ELEC 372 LECTURE NOTES, WEEK 5

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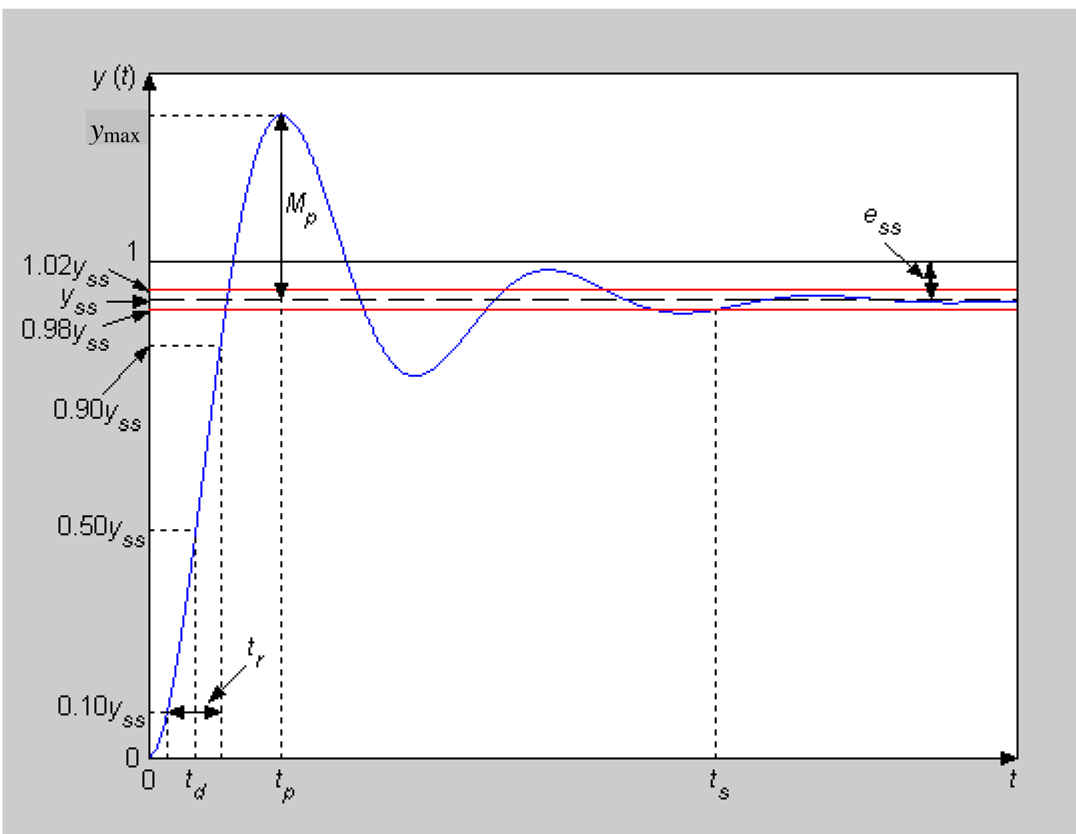
Parts of these notes are adapted from the materials in the following references:

- Modern Control Systems by Richard C. Dorf and Robert H. Bishop, Prentice Hall.
- Feedback Control of Dynamic Systems by Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, Prentice Hall.
- Automatic Control Systems by Farid Golnaraghi and Benjamin C. Kuo, John Wiley & Sons, Inc., 2010.

- Many of the control design objectives are in fact related to the unit step response of the control system.
- The important parameters of the unit step response for an underdamped second-order system are shown in the following figure.



- Maximum overshoot (M_p): Let $y(t)$ be the unit step response, with $y_{\max}$ representing the maximum value of $y(t)$ and y_{ss} representing the steady-state value of $y(t)$. The maximum overshoot of $y(t)$ is defined as:

$$M_p = y_{\max} - y_{ss}$$

- Maximum overshoot is often represented as a percentage of the final value of the step response, that is:

$$\underline{\text{Percentage overshoot}}\ (P.O.) = \frac{M_p}{y_{ss}} \times 100\%$$

Percentage overshoot is equal to:

$$P.O. = 100 e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

- Peak time (t_p): The time required for the step response to reach its maximum value $y_{\max}$. Peak time is equal to:

$$t_p = \frac{\pi}{\omega_d}$$

- Settling time (t_s): The time required for the step response to decrease and stay within a specified percentage of its final value. A frequently used figure is 2%, which results in the following approximate value for the settling time:

$$t_s \cong \frac{4}{\zeta\omega_n}$$

- Delay time (t_d): The delay time t_d is defined as the time required for the step response to reach 50 percent of its final value.

$$t_d \cong \frac{1+0.7\zeta}{\omega_n}$$

- Rise time (t_r): The rise time t_r is defined as the time required for the step response to rise from 10 to 90 percent of its final value. An approximate value for the rise time is given by (R. C. Dorf and R. H. Bishop, *Modern Control Systems*, 12$^{\text{th}}$ Edition, Prentice Hall, 2011):

$$t_r \cong \frac{2.16\zeta + 0.60}{\omega_n}$$

- <u>Steady-state error</u>: The steady-state error (e_{ss}) is the error between the output and reference input when the steady state is reached. For the unit step response:

$$e_{ss} = 1 - y_{ss}$$

- Maximum overshoot, peak time, settling time, delay time, and rise time give a direct measure of the transient characteristic of a second-order underdamped control system in terms of the unit step response.
- Steady-state error gives a direct measure of the steady-state characteristics of a second-order underdamped control system.
- One can define the DC gain of a second-order stable system in a way similar to the first-order system as $H(0)$.
- The DC gain of the standard second-order system $H(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ with $\zeta > 0$ (stable system) is equal to 1.
- The steady-state error of a second-order system due to a constant reference input is equal to zero if and only if its DC gain is unity.
- The most frequently used values for ζ in control systems are 0.5, 0.6 and 0.7 whose corresponding percentage overshoots are:

ζ	***P.O.***
0.7	5%
0.6	10%
0.5	15%

- One can use the Laplace transform techniques to find the unit ramp response of a second-order system. The asymptote of the unit ramp response of the second-order system $H(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ is equal to:

$$y_{ss}(t) = t - \frac{2\zeta}{\omega_n}$$

This means that the steady-state error for the unit ramp signal is $e_{ss} = \dfrac{2\zeta}{\omega_n} \neq 0$.

Decreasing ζ reduces e_{ss} but makes the response more oscillatory. This is a typical trade-off in the design of control systems (small steady-state error versus more stability margin).

- **Example 5.1:** Find the desired pole location for an underdamped second-order system in order to meet the following specifications:

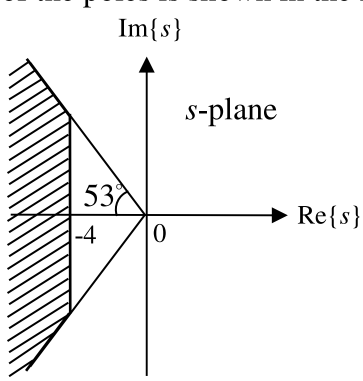
$$t_s \leq 1\,\text{sec}, \quad P.O. \leq 10\%$$

- **Solution:** Using the equations given for t_s and $P.O.$, we will have:

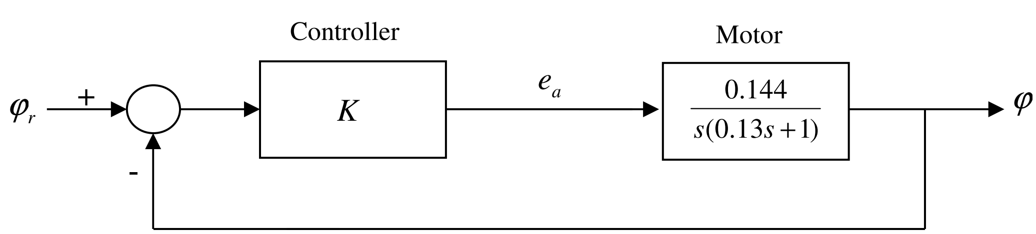
$$t_s \leq 1\,\text{sec} \Rightarrow \frac{4}{\xi\omega_n} \leq 1 \Rightarrow \xi\omega_n \geq 4$$

$$P.O. \leq 10\% \Rightarrow 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \leq 10 \Rightarrow \zeta \geq 0.6 \Rightarrow \theta = \cos^{-1}\zeta \leq 53^\circ$$

The desired location of the poles is shown in the following figure.



- **Example 5.2:** Consider the following servomotor system:



Find the value of K such that $P.O. = 5\%$.

- **Solution:** The transfer function of the closed-loop system is given by:

$$\frac{\varphi(s)}{\varphi_r(s)} = \frac{1.108K}{s^2 + 7.7s + 1.108K}$$

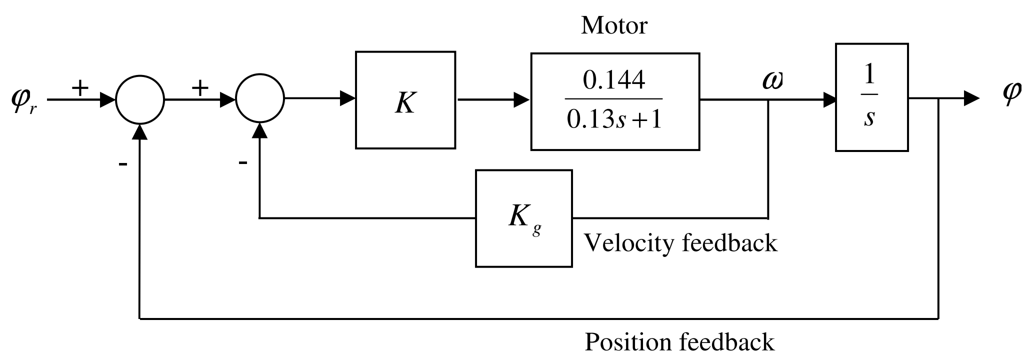
We have:

$$\omega_n = \sqrt{1.108K}$$
$$2\zeta\omega_n = 7.7$$

On the other hand, for $P.O. = 5\%$ we must have $\zeta = 0.7$. This means that:

$$\omega_n = \frac{7.7}{2 \times 0.7} = 5.5 \Rightarrow K = 27.3$$

- In the above example, we only had one degree of freedom K which affects ζ and ω_n but damping factor $\zeta\omega_n$ is fixed and cannot be changed by K. Since the settling time depends on $\zeta\omega_n$, this implies that the settling time cannot be reduced by using a constant controller with position feedback, and is always equal to $\frac{4}{7.7/2} = 1.04$.
- In the next example, we will see how adding a velocity feedback can improve the performance of the position control system.
- **Example 5.3:** Consider the following servomotor system with position and velocity feedback:



Find the values of K and K_g such that $P.O. = 5\%$ and $t_s = 0.5\sec$.

- **Solution:** The transfer function of the closed-loop system is given by:

$$\frac{\varphi(s)}{\varphi_r(s)} = \frac{1.108K}{s^2 + (7.7 + 1.108KK_g)s + 1.108K}$$

Therefore:

$$2\zeta\omega_n = 7.7 + 1.108KK_g \Rightarrow t_s = \frac{4}{\zeta\omega_n} = \frac{4}{3.85 + 0.554KK_g} = 0.5 \Rightarrow KK_g = 7.49$$

$$P.O. = 5\% \Rightarrow \zeta = 0.7 \Rightarrow \frac{4}{\zeta\omega_n} = \frac{4}{0.7\omega_n} = 0.5 \Rightarrow \omega_n = 11.43$$

$$\omega_n = \sqrt{1.108K} \Rightarrow K = 117.9 \Rightarrow K_g = \frac{7.49}{117.9} = 0.0635$$

Dominant poles and model reduction

- Any LTI system with rational transfer function can be modelled as a combination of first-order and second-order systems (in series or in parallel).
- If a real pole is by far closer to the imaginary axis and no zero is around it, then the response is similar to the response of a first-order system having only that pole.
- Similarly, if a pair of complex poles are by far closer to the imaginary axis and no zeros are around them, then the response is similar to the response of a second-order system having only those poles.
- In general, the poles near the imaginary axis of the *s*-plane relative to the other poles of the system are labelled the dominant poles of the system because they dominate the transient response.
- The relative dominance of the complex poles, in a third-order system with a pair of complex conjugate poles, is determined by the ratio of the real pole to the real part of the complex poles. For example if the poles are located at $s_1 = -3$, and $s_2, s_3 = -0.5 \pm j2$, the relative dominance of the complex poles is given by the ratio $\frac{-3}{-0.5} = 6$.

- One can use the concept of dominant poles to find a lower-order approximation for a transfer function, by considering only the poles with a dominance factor of about 5 or higher and neglecting the other ones.
- **Example 5.4:** Consider the following second-order system:

$$H(s) = \frac{1}{s^2 + 4s + 1}$$

Find a first-order approximation for this system.

- **Solution:** For this system we have $\zeta = 2, \omega_n = 1$, and the poles of the transfer function are located at:

$$s_1 = -0.27,\ s_2 = -3.73$$

Partial fraction expansion for $H(s)$ is given by:

$$H(s) = \frac{0.289}{s + 0.27} - \frac{0.289}{s + 3.73}. \tag{5.1}$$

The unit step response of the system will be:

$$y(t) = 1 + 0.077e^{-3.73t} - 1.077e^{-0.27t}, \quad t \geq 0 \tag{5.2}$$

$$\Rightarrow y_{ss} = \lim_{t \to \infty} y(t) = 1 \quad (\text{DC gain} = 1)$$

To approximate $H(s)$ with a first-order model, we neglect the second term in (5.1) which represents fast response in (5.2). In other words, the system will be approximated using the dominant pole:

$$H(s) \cong H_1(s) = \frac{0.289}{s + 0.27}.$$

The step response of the first-order approximating system $H_1(s)$ is given by:

$$y_1(t) = \frac{0.289}{0.27} - \frac{0.289}{0.27} e^{-0.27t}, \quad t \geq 0$$

$$y_{1,ss} = \frac{0.289}{0.27}$$

The steady-state output of the approximating system ($\frac{0.289}{0.27}$) is different from that of the original system (unity). This is due to the fact that the DC gain of the approximating system is different from the original system.

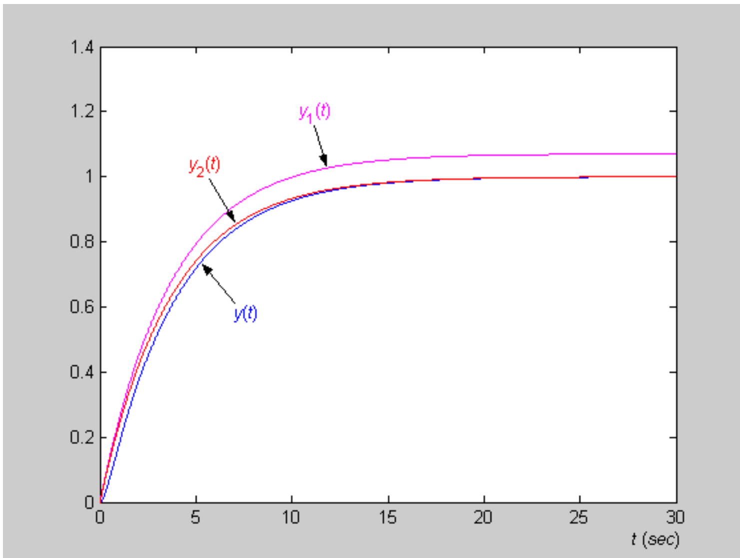
To adjust the DC gain, we can use the following approximation:

$$H(s) \cong H_2(s) = \frac{0.27}{s+0.27}.$$

The corresponding step response is:

$$y_2(t) = 1 - e^{-0.27t}, \quad t \geq 0$$

Step responses for $H(s)$, $H_1(s)$ and $H_2(s)$ are given in the following figure.

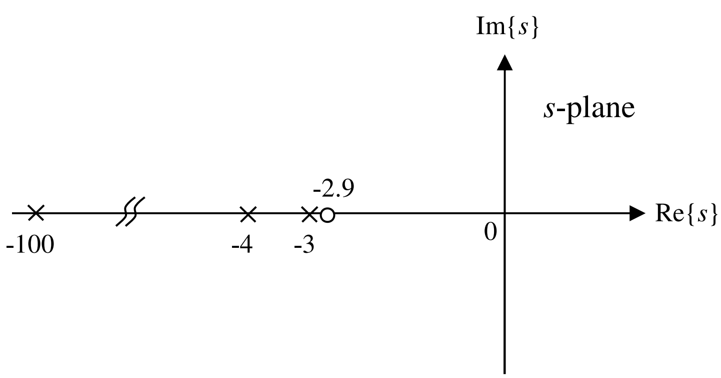


- **Example 5.5:** Consider the following third-order system:

$$H(s) = \frac{200(s+2.9)}{(s+3)(s+4)(s+100)}.$$

Find a lower order approximation for this system.

- **Solution:** The DC gain of the system is equal to $\frac{200 \times 2.9}{3 \times 4 \times 100} = 0.4833$ and the pole-zero configuration is given in the following figure:



Since the pole at $s=-100$ is very far from the imaginary axis compared to the other poles and also there is a zero close to the pole at $s=-3$, the dominant pole of this system is $s=-4$. This can also be verified by using the partial fraction expansion as follows:

$$H(s)=-\frac{0.262}{s+3}+\frac{2.2917}{s+4}-\frac{2.0855}{s+100}$$

Note that the residue of the pole $s=-3$ is very small, which is due to the zero close to that pole. So the first-order approximating model can be obtained by using the second term in the partial fraction expansion and adjusting the corresponding DC gain as follows:

$$H(s)\cong\frac{1.9332}{s+4}$$