ELEC 372 LECTURE NOTES, WEEK 3

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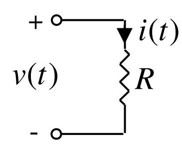
Parts of these notes are adapted from the materials in the following references:

- Modern Control Systems by Richard C. Dorf and Robert H. Bishop, Prentice Hall.
- Feedback Control of Dynamic Systems by Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, Prentice Hall.
- Automatic Control Systems by Farid Golnaraghi and Benjamin C. Kuo, John Wiley & Sons, Inc., 2010.

Components of systems

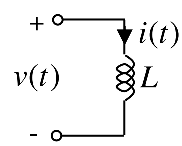
1. Electrical circuits: The elements of LTI electrical circuits are:

- Resistor:



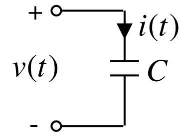
$$v(t) = Ri(t)$$
$$V(s) = RI(s)$$

- Inductor:



$$v(t) = L\frac{di(t)}{dt}$$
$$V(s) = sLI(s)$$

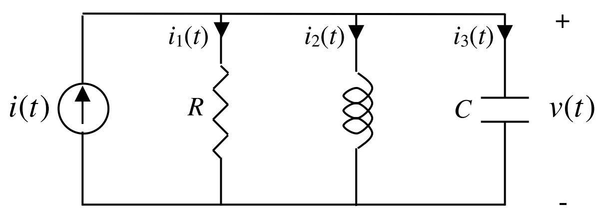
- Capacitor:



$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau, \quad for\ v(0) = 0$$

$$V(s) = \frac{1}{sC}I(s)$$

- **Example 3.1:** Find the transfer function $H(s) = \frac{V(s)}{I(s)}$ of the following circuit:



- **Solution:** Using KVL and KCL we will have:

$$I(s) = I_1(s) + I_2(s) + I_3(s) \Rightarrow \frac{V(s)}{R} + \frac{V(s)}{sL} + sCV(s)$$

$$\Rightarrow H(s) = \frac{V(s)}{I(s)} = \frac{1}{\frac{1}{R} + \frac{1}{sL} + sC} = \frac{sLR}{LCRs^2 + Ls + R}$$

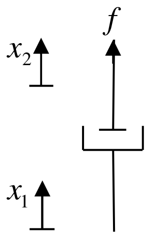
The impulse response of the system $h(t)$ is the inverse Laplace transform of $H(s) = \frac{sLR}{LCRs^2 + Ls + R}$.

- Note that any input and output of a LTI RLC circuit are related through a linear constant-coefficient differential equation.

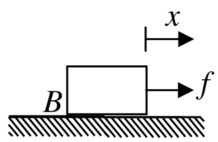
2. Mechanical systems: The elements of LTI translational and rotational mechanical systems are:

- Viscous friction (damper)

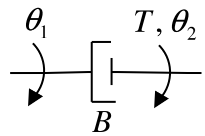
Translational | Rotational



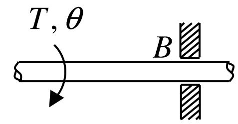
$$f(t) = B\frac{d}{dt}[x_2(t) - x_1(t)]$$
$$F(s) = sB[X_2(s) - X_1(s)]$$



$$f(t) = B\frac{dx(t)}{dt}$$
$$F(s) = sBX(s)$$



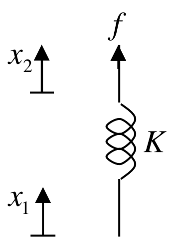
$$T(t) = B\frac{d}{dt}[\theta_2(t) - \theta_1(t)]$$
$$T(s) = sB[\theta_2(s) - \theta_1(s)]$$



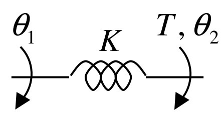
$$T(t) = B\frac{d\theta(t)}{dt}$$
$$T(s) = sB\theta(s)$$

- Spring

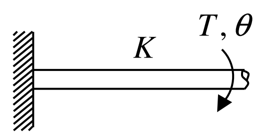
Translational spring | Rotational spring (torsional spring)



$$f(t) = K[x_2(t) - x_1(t)]$$
$$F(s) = K[X_2(s) - X_1(s)]$$



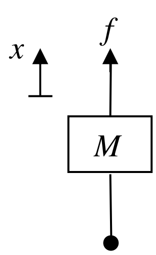
$$T(t) = K[\theta_2(t) - \theta_1(t)]$$
$$T(s) = K[\theta_2(s) - \theta_1(s)]$$



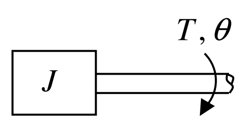
$$T(t) = K\theta(t)$$
$$T(s) = K\theta(s)$$

- Mass

Translational mass | Rotational mass

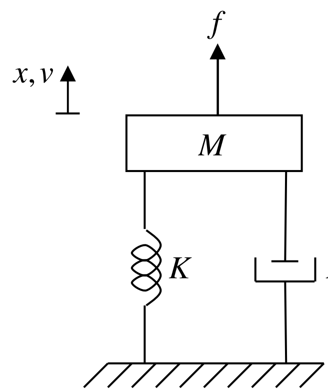


$$f(t) = M\frac{d^2x(t)}{dt^2}$$
$$F(s) = s^2MX(s)$$



$$T(t) = J\frac{d^2\theta(t)}{dt^2}$$
$$T(s) = s^2J\theta(s)$$

- **Example 3.2:** Find the transfer function $H(s)=\dfrac{V(s)}{F(s)}$ of the following system:

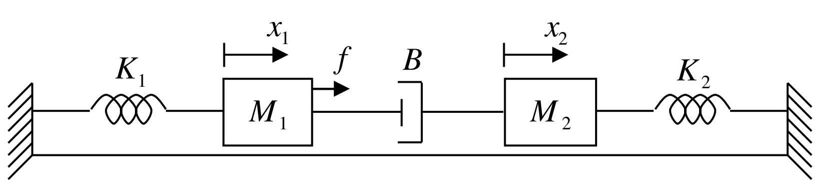


- **Solution:** Using Newton's law we will have:

$$f = M\frac{d^2x}{dt^2} + Kx + B\frac{dx}{dt} \Rightarrow \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

$$v = \frac{dx}{dt} \Rightarrow X(s) = \frac{1}{s}V(s) \Rightarrow H(s) = \frac{V(s)}{F(s)} = \frac{s}{Ms^2 + Bs + K}$$

- **Example 3.3:** Find the differential equations for the following mechanical system and also the transfer function from f to x_1.



- **Solution:** We have:

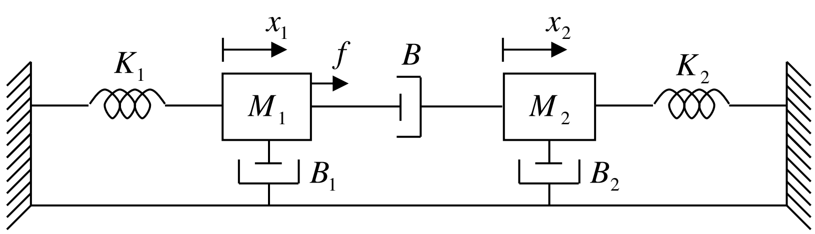
$$\begin{cases} K_1x_1 + M_1\ddot{x}_1 + B(\dot{x}_1 - \dot{x}_2) = f \\ K_2x_2 + M_2\ddot{x}_2 + B(\dot{x}_2 - \dot{x}_1) = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} M_1s^2 + K_1 + Bs & -Bs \\ -Bs & M_2s^2 + K_2 + Bs \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

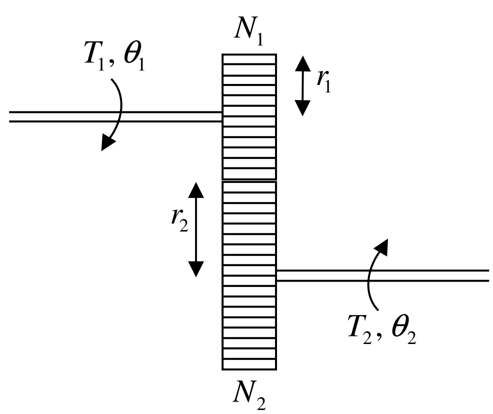
$$\Rightarrow X_1(s) = \frac{\begin{vmatrix} F(s) & -Bs \\ 0 & M_2 s^2 + K_2 + Bs \end{vmatrix}}{\begin{vmatrix} M_1 s^2 + K_1 + Bs & -Bs \\ -Bs & M_2 s^2 + K_2 + Bs \end{vmatrix}}$$

$$\Rightarrow \frac{X_1(s)}{F(s)} = \frac{(M_2 s^2 + Bs + K_2)}{(M_1 s^2 + Bs + K_1)(M_2 s^2 + Bs + K_2) - B^2 s^2}$$

- It is to be noted that in the previous example it is assumed that there is no friction between the masses and the ground. In the presence of friction, the diagram will be as follows:

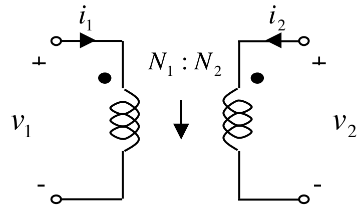


- Gear train: A gear train is a mechanical device that transmits energy from one part of the system to another. Consider the following gear train with the teeth numbers N_1 and N_2:



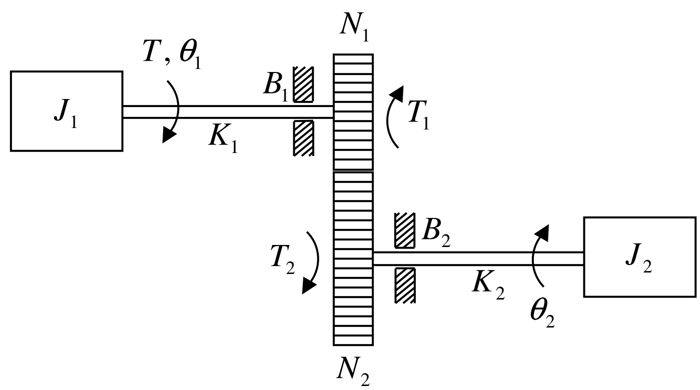
$$\frac{r_2}{r_1} = \frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} \tag{3.1}$$

- A gear train in mechanical systems is similar to a transformer in electrical systems, as follows:



$$\frac{v_2}{v_1} = \frac{i_1}{i_2} = \frac{N_2}{N_1}$$

- In practice, a load is attached to each gear as follows:



- Assuming that the gear train has no backlash or dead zone, and that the inertia and friction between the coupled gear teeth are negligible (ideal gear train), we have:

$$T_2(t) = J_2 \frac{d^2\theta_2(t)}{dt^2} + B_2 \frac{d\theta_2(t)}{dt} + K_2\theta_2(t)\,, \tag{3.2}$$

$$T(t) = J_1 \frac{d^2\theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + K_1\theta_1(t) + T_1(t)\,. \tag{3.3}$$

From (3.1), we have:

$$T_1(t) = \frac{N_1}{N_2} T_2(t), \;\; \theta_2(t) = \frac{N_1}{N_2}\theta_1(t)$$

Multiplying (3.2) by $\frac{N_1}{N_2}$ and substituting from the above relations, one obtains:

$$\Rightarrow \; T_1(t) = \left(\frac{N_1}{N_2}\right)^2 J_2 \frac{d^2\theta_1(t)}{dt^2} + \left(\frac{N_1}{N_2}\right)^2 B_2 \frac{d\theta_1(t)}{dt} + \left(\frac{N_1}{N_2}\right)^2 K_2\theta_1(t) \tag{3.4}$$

By replacing (3.4) in (3.3), we will have:

$$T(t)=\left[J_1+\left(\frac{N_1}{N_2}\right)^2 J_2\right]\frac{d^2\theta_1(t)}{dt^2}+\left[B_1+\left(\frac{N_1}{N_2}\right)^2 B_2\right]\frac{d\theta_1(t)}{dt}+\left[K_1+\left(\frac{N_1}{N_2}\right)^2 K_2\right]\theta_1(t)$$

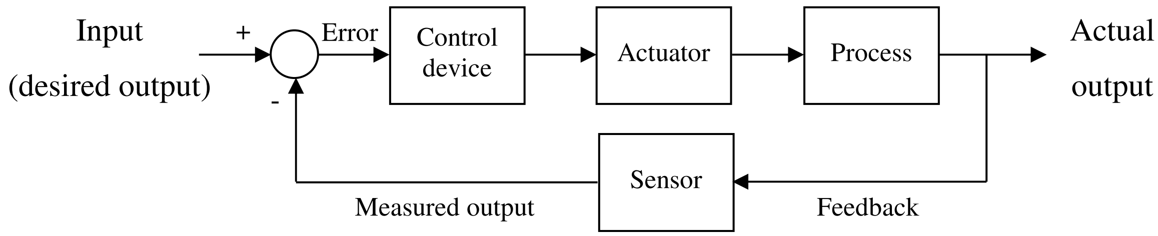
This implies that one can reflect inertia, friction, and spring from one side of a gear train to the other. The above equation indicates that by reflecting gear parameters and variables from gear 2 (with N_2 teeth) to gear 1 (with N_1 teeth), the overall system can be replaced by a single mechanical system with the following equivalent parameters:

$$J_{1e}:=J_1+\left(\frac{N_1}{N_2}\right)^2 J_2\ ,\ \ B_{1e}:=B_1+\left(\frac{N_1}{N_2}\right)^2 B_2\ ,\ \ K_{1e}:=K_1+\left(\frac{N_1}{N_2}\right)^2 K_2$$

- *Analogy of electrical circuits and mechanical systems*: Using the following analogous variables:
  - *Across variables*: velocity $v(t)$ (or angular velocity $\omega(t)$ in rotational motion) and voltage $v(t)$,
  - *Through variables*: force $f(t)$ (or torque $T(t)$ in rotational motion) and current $i(t)$,

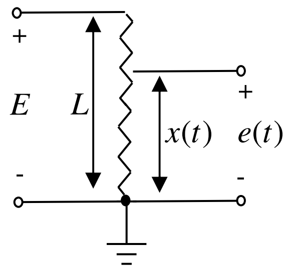
  one can find an electrical analogous model for any mechanical systems (under some conditions). Newton's laws for the mechanical systems and Kirchhoff's laws for the electrical system lead to analogous equations. A resistor in an electrical systems acts similar to a damper in a mechanical system. An inductor in an electrical system acts similar to a spring in a mechanical system, and a capacitor in an electrical system acts similar to a mass in a mechanical system. In addition, a transformer in an electrical system acts similar to a gear train in a mechanical system.

- **Sensors:** A sensor in a control system is a device that measures the output signal.



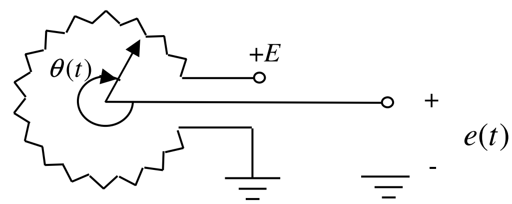
- A potentiometer is an electromechanical transducer that converts mechanical displacement into electrical signal.

Translational displacement



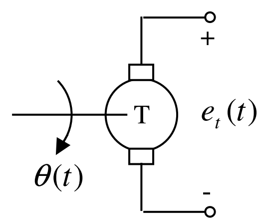
$$e(t) = \frac{E}{L}x(t), \quad 0 \le x(t) \le L$$

Rotational displacement



$$e(t) = \frac{E}{2\pi}\theta(t), \quad 0 \le \theta(t) \le 2\pi$$

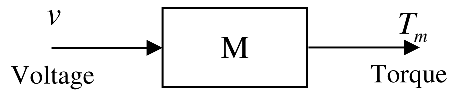
- A tachometer is an electromechanical device that generates a voltage proportional to the magnitude of the angular velocity of the shaft.



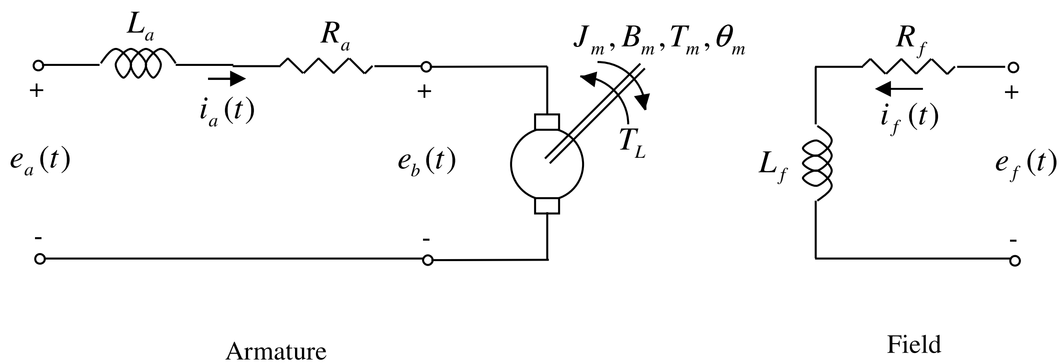
$$\begin{aligned} e_t(t) &\propto \omega(t) \\ &= K_b \frac{d\theta(t)}{dt} \end{aligned}$$

- Potentiometers and tachometers are often used in control systems to measure position and angular velocity.
- **Actuators:** An actuator in a control system is a device that provides the motive power to the process.
- DC motors: DC motors are low-power rotational actuators. There are two different types of DC motors. In DC motors with brushes, the commutation is done mechanically while brushless DC motors employ electrical commutation of the armature current. Brushless DC motors are usually used when a low moment

of inertia is needed.



- The following figure depicts a linear DC motor:



- We have:

$$T_m(t) \propto \phi(t)\, i_a(t)$$
$$= K_1 \phi(t)\, i_a(t),$$

where T_m is the torque generated by the motor, and i_a is the armature current. ϕ is the magnetic flux or air-gap flux generated by the field (in Webers), and is approximately proportional to the field current (by neglecting saturation and hysteresis effects), i.e.:

$$\phi(t) = K_f i_f(t),$$

which will result in:

$$T_m(t) = K_1 K_f i_f(t)\, i_a(t) \tag{3.5}$$

- On the other hand, when the conductor moves in the magnetic field, a back electromotive force is generated across its terminals. The corresponding voltage is proportional to the shaft velocity and tends to oppose the current flow. The back emf is given by:

$$e_b(t) = K_b \omega_m(t)$$
$$\frac{E_b(s)}{\omega_m(s)} = K_b \quad (3.6)$$

where ω_m is the angular velocity of the shaft.

- One can conclude from (3.5) that a DC motor can be controlled in two different ways: by using a fixed field current i_f and changing the armature current i_a, or by using a fixed armature current i_a and changing the field current i_f.

1. Armature-controlled DC motor: Assume $i_f(t)$ is constant. This will result in a constant magnetic field and the motor will be controlled through the armature current.

- The cause-and-effect equations for the motor circuit:

$$e_a(t) = e_b(t) + R_a i_a(t) + L_a \frac{di_a(t)}{dt}$$
$$\Rightarrow E_a(s) - E_b(s) = sL_a I_a(s) + R_a I_a(s)$$
$$\Rightarrow \frac{I_a(s)}{E_a(s) - E_b(s)} = \frac{1}{sL_a + R_a} \quad (3.7)$$

- On the other hand, since the field current is constant, we will have:

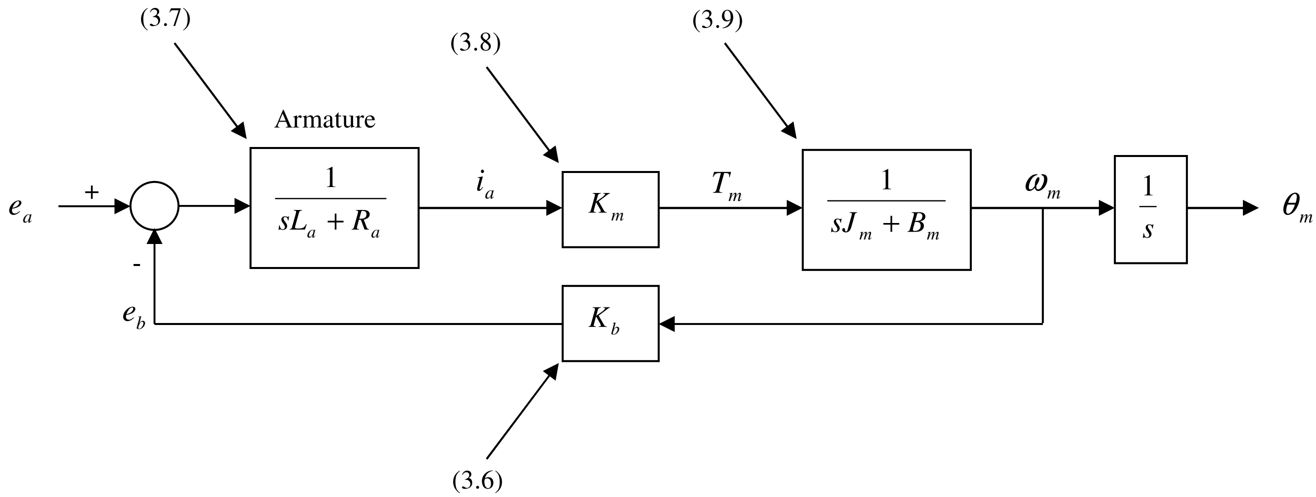
$$T_m(t) = K_1 K_f i_f i_a(t) = K_m i_a(t)$$
$$\Rightarrow \frac{T_m(s)}{I_a(s)} = K_m \quad (3.8)$$

It can be shown that $K_m = K_b$.

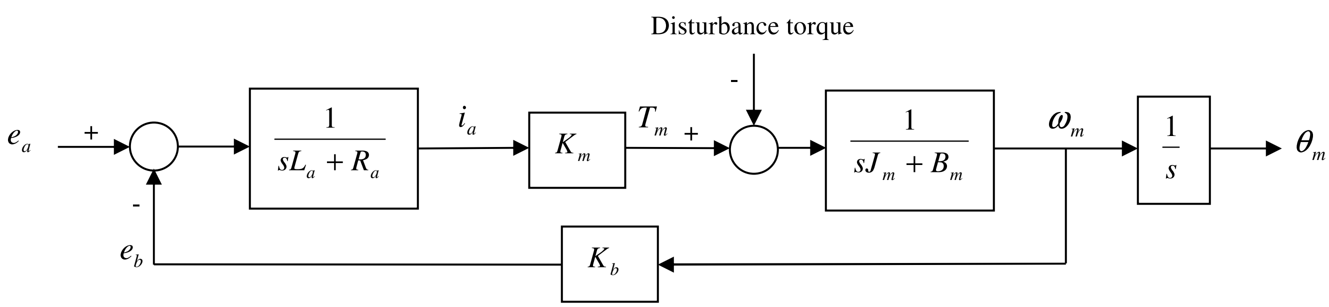
- The load torque for rotating inertia is written as:

$$T_m(t) = J_m \dot{\omega}_m(t) + B_m \omega_m(t)$$
$$\Rightarrow T_m(s) = sJ_m \omega_m(s) + B_m \omega_m(s)$$
$$\Rightarrow \frac{\omega_m(s)}{T_m(s)} = \frac{1}{sJ_m + B_m} \quad (3.9)$$

- The block diagram of the armature-controlled DC motor is as follows:



- This block diagram shows that the armature-controlled DC motor has a built-in feedback loop caused by the back emf.
- In the presence of disturbance torque, the block diagram of the armature-controlled DC motor will be as follows:



- Assuming that there is no disturbance torque, and that $L_a \cong 0$, the transfer function of the overall system will be as follows:

$$\frac{\theta_m(s)}{E_a(s)} \cong \frac{K}{s(s\tau + 1)}$$

where:

$$K := \frac{K_m}{R_a B_m + K_b K_m},$$
$$\tau := \frac{R_a J_m}{R_a B_m + K_b K_m}.$$

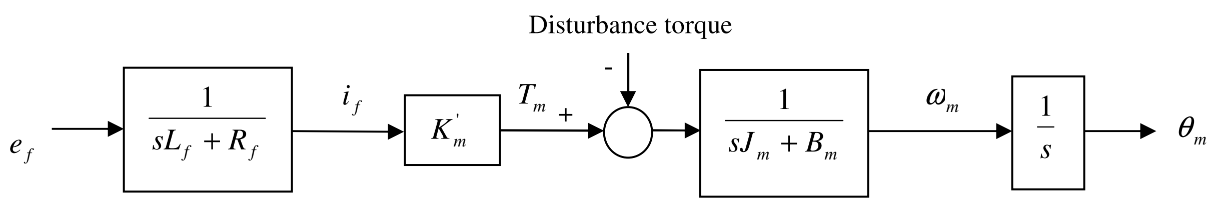
- Similarly, the transfer function of the armature-controlled DC motor from e_a to ω_m (under the above assumptions) is:

$$\frac{\omega_m(s)}{E_a(s)} \cong \frac{K}{s\tau + 1}$$

- Armature-controlled DC motors are very popular in control system applications.

2. **Field current controlled DC motor:** Assume now that $i_a(t)$ is constant. Using the set of equations derived for the armature-controlled DC motor in a similar way, the transfer function of the field current controlled motor can be obtained.

- The block diagram for the field current controlled DC motor is given below:



$$\frac{\omega_m(s)}{E_f(s)} = \frac{K_m'}{(sJ_m + B_m)(L_f s + R_f)}.$$

- where K_m' can be obtained from (3.5) and is equal to $K_m' := K_1 K_f i_a$. Note that in the field current controlled motor there is no internal feedback in the system. So, the armature-controlled motor is more popular in control systems.
- DC motors are used in speed control and position control systems.