_2 - \dfrac{1}{\sqrt{y_1}} + \dfrac{x}{y_1}\end{bmatrix}.$$

One can now use equation (2.1) to find the linearized model for (2.2).

2. Direct substitution: An easier approach to solve this problem would be finding the linear approximation directly by replacing the static nonlinear functions in (2.2) with their first-order approximations, and neglecting products of two or more first-order terms $\Delta x(t)$, $\Delta z(t)$, $\dfrac{d\Delta z(t)}{dt}$, and $\dfrac{d^2\Delta z(t)}{dt^2}$, as follows:

$$x(t) = x_0(t) + \Delta x(t), \quad z(t) = z_0(t) + \Delta z(t),$$

$$(z_0 + \Delta z)\frac{d^2}{dt^2}(z_0 + \Delta z) + (x_0 + \Delta x)^2\frac{d}{dt}(z_0 + \Delta z) + \sqrt{z_0 + \Delta z} = x_0 + \Delta x,$$

$$\sqrt{z_0+\Delta z} \cong \sqrt{z_0} + \frac{1}{2\sqrt{z_0}}\Delta z, \quad (x_0+\Delta x)^2 \cong x_0^{\,2} + 2x_0\Delta x,$$

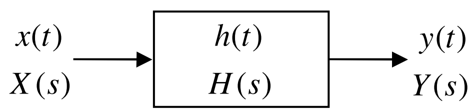
$$\underline{z_0\frac{d^2 z_0}{dt^2}} + \underline{\Delta z\frac{d^2 z_0}{dt^2}} + z_0\frac{d^2\Delta z}{dt^2} + \underline{\underline{\Delta z\frac{d^2\Delta z}{dt^2}}} + \underline{x_0^{\,2}\frac{dz_0}{dt}} + \underline{2x_0\Delta x\frac{dz_0}{dt}} + x_0^{\,2}\frac{d\Delta z}{dt}$$

$$+\underline{\underline{2x_0\Delta x\frac{d\Delta z}{dt}}} + \underline{\underline{\underline{\sqrt{z_0}}}} + \frac{1}{2\sqrt{z_0}}\Delta z \cong \underline{\underline{\underline{x_0}}} + \Delta x,$$

$$\Rightarrow z_0\frac{d^2\Delta z}{dt^2} + x_0^{\,2}\frac{d\Delta z}{dt} + \frac{1}{2\sqrt{z_0}}\Delta z \cong \Delta x\,.$$

Block diagram representation

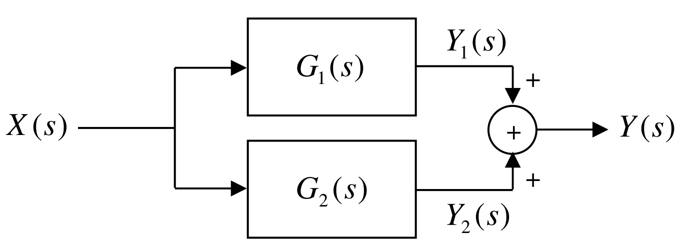
- An LTI system is usually represented by a block as follows:



$$y(t) = h(t) * x(t)$$

$$Y(s) = H(s)X(s)$$

- A system may consist of several interconnected subsystems.
- One may use different techniques to simplify the topology of a block diagram and find the overall transfer function of the system. This is called block diagram reduction.
- **Elementary block diagrams:**
  - **Interconnection in parallel:**

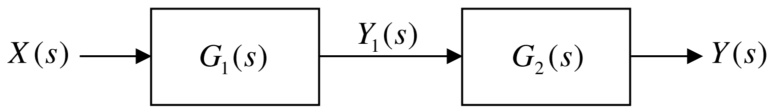


$$Y(s) = Y_1(s) + Y_2(s)$$
$$= G_1(s)X(s) + G_2(s)X(s)$$

$$= \underbrace{(G_1(s) + G_2(s))}_{G(s)} X(s)$$

This implies that the parallel interconnection of two systems with the transfer functions $G_1(s)$ and $G_2(s)$ is equivalent to one system with the transfer function $G(s) = G_1(s) + G_2(s)$.

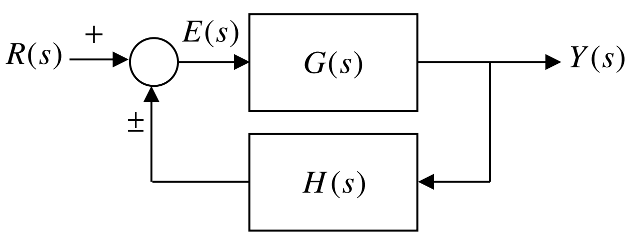
- **Interconnection in series:**



$$Y(s) = G_2(s)Y_1(s)$$
$$= \underbrace{G_2(s)G_1(s)}_{G(s)} X(s)$$

This implies that interconnection of two systems in series with the transfer functions $G_1(s)$ and $G_2(s)$ is equivalent to one system with the transfer function $G(s) = G_2(s)G_1(s)$.

- **Feedback interconnection:**

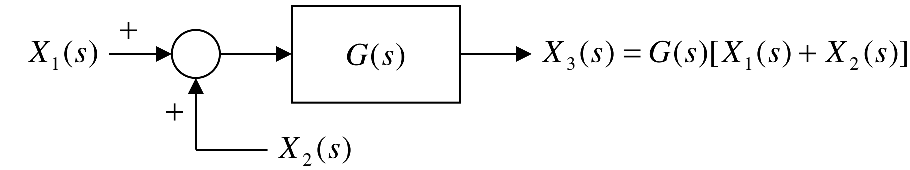


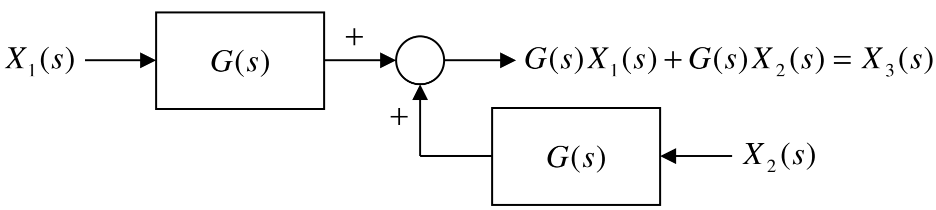
$$Y(s) = G(s)E(s)$$
$$= G(s)[R(s) \pm H(s)Y(s)]$$
$$\Rightarrow [1 \mp G(s)H(s)]Y(s) = G(s)R(s)$$
$$\Rightarrow Y(s) = \frac{G(s)}{1 \mp G(s)H(s)} R(s)$$

This implies that the feedback interconnection of two systems with the transfer functions $G(s)$ and $H(s)$ is equivalent to one system with the transfer function $G_{cl}(s) = \dfrac{G(s)}{1 - G(s)H(s)}$ for positive feedback or $G_{cl}(s) = \dfrac{G(s)}{1 + G(s)H(s)}$ for negative feedback.

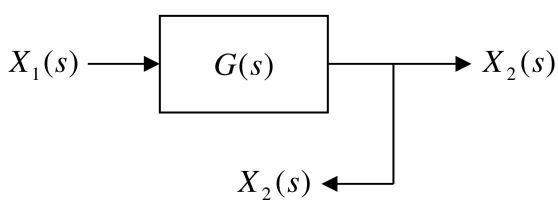
- **Other useful block diagram reduction techniques** (Modern Control Systems by Richard C. Dorf and Robert H. Bishop, Prentice Hall)**:**

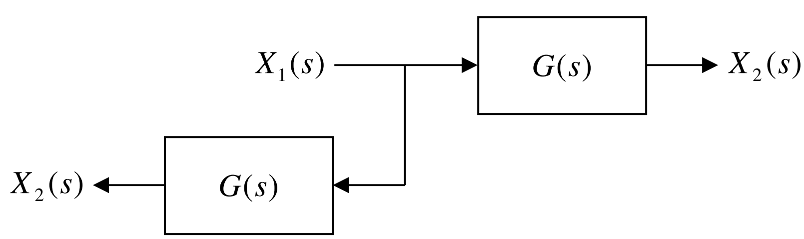
1) Moving a summing point behind a block:



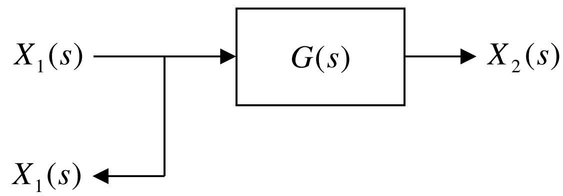


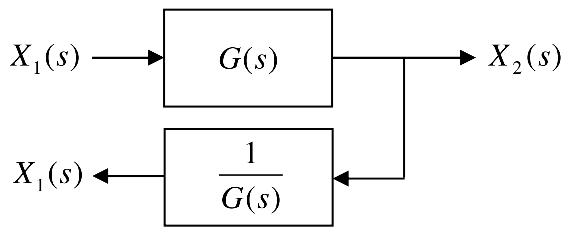
2) Moving a pickoff point ahead of a block:



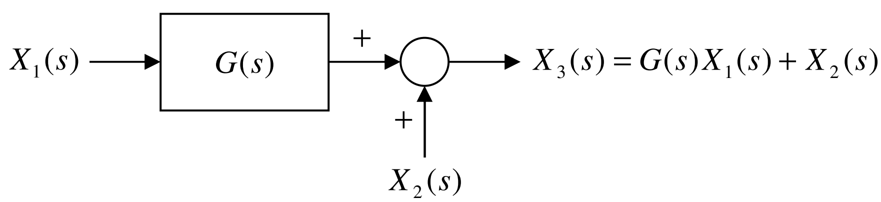


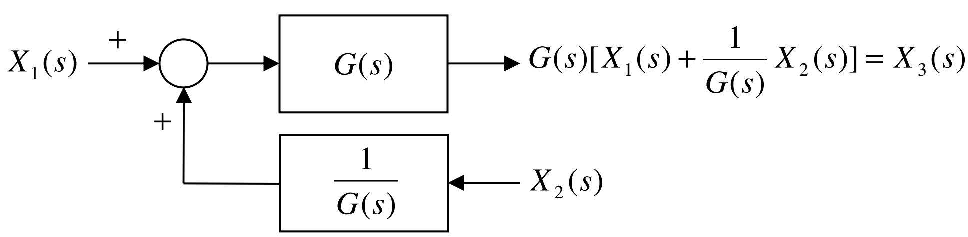
3) Moving a pickoff point behind a block:





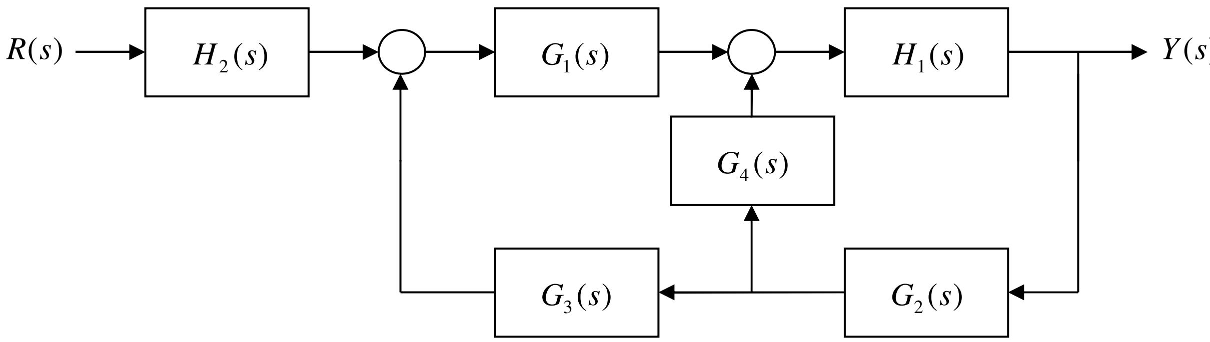
4) Moving a summing point ahead of a block:





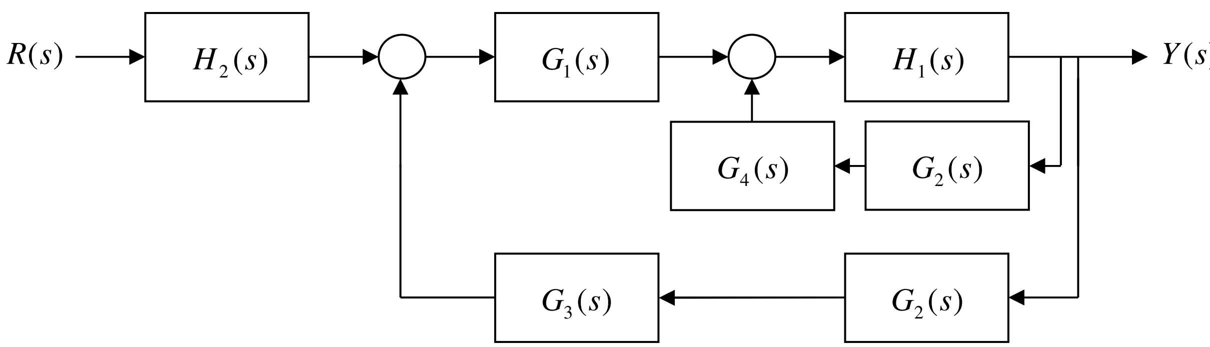
- One may use the Mason's law to find the overall transfer function of any complicated block diagram. However, using the block diagram algebra (the four reduction techniques given above and the three elementary block diagrams) one can simplify the topology of any complicated block diagram.

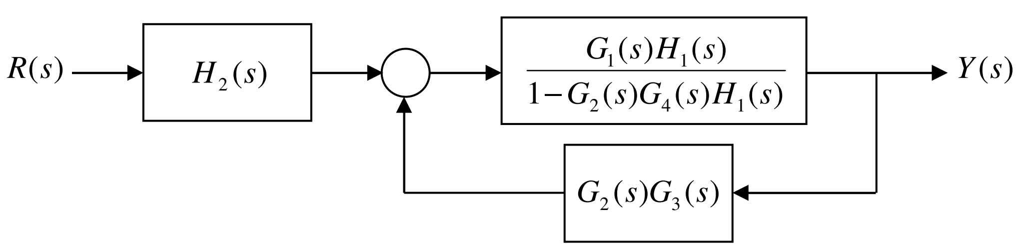
- **Example 2.4:** Find the transfer function $\frac{Y(s)}{R(s)}$ in the following block diagram:



- **Solution:** We will use two different approaches as follows:

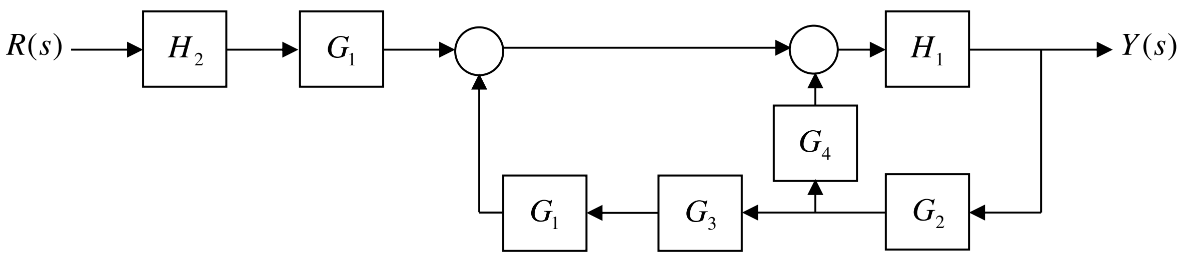
1) Separating the two feedback loops by using the second trick (moving a pickoff point ahead of a block):



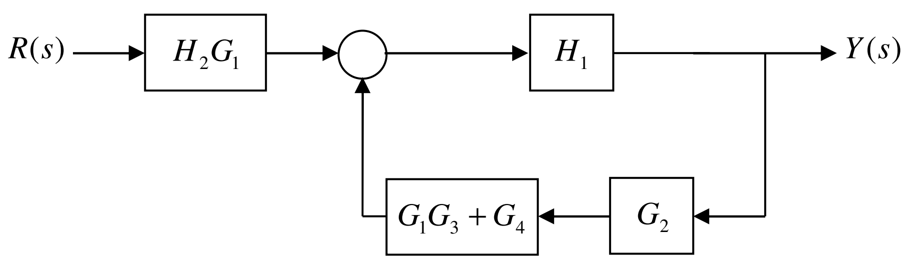


$$\frac{Y}{R} = H_2 \frac{\dfrac{G_1 H_1}{1 - G_2 G_4 H_1}}{1 - G_2 G_3 \dfrac{G_1 H_1}{1 - G_2 G_4 H_1}} = \frac{G_1 H_1 H_2}{1 - G_2 G_4 H_1 - G_1 G_2 G_3 H_1}$$

2) Moving a summing point behind a block ($G_1(s)$):

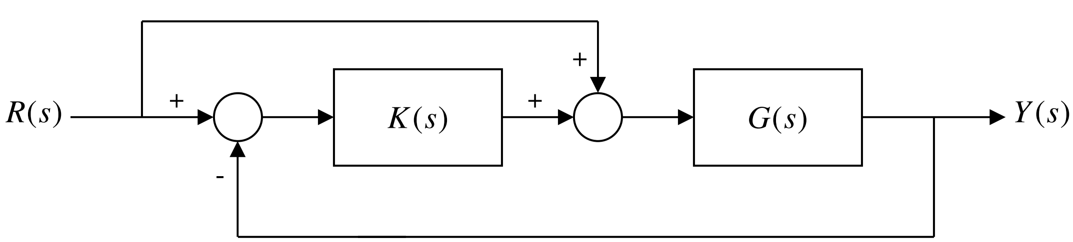


- Replacing the cascade interconnections and the parallel interconnection with single blocks, we will have:

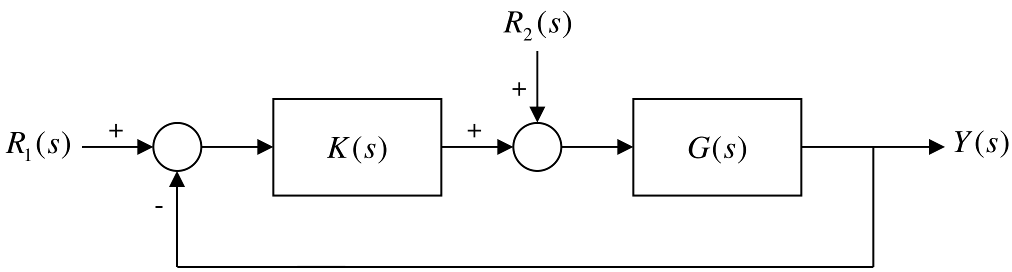


$$\frac{Y}{R} = H_2 G_1 \frac{H_1}{1 - H_1 G_2 (G_1 G_3 + G_4)} = \frac{G_1 H_1 H_2}{1 - G_2 G_4 H_1 - G_1 G_2 G_3 H_1}$$

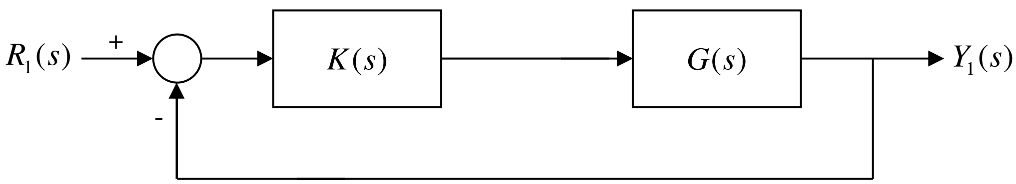
- One can also use other tricks such as superposition to simplify the topology of a block diagram and find the transfer function of the overall system.
- **Example 2.5:** Simplify the following block diagram:



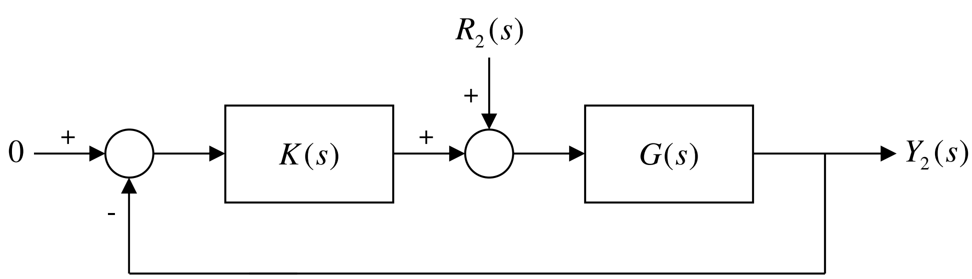
- **Solution:** We will use the principle of superposition as follows:
  - Write the signal $r(t)$ applied to the two adders as two separate inputs $r_1(t)$ and $r_2(t)$.

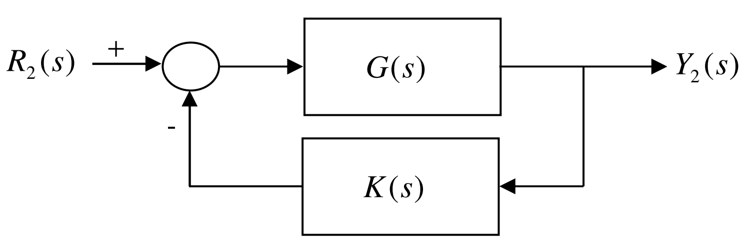


  - Find the output of the system to the input $r_1(t)$ when $r_2(t) = 0$. This output is denoted by $Y_1(s)$ in the s-domain.



  - Find the output of the system to the input $r_2(t)$ when $r_1(t) = 0$. This output is denoted by $Y_2(s)$ in the s-domain.





- Find the overall transfer function by applying the principle of superposition $Y(s) = Y_1(s) + Y_2(s)$, and substituting $R_1(s) = R_2(s) = R(s)$.

$$\begin{aligned} Y(s) &= Y_1(s) + Y_2(s) \\ &= \frac{K(s)G(s)}{1+K(s)G(s)} R_1(s) + \frac{G(s)}{1+K(s)G(s)} R_2(s) \\ &= \frac{(K(s)+1)G(s)}{1+K(s)G(s)} R(s) \end{aligned}$$