

ELEC 372 LECTURE NOTES, WEEK 6

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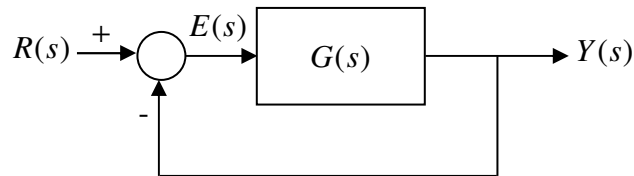
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Parts of these notes are adapted from the materials in the following references:

- Modern Control Systems by Richard C. Dorf and Robert H. Bishop, Prentice Hall.
- Feedback Control of Dynamic Systems by Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, Prentice Hall.
- Automatic Control Systems by Farid Golnaraghi and Benjamin C. Kuo, John Wiley & Sons, Inc., 2010.

Steady-state error in feedback systems

- Consider the following unity negative feedback control system and assume that the overall system is stable (this is the main assumption).



- In this block diagram:

$G(s)$: Open loop transfer function

$R(s)$: Reference input

$Y(s)$: Output signal

$E(s)$: Error signal

- We have:

$$Y(s) = \frac{G(s)}{1+G(s)} R(s) \Rightarrow E(s) = R(s) - Y(s) = \frac{1}{1+G(s)} R(s)$$

- Assume that all of the poles of $\frac{sR(s)}{1+G(s)}$ are located in the LHP. Using the final

value theorem, we will have:

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}.$$

- It is desired to find the steady-state error due to different reference inputs:

1. Step input: $r(t) = Au(t) \Rightarrow R(s) = \frac{A}{s}$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \frac{A}{s}}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{A}{1 + G(s)} = \frac{A}{1 + K_p}, \quad K_p := \lim_{s \rightarrow 0} G(s)$$

- K_p is called the “position constant” or “position error constant” or “static position error constant” or “step-error constant”.
- For zero steady-state error due to a step reference input, we must have $K_p = \infty$ which implies that $G(s)$ must have at least one integrator.
- Note that the stability of the closed loop system ensures that the final value theorem can be used in this case.
- A closed loop system with N integrators in the open loop transfer function, i.e., $G(s) = \frac{p(s)}{s^N q(s)}$, where $p(s)$ and $q(s)$ are polynomials in s and $p(0) \neq 0$ and $q(0) \neq 0$ is called a type N system (factor out s from the polynomials in the numerator and denominator, if necessary, and simplify them as one term in the denominator only).
- For zero steady-state error due to a step reference input we need at least a type 1 system.

2. Ramp input: $r(t) = Atu(t) \Rightarrow R(s) = \frac{A}{s^2}$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \frac{A}{s^2}}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{A}{s[1 + G(s)]} = \lim_{s \rightarrow 0} \frac{A}{sG(s)} = \frac{A}{K_v}, \quad K_v := \lim_{s \rightarrow 0} sG(s)$$

- K_v is called the “velocity constant” or “velocity error constant” or “static velocity error constant” or “ramp-error constant”.

- For zero steady-state error due to a ramp reference input, we must have $K_v = \infty$ which requires $G(s)$ to have at least two integrators.
- For zero steady-state error due to a ramp reference input we need at least a type 2 system.

3. Parabolic input: $r(t) = \frac{1}{2}At^2u(t) \Rightarrow R(s) = \frac{A}{s^3}$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \frac{A}{s^3}}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{A}{s^2[1 + G(s)]} = \lim_{s \rightarrow 0} \frac{A}{s^2 G(s)} = \frac{A}{K_a}, \quad K_a := \lim_{s \rightarrow 0} s^2 G(s)$$

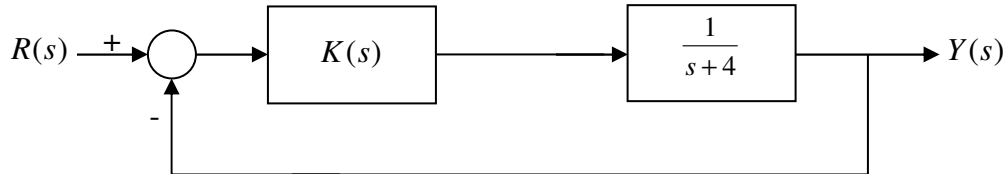
- K_a is called the “acceleration constant” or “acceleration error constant” or “static acceleration error constant” or “parabolic-error constant”.
- For zero steady-state error due to a parabolic reference input, we must have $K_a = \infty$ which requires $G(s)$ to have at least three integrators.
- For zero steady-state error due to a parabolic reference input we need at least a type 3 system.
- The following table summarizes the results for the steady-state error:

Input Type	A/s	A/s^2	A/s^3	...	A/s^i
0	$A/(1+K_p)$	∞	∞	...	∞
1	0	A/K_v	∞	...	∞
2	0	0	A/K_a	...	∞
3	0	0	0	...	∞
...
$i - 2$	0	0	0	...	∞
$i - 1$	0	0	0	...	A/K_t ← $K_t := \lim_{s \rightarrow 0} s^{i-1} G(s)$
i	0	0	0	...	0

- In other words, to have zero steady-state error for a stable closed loop system with a reference input of the form given in the above table, the open loop transfer

function $G(s)$ must have at least as many integrators as the number of input poles in the origin.

- **Example 6.1:** Consider the following closed loop control system:



Design $K(s)$ such that:

- There is no steady state error for the step input.
- The percentage overshoot for the step input is $P.O. \leq 5\%$.

- **Solution:** Using $K(s) = \frac{K}{s}$ and choosing the parameter K such that the closed loop system is stable, condition (i) will be satisfied. With this controller, we will have:

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + 4s + K}.$$

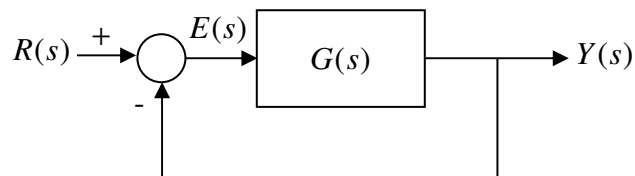
For stability, we must have a positive value for K .

For condition (ii), on the other hand, we must have $\zeta \geq 0.7$. This implies that:

$$2\zeta\omega_n = 4, \quad \zeta \geq 0.7 \Rightarrow \omega_n \leq 2.857$$

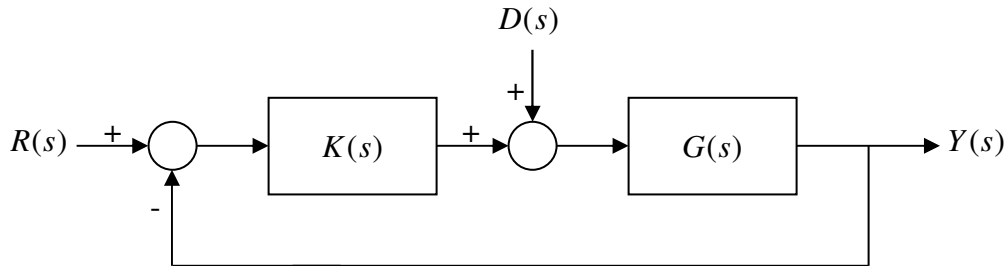
$$\omega_n^2 = K \Rightarrow K \leq 8.163 \Rightarrow K(s) = \frac{K}{s}, \quad 0 < K \leq 8.163$$

- In general, one can use the internal model principle for the steady state analysis of a unity feedback control system as follows.
- Internal model principle: Consider the following unity feedback control system:



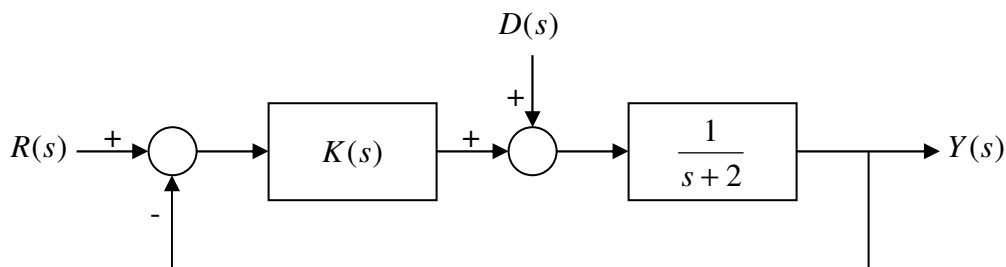
Assume that the closed loop system is stable. Assume also that $R(s)$ is a rational function of s and has no poles in the open LHP. Then the steady-state error e_{ss} is zero if and only if the poles of $G(s)$ include all poles of $R(s)$.

- For the effect of disturbance in the steady state, one can use the following extension of the internal model principle.
- Consider the following unity feedback control system:



Assume that the closed loop system is stable. Assume also that $G(s)$ has no zeros in the RHP or on the $j\omega$ axis, and that $D(s)$ is a rational function of s with no poles in the open LHP. Then, the effect of the disturbance $d(t)$ on the output $y(t)$ will go to zero as $t \rightarrow \infty$, if and only if the poles of $K(s)$ include all poles of $D(s)$.

- Note that in the case of reference input, we want the output to approach the input as $t \rightarrow \infty$ so that the steady-state error goes to zero but in the case of disturbance input, we want the output to go to zero as $t \rightarrow \infty$ (so that the effect of disturbance in the output is vanished with time).
- **Example 6.2:** Consider the following control system:



Design a controller $K(s)$ such that:

- i) The steady-state error for the step reference input is zero.
 - ii) The effect of the step disturbance input on the output goes to zero as $t \rightarrow \infty$.
 - iii) The percentage overshoot for a step reference input is 10%.
- **Solution:** Conditions (i) and (ii) can be satisfied by using $K(s) = \frac{K}{s}$ with a positive value for K (note that only for positive values of K the closed loop system will be stable). For this controller, we will have:

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + 2s + K}$$

$$\Rightarrow \zeta\omega_n = 1 \Rightarrow \omega_n = \frac{1}{\zeta} = \frac{1}{0.6} = 1.67$$

$$\Rightarrow K = \omega_n^2 = 2.78 \Rightarrow K(s) = \frac{2.78}{s}$$

- In order to take the history of the error into account (not just the steady state error), one can use one of the following performance indices:

$$\text{Integral of the square of the error : ISE} = \int_0^T e^2(t)dt$$

$$\text{Integral of the absolute magnitude of the error : IAE} = \int_0^T |e(t)|dt$$

$$\text{Integral of time multiplied by the absolute error : ITAE} = \int_0^T t|e(t)|dt$$

$$\text{Integral of time multiplied by the squared error : ITSE} = \int_0^T te^2(t)dt$$

- T is a sufficiently large finite number and is usually convenient to choose it as the settling time t_s .
- ISE and IAE are often used in the optimal control of practical systems.

The stability of linear feedback systems

- The Routh-Hurwitz (RH) stability criterion: The RH criterion is used to find the number of roots of a given polynomial in the RHP (including the $j\omega$ axis).
- Given a polynomial $q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_1 s + a_0$, we form the Routh table as follows:

$$\begin{array}{l|llll}
 s^n & a_n & & a_{n-2} & a_{n-4} & \dots \\
 s^{n-1} & a_{n-1} & & a_{n-3} & a_{n-5} & \dots \\
 s^{n-2} & b_{n-1} := \frac{a_{n-1}a_{n-2} - a_{n-3}a_n}{a_{n-1}} & b_{n-3} := \frac{a_{n-1}a_{n-4} - a_{n-5}a_n}{a_{n-1}} & & & \dots \\
 s^{n-3} & c_{n-1} := \frac{b_{n-1}a_{n-3} - b_{n-3}a_{n-1}}{b_{n-1}} & & & & \dots \\
 \vdots & \vdots & & & & \vdots \\
 \vdots & \vdots & & & & \vdots
 \end{array}$$

- Number of sign changes in the first column of the Routh table is equal to the number of the RHP roots of $q(s)$.
- **Example 6.3:** Consider the following polynomial:

$$q(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$$

Find the number of roots of the equation $q(s) = 0$ in the RHP.

- **Solution:** The Routh table for this polynomial is as follows:

$$\begin{array}{l|lll}
 s^4 & 1 & 3 & 5 \\
 s^3 & 2 & 4 & 0 \\
 + \curvearrowright s^2 & \frac{2 \times 3 - 4 \times 1}{2} = 1 & \frac{2 \times 5 - 0}{2} = 5 & 0 \\
 - \curvearrowleft s^1 & \frac{1 \times 4 - 5 \times 2}{1} = -6 & 0 & 0 \\
 + \curvearrowleft s^0 & \frac{-6 \times 5 - 0}{-6} = 5 & 0 & 0
 \end{array}$$

There are two changes of sign in the first column and so two of the roots are located in the RHP. The roots of this polynomial (obtained by using MATLAB) are in fact located at $0.2878 \pm j1.4161$ and $-1.2878 \pm j0.8579$.

- Note that the RH method only gives the number of RHP roots of a polynomial not the exact location of the roots. To check the stability of a system, it is sufficient to know if there are any roots in the RHP but for relative stability, it is important to know how far the roots of a stable system are from the imaginary axis. The RH method does not give such information.
- **Example 6.4:** Consider the following second order equation:

$$a_2s^2 + a_1s + a_0 = 0$$

Under what conditions are all of the roots of the above equation in the LHP?

- **Solution:** The Routh table for this equation is as follows:

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & \frac{a_1 \times a_0 - 0}{a_1} = a_0 & 0 \end{array}$$

For all of the roots to be in the LHP, a_2 , a_1 , and a_0 must have the same sign.

This result can be used for all second order equations.

- **Example 6.5:** Consider the following third order equation:

$$s^3 + a_2s^2 + a_1s + a_0 = 0$$

Under what conditions are all of the roots of the above equation in the LHP?

- **Solution:** The Routh table for this equation is as follows:

$$\begin{array}{c|cc} s^3 & 1 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & \frac{a_2 \times a_1 - a_0}{a_2} & 0 \\ s^0 & a_0 & 0 \end{array}$$

For all of the roots to be in the LHP, we must have:

$$\begin{aligned} a_2 &> 0, \\ \frac{a_2 a_1 - a_0}{a_2} &> 0 \Rightarrow a_2 a_1 > a_0, \\ a_0 &> 0. \end{aligned}$$

Special cases: In two special cases, the Routh table terminates prematurely.

- Special case 1: First element of a row is zero, but the entire row is not zero. In this case, we replace the zero in the first column with $\varepsilon > 0$ and continue the table to the end. Then, we let $\varepsilon \rightarrow 0$ and count the change of signs in the first column.
- **Example 6.6:** Find the number of RHP roots of the following equation.

$$s^4 + s^3 + 2s^2 + 2s + 3 = 0$$

Find the number of roots of the equation $q(s) = 0$ in the RHP.

- **Solution:** The Routh table for this equation is as follows:

s^4	1	2	3
s^3	1	2	0
s^2	ε	3	0
s^1	$\frac{2\varepsilon - 3}{\varepsilon}$	0	0
s^0	3	0	0

For $\varepsilon = 0^+$ the first column of the Routh table will be as follows:

1
1
0^+
$-\infty$
3

Since there are two sign changes in the first column for $\varepsilon = 0^+$, the equation has two roots in the RHP. The roots of this equation (obtained by using MATLAB) are in fact located at $-0.9057 \pm j0.902$ and $0.4057 \pm j1.2928$.

- Special case 1 always results in at least one root in the RHP.
- Another approach for special case 1: Roots of

$$q_1(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_1 s + a_0 = 0$$

and

$$q_2(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n = 0$$

do not change half planes. In other words, $q_1(s) = 0$ and $q_2(s) = 0$ have the same number of RHP roots. To verify this claim, define:

$$q_1(x) = q_1(s) \Big|_{s=1/x}.$$

We will have:

$$q_1(x) = \frac{a_n}{x^n} + \frac{a_{n-1}}{x^{n-1}} + \dots + \frac{a_2}{x^2} + \frac{a_1}{x} + a_0 = x^{-n} (a_n + a_{n-1}x + \dots + a_2 x^{n-2} + a_1 x^{n-1} + a_0 x^n)$$

Since the real part of s and $x = 1/s$ have the same sign, the number of RHP roots of $q_1(s) = 0$ and $q_2(s) = 0$ are the same.

- Special case 2: The entire row is zero. In this case, we identify an auxiliary equation from the coefficients of the row right above the zero row and proceed by differentiating this equation.
- **Example 6.7:** Consider the following equation:

$$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$$

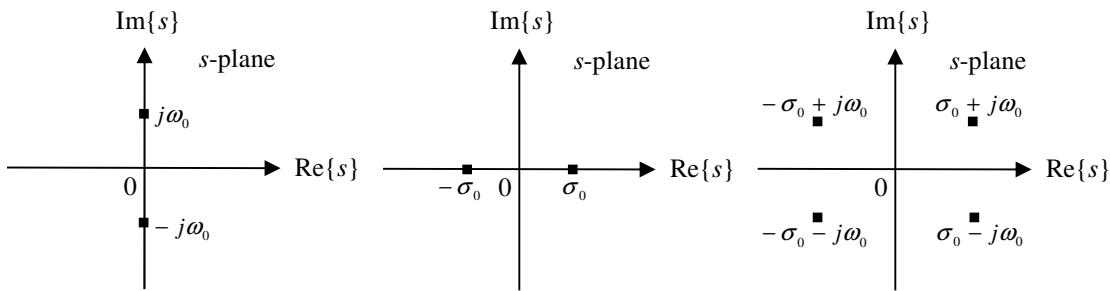
Use the RH method to find the number of RHP roots of this equation.

- **Solution:** The Routh table for this equation is as follows:

s^5	1	8	7	
s^4	4	8	4	
s^3	6	6	0	
s^2	4	4	0	Aux. eq'n : $A(s) = 4s^2 + 4$
s^1	\emptyset 8	0	0	$\frac{dA(s)}{ds} = 8s$
s^0	4	0	0	

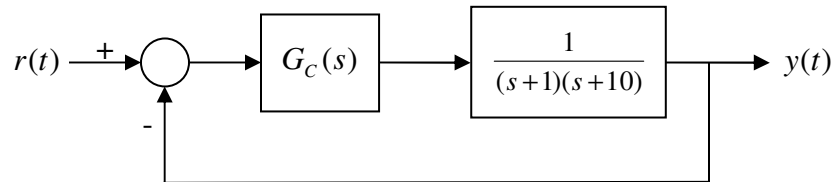
Note that to proceed with the table, we replace the zero row with the coefficients of $\frac{dA(s)}{ds} = 8s$. Since there is no change of sign in the first column, there are no roots in the RHP.

- Roots of the auxiliary equation $A(s) = 0$ are always the roots of the original polynomial. For instance, in Example 6.7, $s = \pm j$ are the roots of the auxiliary equation and also the roots of the original equation. Therefore, although this equation does not have any RHP roots, it does have roots on the imaginary axis.
- Roots of $A(s) = 0$ are always symmetrical with respect to the origin. For example they can have one of the following forms:



- For the case when the roots of the auxiliary equation are located at $\pm j\omega_0$ (no repeated roots), and all other roots are located in the LHP, the output of the corresponding system due to a zero input (and nonzero initial conditions) will have undamped sinusoidal oscillations with the frequency ω_0 , as t increases.

- Since the roots of a polynomial can be easily obtained by using a computer, the Routh-Hurwitz criterion is more useful in finding the dependency of the RHP poles to the parameters of the equation.
- **Example 6.8:** Consider the following closed-loop system:



Design the controller $G_C(s)$ such that the steady-state error due to a step input is zero.

- **Solution:** In order to meet the given design specifications, we must have a stable closed-loop system with at least one integrator in the forward-path transfer function. So, we will choose $G_C(s) = \frac{K}{s}$ and will try to set the value of K such that the roots of the characteristic equation are all in the LHP. The characteristic equation of the closed-loop system using the above controller is:

$$s^3 + 11s^2 + 10s + K = 0.$$

The Routh table will be as follows:

s^3	1	10
s^2	11	K
s^1	$\frac{110 - K}{11}$	0
s^0	K	0

So, as we had seen before, the condition for the stability of the closed-loop system is:

$$K > 0, 110 - K > 0 \Rightarrow 0 < K < 110.$$

For $K = 110$, we will have the second special case and the corresponding auxiliary equation will be $s^2 + 10 = 0$. Therefore the closed-loop system will have

two roots at $\pm j\sqrt{10}$ and the other root (which can be obtained by dividing the original equation by the auxiliary equation) will be at $s = -11$. This implies that for $K = 110$, the system will oscillate with the frequency $\omega = \sqrt{10}$ rad/sec.