ELEC 372 LECTURE NOTES, WEEK 7

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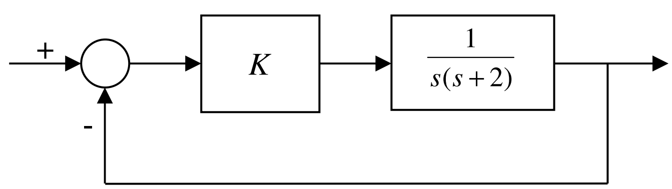
Root-locus technique

- It is a method to investigate the trajectories of the roots of the characteristic equation when a certain system parameter varies.
- **Example 7.1:** Plot the root locus for the characteristic equation of the following system as K varies.



- **Solution:** The characteristic equation for this system is given by:

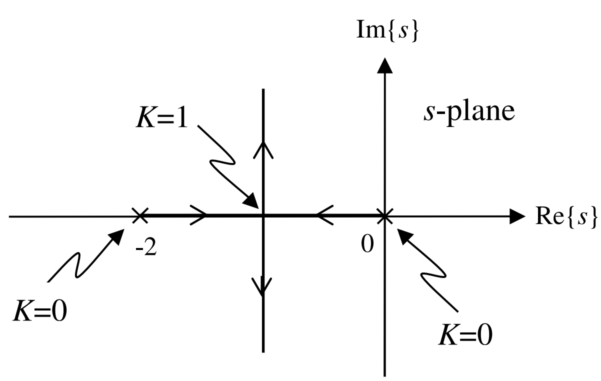
$$1 + K\frac{1}{s(s+2)} = 0 \Rightarrow s^2 + 2s + K = 0$$

So, the poles of the closed loop system will be:

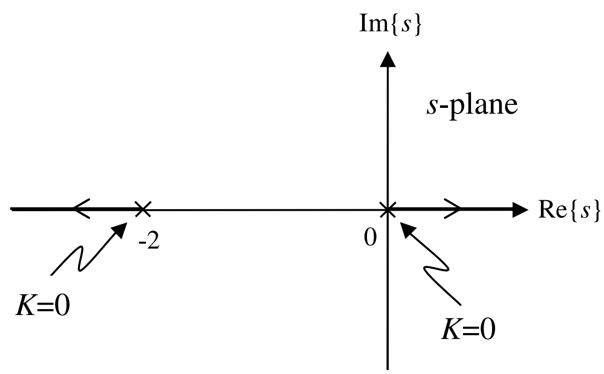
$$s_1,\ s_2 = -1 \pm \sqrt{1-K}$$

1) Root locus (RL) for $K \geq 0$:

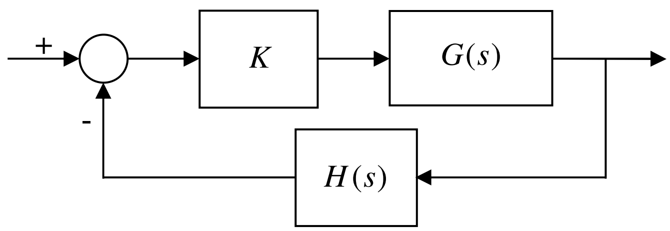
In this case, we will have the following trajectory:



2) Complementary root locus (CRL) or inverse root locus (IRL) for $K \leq 0$:



- We will now introduce a procedure to sketch the root locus for $-\infty \leq K \leq +\infty$ from the location of the zeros and poles of the open loop transfer function.
- Consider the following system:



- Here $G(s)$ and $H(s)$ are rational functions of s.
- Characteristic equation of this closed loop system is given by:

$$1 + KG(s)H(s) = 0$$

- $KG(s)H(s)$ is called the open loop transfer function or simply loop transfer function.
- Let $G(s)H(s)$ be equal to $\dfrac{b(s)}{a(s)}$. So the characteristic equation will be:

$$1 + K\frac{b(s)}{a(s)} = 0 \tag{7.1}$$

- Assume that the coefficients of the highest order term in $b(s)$ and $a(s)$ are equal to 1 (if this is not the case, we can multiply K by a constant to achieve this).

- If s_0 is a root of the characteristic equation, we must have:

$$K\frac{b(s_0)}{a(s_0)}=-1.$$

- This implies that the following two conditions must be satisfied:
  1) Condition on magnitude:

$$|K|\frac{|b(s_0)|}{|a(s_0)|}=1$$

  2) Condition on angles:

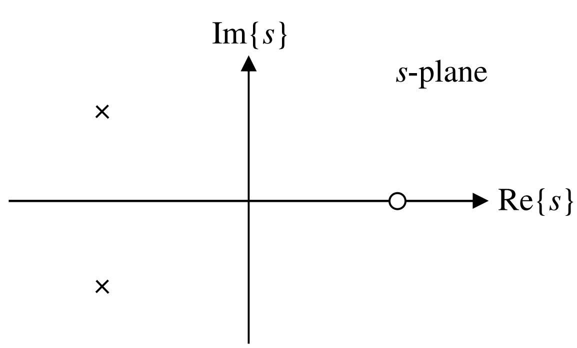
$$\begin{cases}\angle b(s_0)-\angle a(s_0)=(2l+1)\pi, & K\geq 0\\ \angle b(s_0)-\angle a(s_0)=2l\pi, & K\leq 0\end{cases},\quad l=0,\pm 1,\pm 2,\dots$$

- If condition on angles for a given point s_0 is met, then the corresponding point is on the root locus. From condition on magnitude, one can find the value of the gain K for which s_0 is on the root locus.

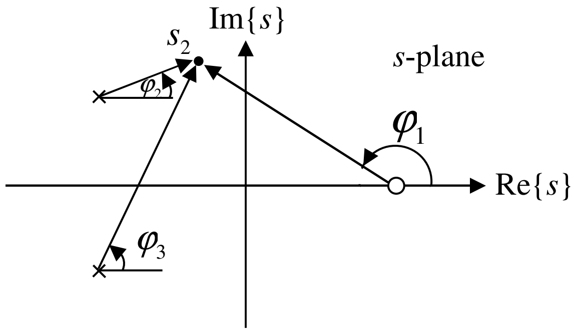
- Assume that $b(s)=\prod_{i=1}^{m}(s-z_i)$ and $a(s)=\prod_{j=1}^{n}(s-p_j)$, where p_j and z_i are the poles and zeros of the loop transfer function. If s_0 is on the root locus, we must have:

$$\sum_{i=1}^{m}\angle(s_0-z_i)-\sum_{j=1}^{n}\angle(s_0-p_j)=\begin{cases}(2l+1)\pi, & K\geq 0\\ 2l\pi, & K\leq 0\end{cases}$$

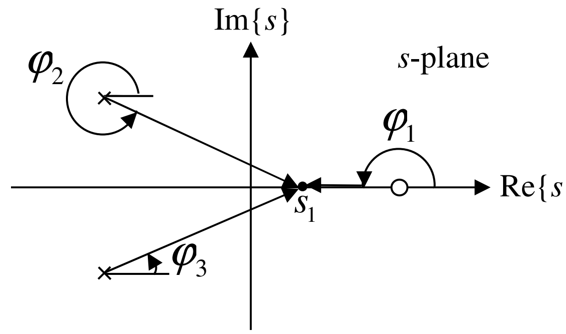
- For example, consider the following pole-zero configuration for the loop transfer function:



- Condition on angles for this system is illustrated in the following figures:



$\varphi_1 - \varphi_2 - \varphi_3 \neq l\pi$



$\varphi_1 - \varphi_2 - \varphi_3 = 180^{\circ} - 360^{\circ} = -180^{\circ}$

- So s_1 can be a point on the RL but s_2 cannot.
- In general, it can be easily verified that all points on the real axis belong to either RL or CRL because at any point on the real axis, the angles of the vectors drawn from the complex-conjugate poles or zeros add up to zero (or an even multiple of 180°). The angles of the vectors drawn from the real poles or zeros to any point on the real axis is a multiple of 180°.
- Construction of the root loci: Assume that the loop transfer function is proper ($n \geq m$). The following steps can be used to draw the RL or CRL trajectories.

1) For $K \to 0$, the characteristic equation given by (7.1) can be written as:

$$1 + K\frac{b(s)}{a(s)} = 0 \Rightarrow a(s) + Kb(s) = 0 \Rightarrow a(s) = 0.$$

This means that for $K \to 0$ the roots of the characteristic equation are the roots of $a(s) = 0$, which are the poles of the loop transfer function.

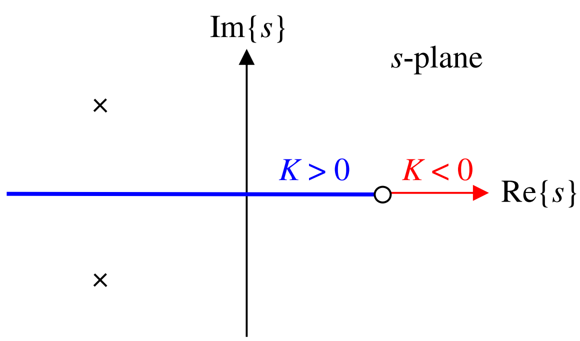
2) For $|K| \to \infty$, the characteristic equation given by (7.1) can be written as:

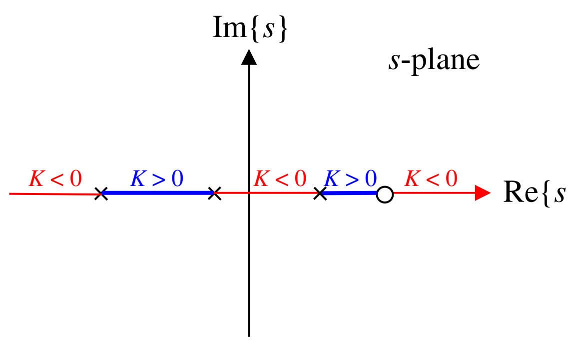
$$1 + K\frac{b(s)}{a(s)} = 0 \Rightarrow a(s) + Kb(s) = 0 \Rightarrow \frac{1}{K}a(s) + b(s) = 0 \Rightarrow b(s) = 0.$$

This means that for $K \to \infty$ the roots of the characteristic equation are the roots of $b(s) = 0$, which are the zeros of the loop transfer function. Note that the characteristic equation has n roots but the loop transfer function has only m finite zeros. However, a strictly proper transfer function also has $n - m$ zeros at infinity. So, for $|K| \to \infty$, m roots of the characteristic equation will go to the finite zeros of the loop transfer function and $n-m$ roots will go to infinity.

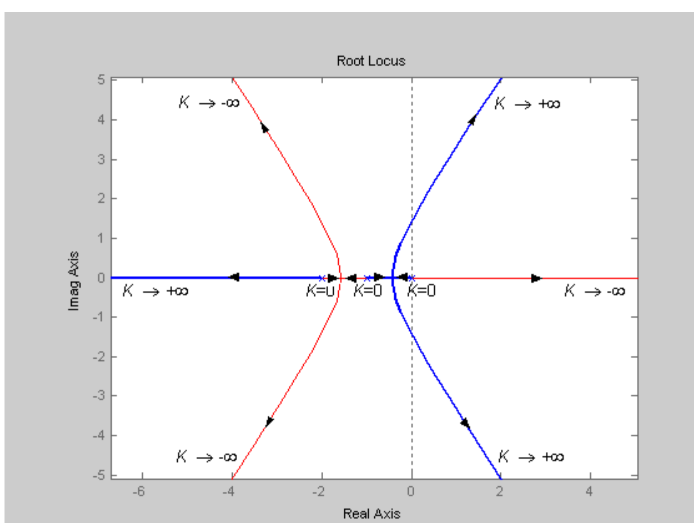
3) The entire real axis of the s-plane is occupied by the root loci.
   - On a given section of the real axis, if the total number of the poles and zeros of the loop transfer function to the right of the section is odd, that section corresponds to RL ($K \geq 0$).
   - On a given section of the real axis, if the total number of the poles and zeros of the loop transfer function to the right of the section is even, that section corresponds to CRL ($K \leq 0$).

For example, sections of the root loci on the real axis are given for $K \leq 0$ and $K \geq 0$ for the following pole-zero configurations:





It is to be noted that the RL and CRL can have branches other than the real axis. For example, we can have trajectories such as the following:



The following steps are very helpful in plotting the other branches of the RL and CRL.

4) As pointed out in step (2), for $|K| \to \infty$, $n-m$ roots of the characteristic equation will go to infinity. So, for $n > m$, the direction of the roots going towards infinity can be determined by the corresponding asymptotes. The intersect of the asymptotes (asymptote centroid) is given by:

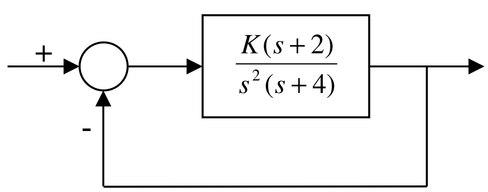
$$\sigma = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m}$$

where p_j and z_i represent the poles and zeros of the loop transfer function. The angles of the asymptotes are given by:

$$\theta_r = \begin{cases} \dfrac{(2r+1)\pi}{n-m}, & K > 0 \\ \dfrac{2r\pi}{n-m}, & K < 0 \end{cases}, \quad r = 0, 1, \ldots, n-m-1$$

Note that if the loop transfer function is not strictly proper, there will be no asymptotes. In other words, the asymptotes exist if and only if $n > m$.

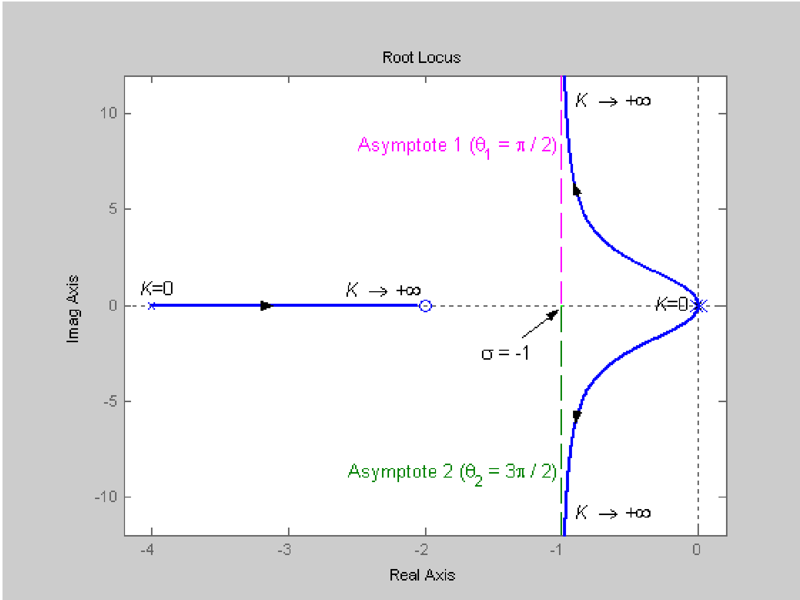
- **Example 7.2:** Plot the root locus of the characteristic equation of the following system for positive values of K.



- **Solution:** We have:

$$\sigma = \frac{-4+0+0-(-2)}{3-1} = -1$$

$$\theta_r = \frac{\pi}{2}, \frac{3\pi}{2} \quad K > 0$$



5) Breakaway points: Breakaway points on the root loci correspond to multiple-order roots.

In Example 7.1, the point $s = -1$ (which corresponds to $K = 1$) is a breakaway point. Also, in Example 7.2, the point the origin of the *s*-plane (which corresponds to $K = 0$) is a breakaway point.

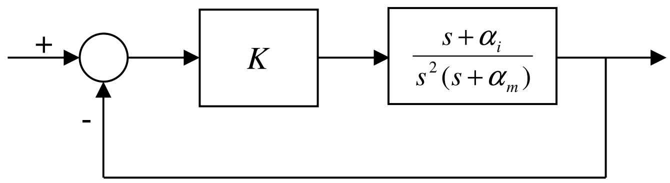
The breakaway point on the root loci of $1 + KG(s)H(s) = 0$ must satisfy the following equation:

$$\frac{d[G(s)H(s)]}{ds} = 0 \,. \tag{7.2}$$

It is to be noted that not all roots of (7.2) are necessarily the breakaway points. Only those roots of (7.2) that also satisfy the characteristic equation for a real K represent breakaway points. Sometimes it can be easily seen that a breakaway point is required in a branch of the root loci. For example, a branch between two poles or two zeros, must have a breakaway point so that when K changes between 0 and $\pm\infty$, the trajectory goes from a pole to a zero or to infinity. In Example 7.1, a breakaway point is expected to exist between $s = 0$ and $s = -2$ and there is no need to check if the root of the equation for the breakaway point at $s = -1$ satisfies the characteristic equation for a real K.

In Examples 7.1 and 7.2 the breakaway points are located on the real axis but in general, they can be anywhere on the root loci, including the complex points.

- **Example 7.3:** Plot the root locus for the characteristic equation of the following system for different values of α_i and α_m. Assume that $K > 0$.



- **Solution:** We have:

$$\sigma = \frac{-\alpha_m - (-\alpha_i)}{3-1} = \frac{\alpha_i - \alpha_m}{2}$$

$$\theta_r = \frac{\pi}{2}, \frac{3\pi}{2} \quad (K > 0)$$

Equation for the breakaway points:

$$\frac{d\left[\frac{s+\alpha_i}{s^2(s+\alpha_m)}\right]}{ds} = 0$$

$$\Rightarrow s^2(s+\alpha_m)\frac{d(s+\alpha_i)}{ds} - (s+\alpha_i)\frac{d[s^2(s+\alpha_m)]}{ds} = 0$$

$$\Rightarrow 2s^3 + (3\alpha_i + \alpha_m)s^2 + 2\alpha_i\alpha_m s = 0 \tag{7.3}$$

In this example, the form of root loci will be different for $\alpha_m = 9\alpha_i$, $\alpha_i < \alpha_m < 9\alpha_i$, and $\alpha_m > 9\alpha_i$.

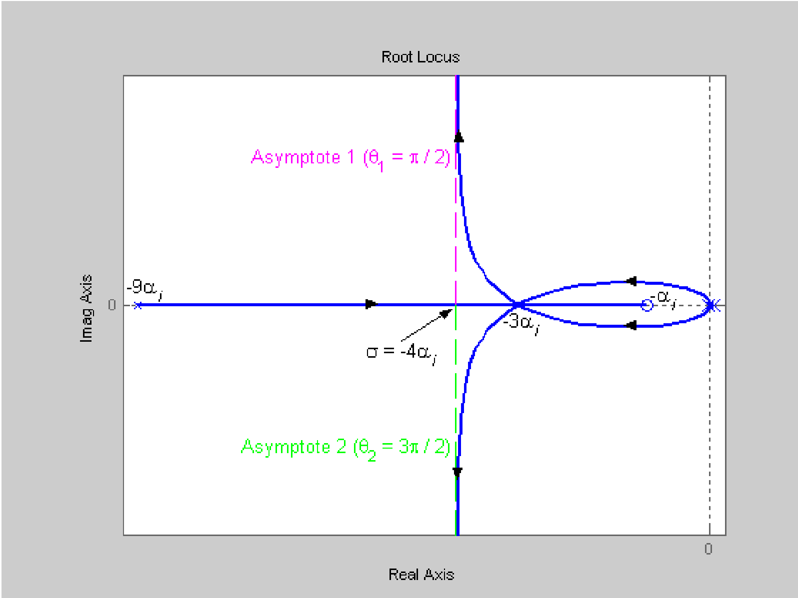
i) For $\alpha_m = 9\alpha_i$, the asymptote centriod will be:

$$\sigma = -4\alpha_i$$

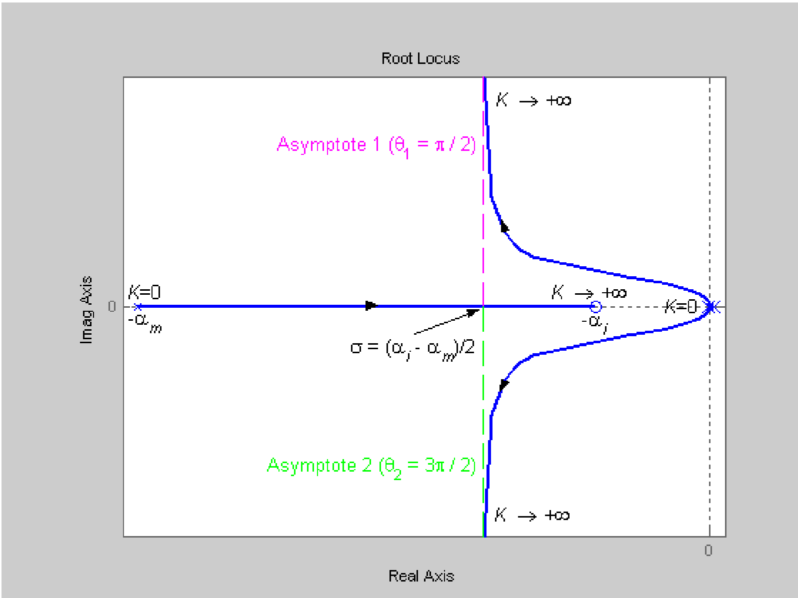
and the roots of the equation for the breakaway points will be:

$$s_1 = 0,\ \ s_2, s_3 = -3\alpha_i.$$

It can be easily verified that $s_1 = 0$ is a root of the characteristic equation $s^2(s+9\alpha_i) + K(s+\alpha_i) = 0$ for $K = 0$. $s_2, s_3 = -3\alpha_i$ are also roots of the characteristic equation for $K = 27\alpha_i^2$. This means that s_1, s_2, s_3 are all breakaway points. The root loci for $\alpha_m = 9\alpha_i$ and $K > 0$ are sketched in the following figure.



ii) For $\alpha_i < \alpha_m < 9\alpha_i$, the root loci will be as follows:

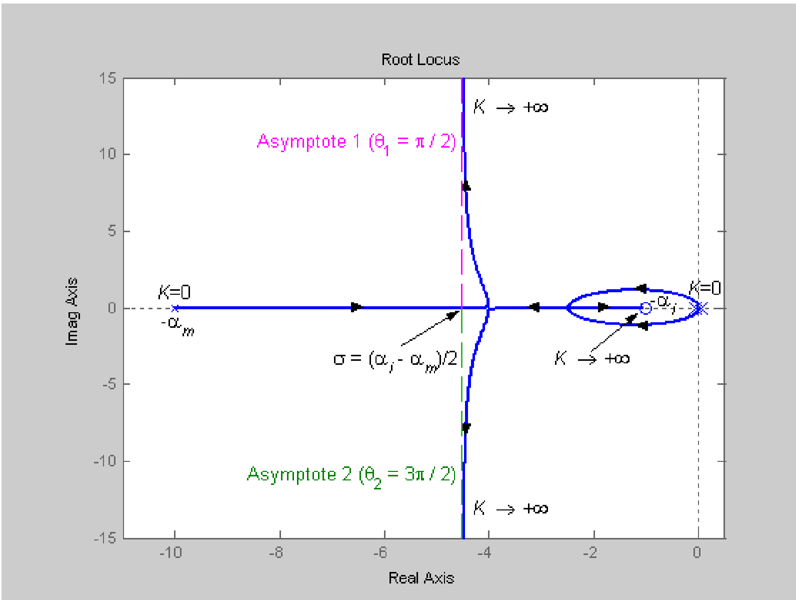


In this case, the roots of the equation (7.3) for the breakaway points are:

$$s_1 = 0, \;\; s_2, s_3 = \frac{-3\alpha_i - \alpha_m \pm \sqrt{9\alpha_i^2 + \alpha_m^2 - 10\alpha_i\alpha_m}}{4}$$

but only $s_1 = 0$ satisfies the characteristic equation for a real K. This means that s_2 and s_3 are not breakaway points in this case (for example this can be verified by choosing $\alpha_m = 5\alpha_i$).

iii) For $\alpha_m > 9\alpha_i$, the root loci will be as follows:



Note that in this case the equation for the breakaway points has three real roots and all of them satisfy the characteristic equation for real values of K and as it is shown in the figure, all of them are breakaway points.

- The following tips can also be very helpful in plotting the RL and CRL in general.
  - The root loci for all K are symmetrical with respect to the real axis of the s-plane and any axes of symmetry of the pole-zero configuration of the loop transfer function. For instance, the RL and CRL in Examples 7.1, 7.2 and 7.3 are all symmetrical with respect to the real axis and in Example 7.1 they are also symmetrical with respect to the line $\text{Re}\{s\} = -1$, which is an axis of symmetry for the poles and zeros of the loop transfer function.