ELEC 372 LECTURE NOTES, WEEK 4

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Parts of these notes are adapted from the materials in the following references:

- Modern Control Systems by Richard C. Dorf and Robert H. Bishop, Prentice Hall.
- Feedback Control of Dynamic Systems by Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, Prentice Hall.
- Automatic Control Systems by Farid Golnaraghi and Benjamin C. Kuo, John Wiley & Sons, Inc., 2010.

- Hydraulic actuator: A hydraulic actuator is used for the linear positioning of a mass and can provide large power amplification.
- Figure 4.1 shows the operation of a hydraulic actuator.

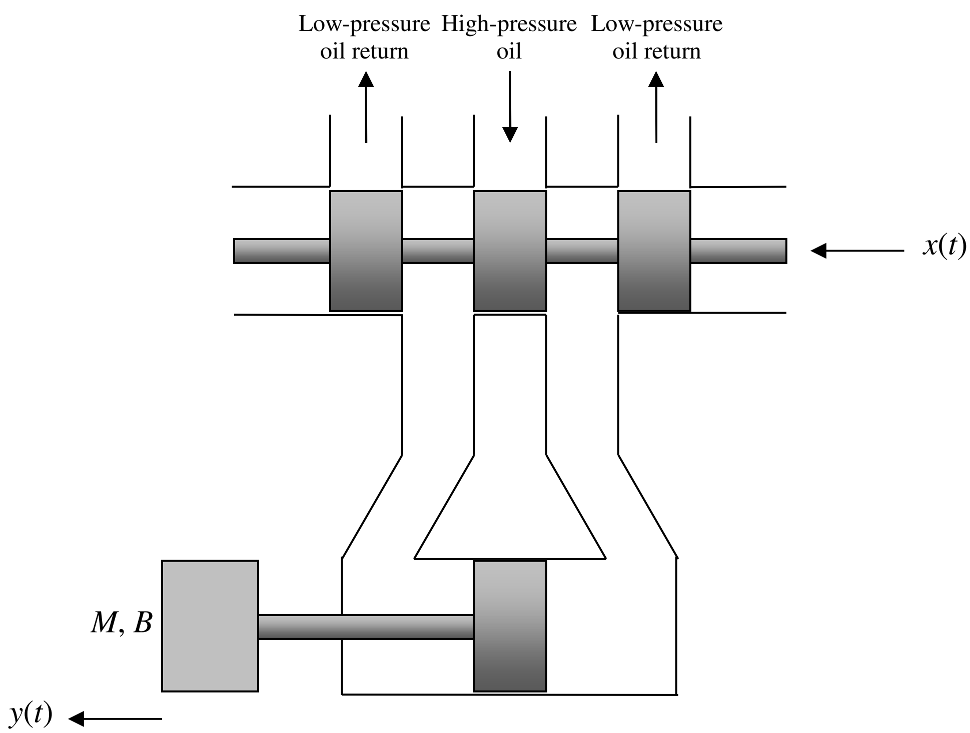


Figure 4.1: A hydraulic actuator

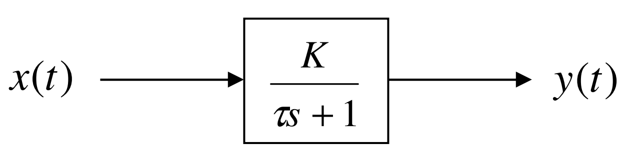
- When $x(t) > 0$, the high-pressure oil enters the right side of the large piston chamber, forcing the piston to the left. This causes the low-pressure oil to flow out of the valve chamber from the leftmost channel. Similarly, when $x(t) < 0$, the high-pressure oil enters the left side of the large piston chamber, forcing the piston to the right. This causes the low-pressure oil to flow out of the valve chamber from the rightmost channel.
- To obtain a model for the hydraulic actuator, it is assumed that the compressibility of the oil is negligible (in practice, the compressibility of oil may cause some resonance because it acts like a stiff spring). It is also assumed that the high-pressure hydraulic oil is provided by a constant pressure source.
- The input $x(t)$ and the output $y(t)$ are related through a second-order nonlinear differential equation and after linearization around $x(t) = 0$ and simplification, we will have the following transfer function for a hydraulic actuator:

$$\frac{Y(s)}{X(s)} = \frac{K}{s(Ms + B)}.$$

  M is the mass of the piston and the attached load. K and B are functions of the piston area, friction, and the flowing oil.
- The transfer function of the hydraulic actuator is similar to that of the electric motor (armature-controlled DC motor) given by $\frac{\theta_m(s)}{E_a(s)} \cong \frac{K}{s(s\tau + 1)}$.

Time domain analysis

1. **First-order systems**: The transfer function of a first-order system is as follows:



- We have:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{K}{s\tau + 1}$$
$$\Rightarrow \tau \frac{dy(t)}{dt} + y(t) = Kx(t)$$

- The impulse response of the system is:

$$h(t) = \frac{K}{\tau} e^{-\frac{t}{\tau}} u(t).$$

- The step response of the system is:

$$Y(s) = \frac{K}{s(s\tau + 1)} = \frac{K}{s} - \frac{K}{s + \frac{1}{\tau}}$$
$$\Rightarrow y(t) = K(1 - e^{-\frac{t}{\tau}})u(t)$$

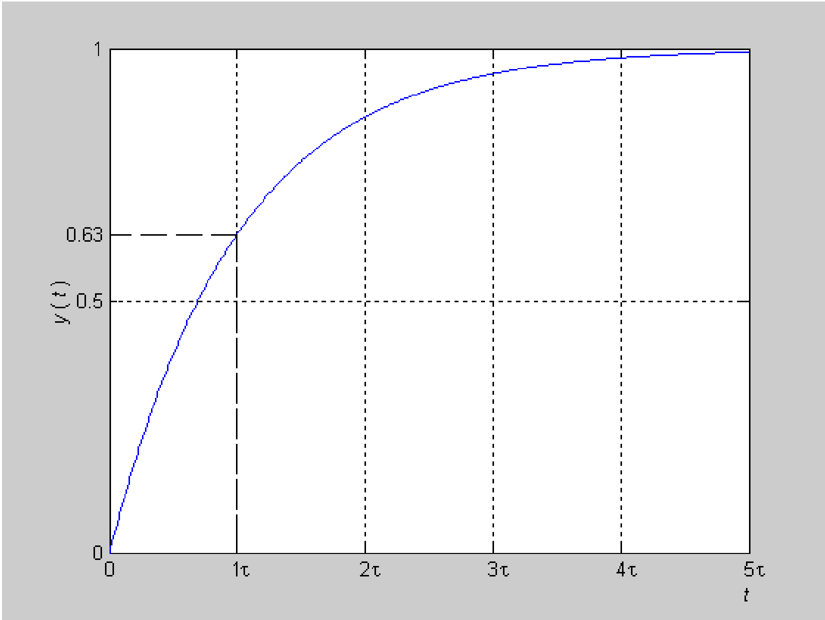
- For $\tau > 0$ we will have the following steady state value for the step response:

$$y_{ss} = \lim_{t \to \infty} y(t) = K$$

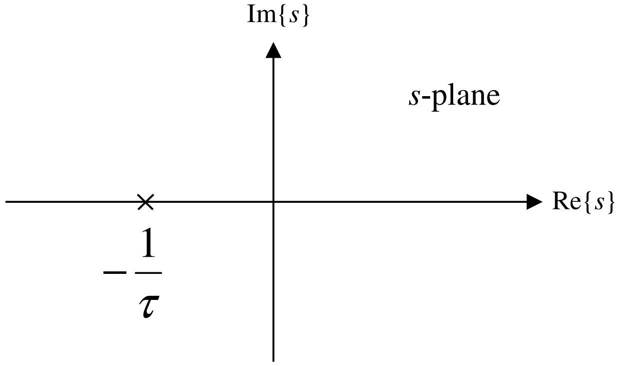
- Note that in general, if all poles of $H(s)$ are in the LHP, y_{ss} can be found using the final-value theorem as shown below:

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s\frac{1}{s}H(s) = H(0)$$

- $H(0)$ is called the DC gain of the system (for a stable system).
- Since for a first-order system $y_{ss} = H(0) = K$, this means that in order to have no steady-state error for the step input, K must be equal to 1.
- The step response of a first-order system for $K = 1$ is given in the following figure ($H(s) = \frac{1}{s\tau + 1}$):



- τ is called the time constant of the system and a smaller τ means a faster system.
- The pole of the first-order system is located at $s = -\frac{1}{\tau}$ and is indicated in the following figure:



- In general, poles closer to the imaginary axis represent slower time response.
- The settling time t_s is the time it takes the system transients to decay. More precisely, it is the time required for the system output to settle within a certain percentage of its steady-state value. The most commonly used percentages are 1%, 2% and 5%.
- For the first-order system with 2% measure we have $t_s = 4\tau$ and for 5% measure we have $t_s = 3\tau$. We will use the 2% measure for the settling time in this course.

- Small settling time is desirable in the design of control systems.
- The ramp response is the response of the system to a unit ramp signal $x(t) = tu(t)$ when the initial conditions are zero. We will have:

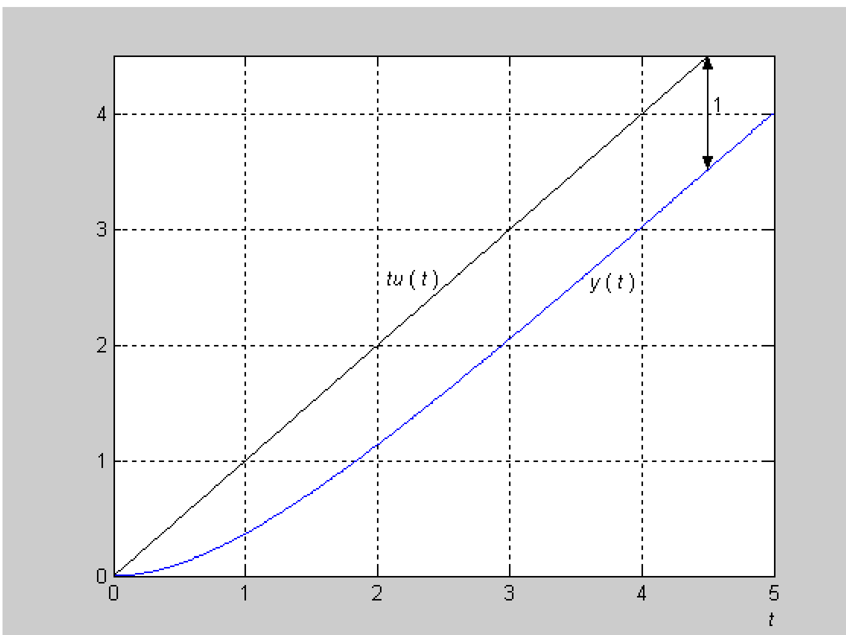
$$X(s) = \frac{1}{s^2} \Rightarrow Y(s) = \frac{K}{s^2(s\tau + 1)} = -\frac{K\tau}{s} + \frac{K}{s^2} + \frac{K\tau}{s + 1/\tau}$$

$$\Rightarrow y(t) = K(t - \tau + \tau e^{-\frac{t}{\tau}})u(t)$$

- The steady-state error for $k = 1$ can be obtained when $t \to \infty$, and is given by:

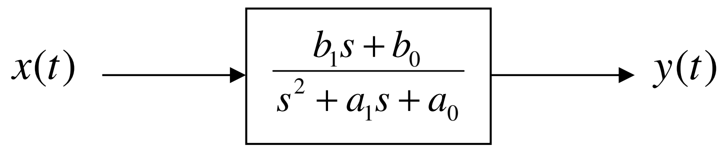
$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} tu(t) - y(t) = \lim_{t \to \infty} t - t + \tau = \tau$$

- The ramp response of a first-order system for $K = 1$ and $\tau = 1$ is given in the following figure ($H(s) = \dfrac{1}{s+1}$):



- From the results obtained for unit step response and unit ramp response, it can be concluded that a stable first-order system with unit DC gain $H(s) = \dfrac{1}{s\tau + 1}$ has zero steady-state error for the step input and constant steady-state error for the ramp input.

2. **Second-order systems**: The transfer function of a second-order system is:



- The differential equation relating the output to the input is given by:

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_1 \frac{dx(t)}{dt} + b_0 x(t) .$$

- Let us assume that $b_1 = 0$, which means that the system has no zeros. Then:

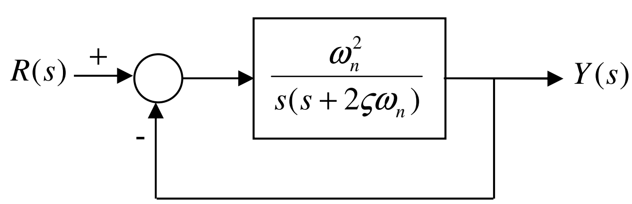
$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0}{s^2 + a_1 s + a_0} .$$

- Usually it is simpler to normalize the second-order transfer function such that the DC gain is one ($b_0 = a_0$) and then use the following standard form to describe the system:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- ω_n is equal to $\sqrt{a_0}$ and ζ is equal to $\frac{1}{2}\frac{a_1}{\sqrt{a_0}}$.

- Note that a second-order system with the standard transfer function can be resulted from the following closed-loop system:



- Many of the practical second-order systems (such as a closed-loop position control system with a DC motor) have in fact the above closed-loop structure.

- The poles of the second-order transfer function $H(s)$ are located at:

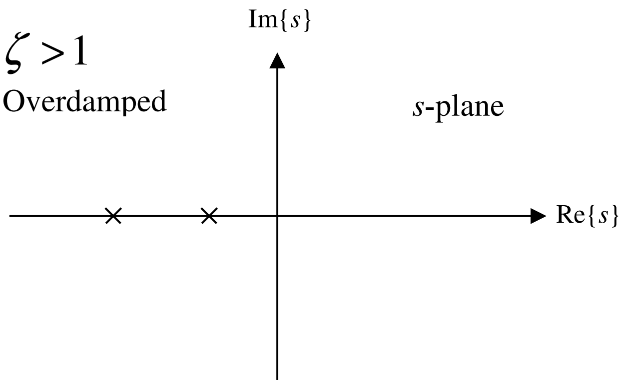
$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}\,.$$

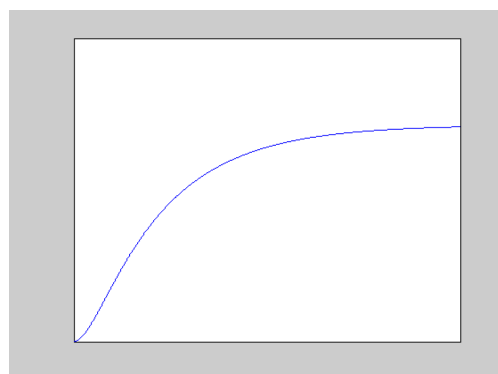
- For $|\xi| \geq 1$ we have two real poles.
- For $|\xi| < 1$ we have two complex poles which always come in complex conjugate pairs.
- The second-order system is stable if and only if $\zeta > 0$ (which results in two poles in the LHP).
- The behaviour of a second-order system depends highly on ζ .

- Stable second-order systems ($\zeta > 0$):

$\zeta > 1$

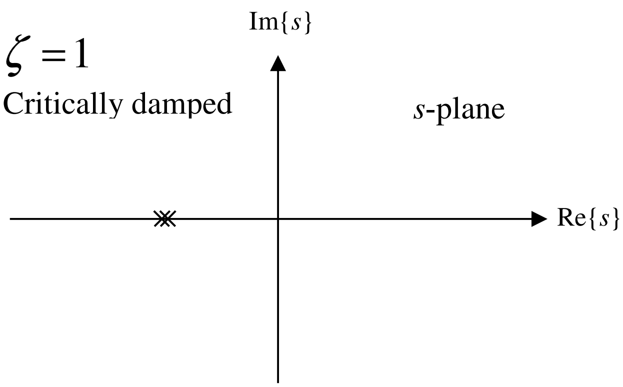
Overdamped

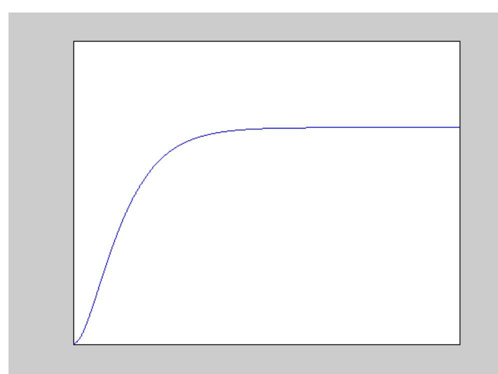


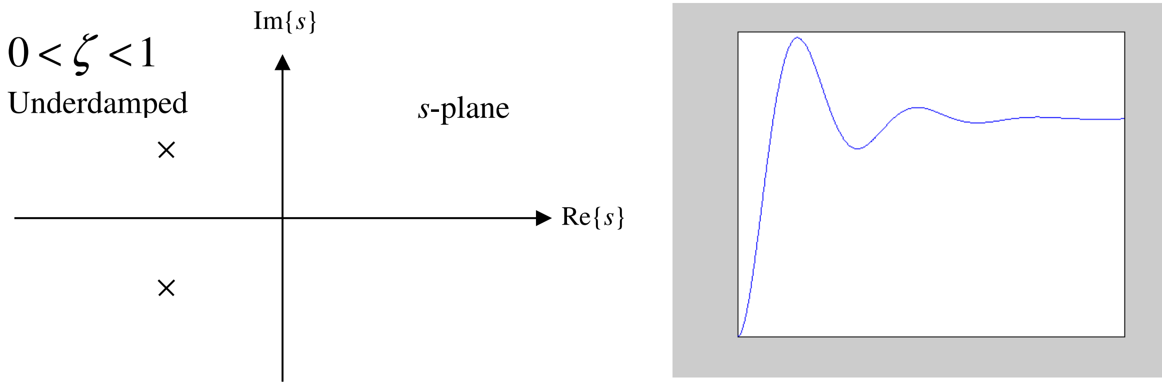


$\zeta = 1$

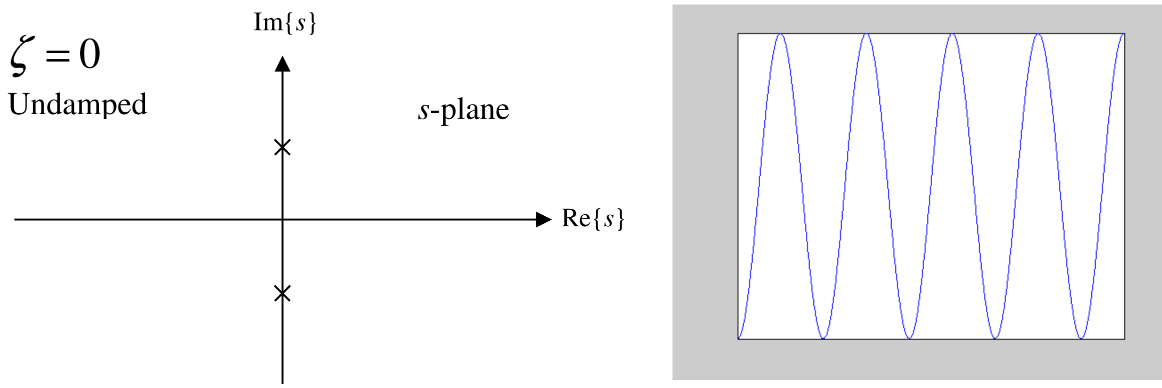
Critically damped

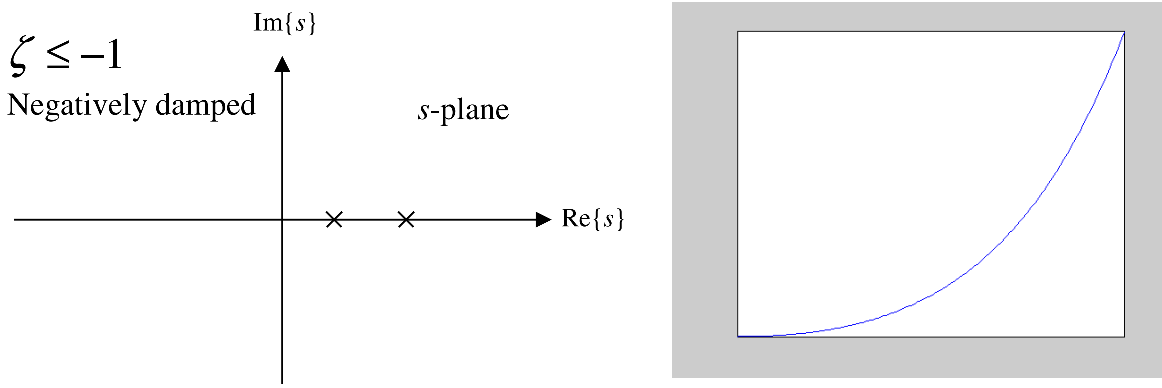


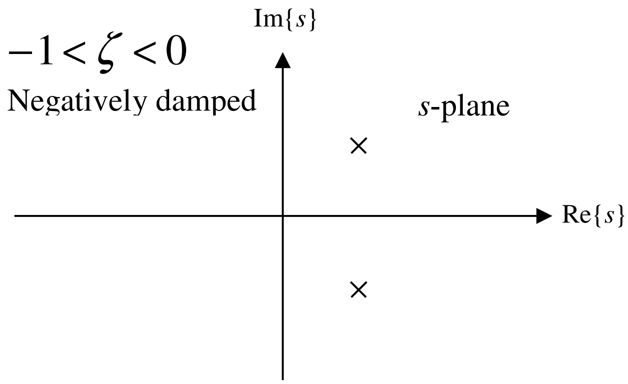


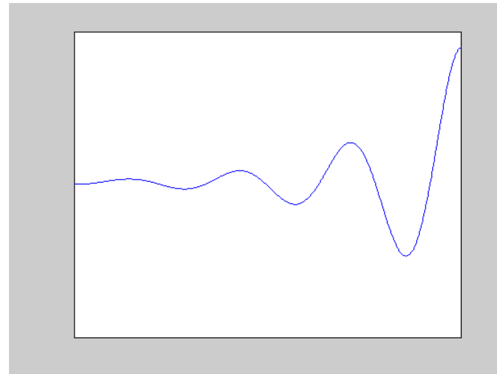


- Unstable second-order system ($\zeta \leq 0$):









- We are only interested in stable second-order systems: overdamped, critically damped and underdamped.
- **Overdamped systems:** An overdamped second-order system has two real poles in the LHP and so it can be considered as the parallel interconnection of two first-order systems.
- **Underdamped systems:** An underdamped second-order system has a pair of complex conjugate poles:

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_d$$

- ω_n is called natural frequency or natural undamped frequency.
- ζ is called damping ratio.
- $\omega_n\sqrt{1-\zeta^2}$ is called the damped natural frequency, or damped frequency, or conditional frequency and is denoted by ω_d. This is, in fact, the frequency of the decaying oscillations in the step response, as we will see in the following pages.
- $\zeta\omega_n$ is called the damping factor or damping constant (because it determines the rate of rise or decay of the step response, as discussed later) and is denoted by α.

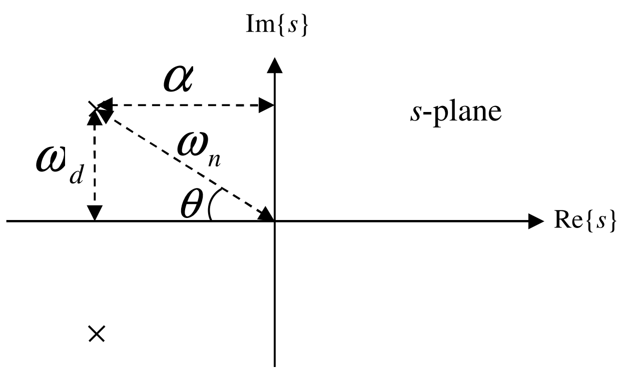


Figure 4.2

- The unit step response of the second-order system $H(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ is:

$$For\ 0 < \zeta < 1: \quad y(t) = 1 - \frac{\omega_n}{\omega_d} e^{-\alpha t} \sin(\omega_d t + \theta), \quad \theta = \cos^{-1}\zeta$$

$$For\ \zeta = 0: \quad y(t) = 1 - \sin(\omega_n t + \frac{\pi}{2}) = 1 - \cos(\omega_n t)$$

$$For\ \zeta = 1: \quad y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$