ELEC 372 LECTURE NOTES, WEEK 8

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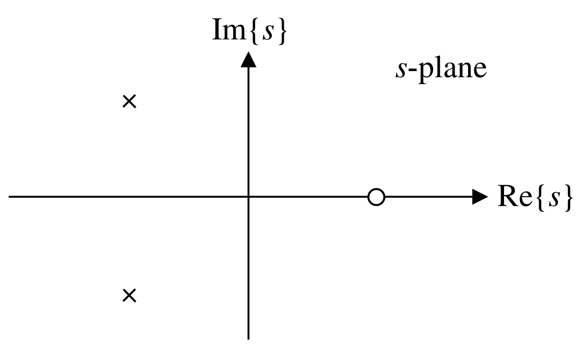
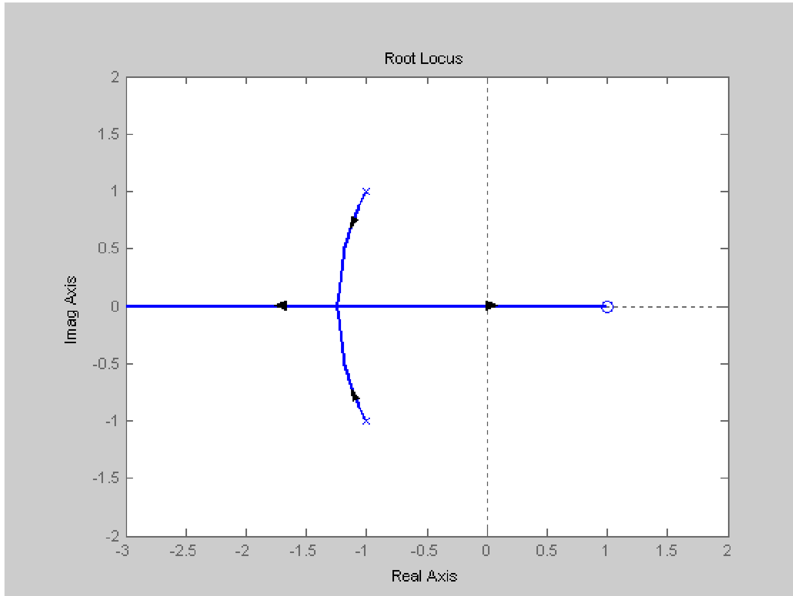
Root-locus technique (cont'd)

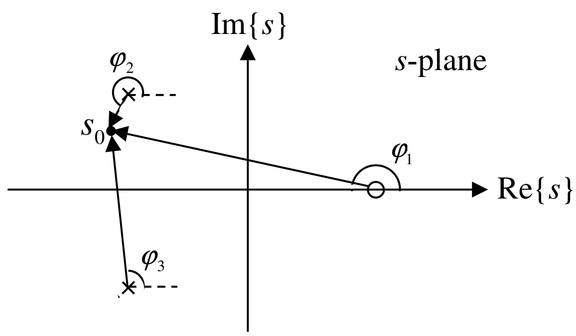
- The following tips can also be very helpful in plotting the RL and CRL in general.
    - The root loci for all K are symmetrical with respect to the real axis of the s-plane and any axes of symmetry of the pole-zero configuration of the loop transfer function. For instance, the RL and CRL in Examples 7.1, 7.2 and 7.3 are all symmetrical with respect to the real axis and in Example 7.1 they are also symmetrical with respect to the line $\mathrm{Re}\{s\} = -1$, which is an axis of symmetry for the poles and zeros of the loop transfer function.
    - The intersect of the branches of RL or CRL with the imaginary axis can be determined using the Routh-Hurwitz method and finding the value of K for which special case 2 is satisfied with roots of the auxiliary equation on the imaginary axis. Those roots will in fact be the intersection of the RL or CRL with the imaginary axis.
    - The angle of departure of the locus from a pole and the angle of arrival of the locus at a zero can be determined using the condition on angles. For example, consider the following pole-zero configuration for the loop transfer function:



Assume that the zero of the loop transfer function is located at $s = 1$ and the poles are located at $s_1, s_2 = -1 \pm j$. The RL for this example is given below.



It is desired to find the angle of locus departure from the pole $s_1 = -1 + j$. Consider a point s_0 on the RL, very close to s_1 as shown below.



From the condition on angles, we will have:

$$\varphi_1 - \varphi_2 - \varphi_3 = \pi$$

Since s_0 is very close to s_1, we can write:

$$\varphi_1 = \pi - \tan^{-1}\left(\frac{1}{2}\right),$$

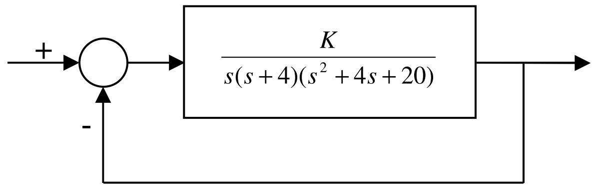
$$\varphi_3 = \frac{\pi}{2}.$$

This implies that:

$$\varphi_2 = \varphi_1 - \varphi_3 - \pi = \pi - \tan^{-1}\left(\frac{1}{2}\right) - \frac{\pi}{2} - \pi \equiv -116.6^\circ \equiv 243.4^\circ$$

In general, one can use the above method to find the angle of departure of the locus from a pole and the angle of arrival of the locus at a zero.

- **Example 8.1:** Plot the root locus of the characteristic equation of the following system for all values of K.



- **Solution:** Open loop poles are located at $0,\ -4,\ -2 \pm j4$. Hence, we have:

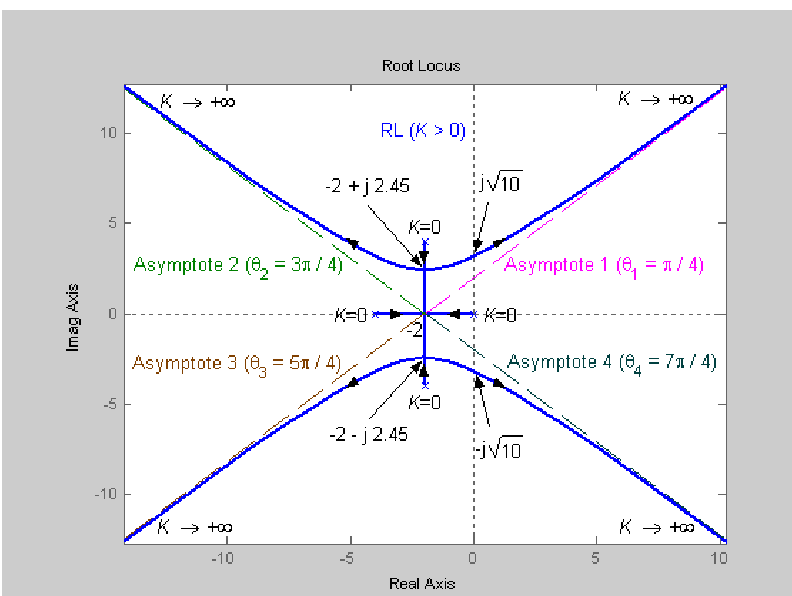
$$\sigma = \frac{-4-2-2}{4} = -2$$

$$\theta_r = \begin{cases} \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} & K > 0 \\ 0, \frac{2\pi}{4}, \frac{4\pi}{4}, \frac{6\pi}{4} & K < 0 \end{cases}$$

Breakaway points:

$$\frac{dG(s)}{ds} = 0 \Rightarrow s_1 = -2,\ \ s_2, s_3 = -2 \pm j2.45$$

So, for $K > 0$ we will have:



To find the point at which the trajectory crosses the imaginary axis, we can use the RH method as follows:

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

s^4	1	36	K
s^3	8	80	0
s^2	$\dfrac{8\times 36 - 80\times 1}{8} = 26$	K	0
s^1	$\dfrac{26\times 80 - 8K}{26}$	0	0
s^0	K	0	0

In order to have roots on the imaginary axis, all elements of one row must be zero, which implies that $K = 260$. The corresponding auxiliary equation will be:

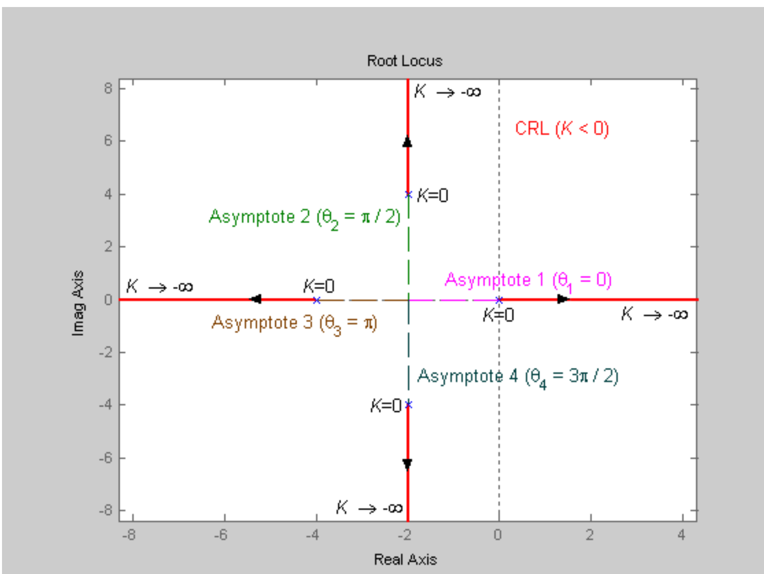
$$26s^2 + 260 = 0$$

Since the roots of the auxiliary equation are the roots of the characteristic equation too, the following points will be on the RL:

$$s_1,\, s_2 = \pm j\sqrt{10}$$

In other words, the RL will intersect the imaginary axis at $\pm j\sqrt{10}$ for $K = 260$.

For $K < 0$ we will have:



Note that the root loci are symmetrical w.r.t. the line $\mathrm{Re}\{s\} = -2$ (and the real axis) as this line is the axis of symmetry for the open loop poles.

- Using the root locus method we can find out how the location of the roots of a polynomial varies in terms of the coefficients of the polynomial.
- **Example 8.2:** Find the location of the roots of the following polynomial as α varies.

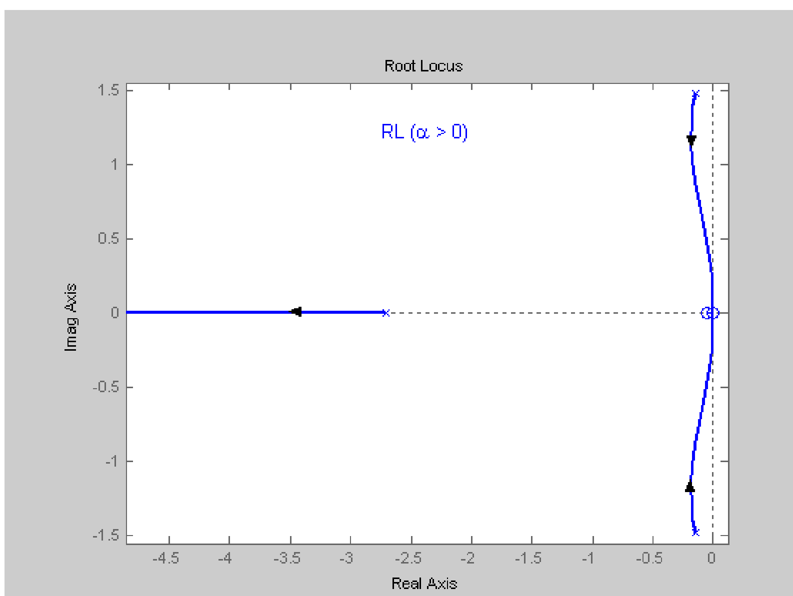
$$s^3 + (3+\alpha)s^2 + 3s + 6 = 0$$

- **Solution:** We have:

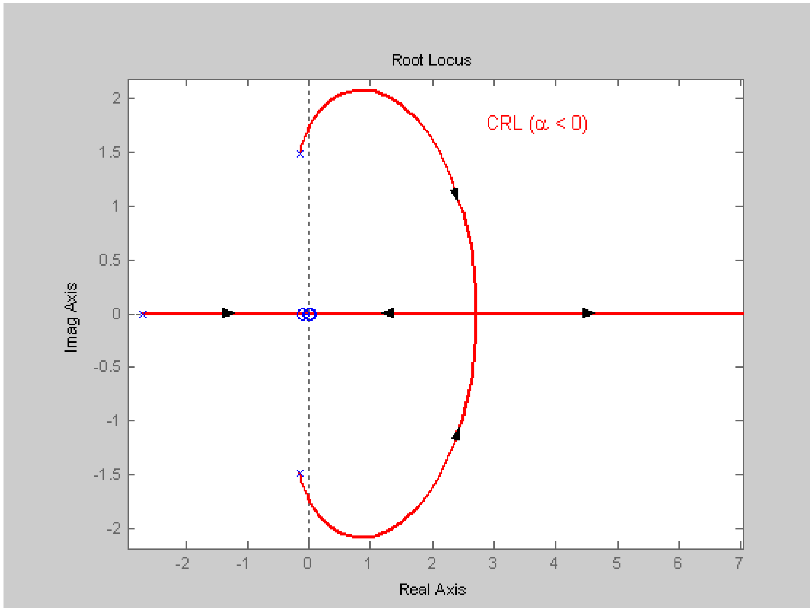
$$s^3 + 3s^2 + 3s + 6 + \alpha s^2 = 0 \Rightarrow 1 + \alpha \frac{s^2}{s^3 + 3s^2 + 3s + 6} = 0$$

We can now assume that we have a virtual open loop system $G(s) = \dfrac{s^2}{s^3 + 3s^2 + 3s + 6}$ with the controller gain α, and use the root locus method.

For $\alpha > 0$ (RL):

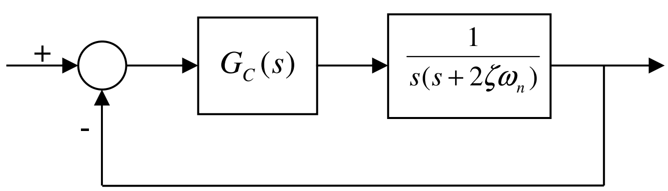


For $\alpha < 0$ (CRL):

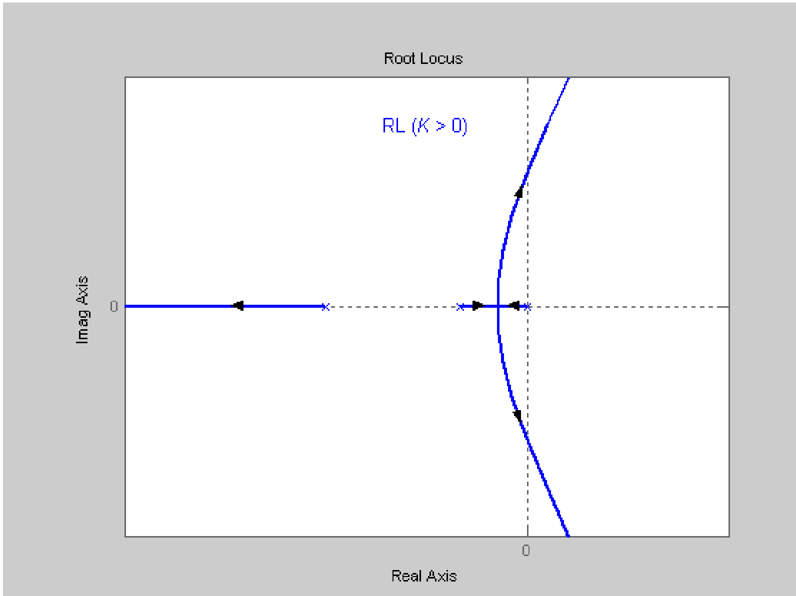


Effects of adding poles and zeros to the forward-path transfer function

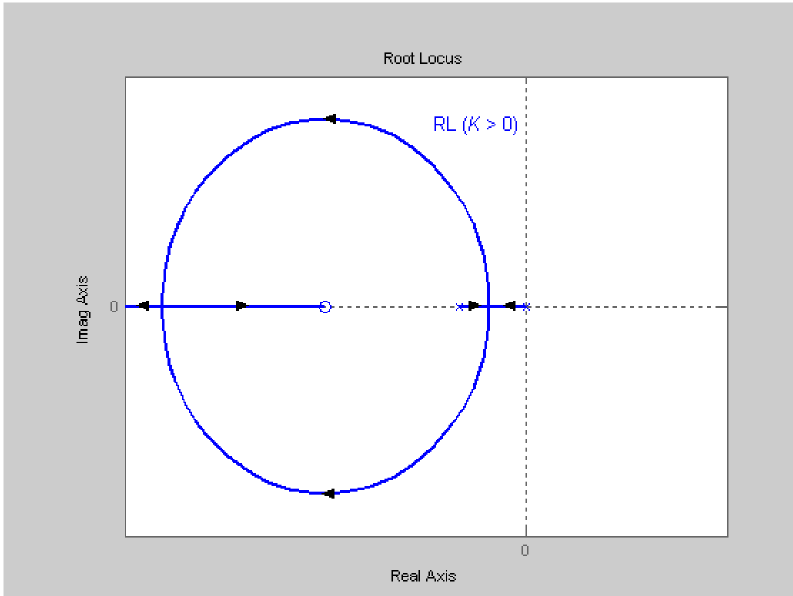
- Consider the following feedback control system with a second order plant:



- Effect of adding a LHP pole to the forward-path transfer function: Consider the controller $G_C(s) = \dfrac{K}{s+p}$. The RL in this case will be as follows:



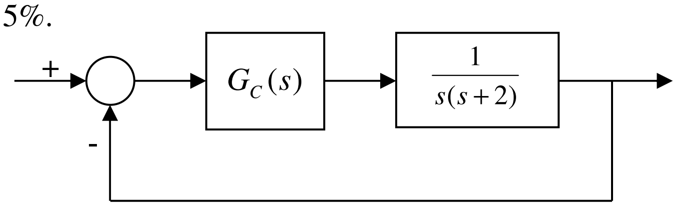
- Note that without the added pole, the trajectory for RL would be two straight lines, one on the real axis, and one parallel to the imaginary axis but with the added pole, the trajectory parallel to the imaginary axis will be pushed toward the RHP and a new section on the real axis is added to the RL.
- By increasing the gain K, two of the roots of the characteristic equation will move to the RHP and the closed loop system will be unstable.
- The shape of the trajectory depends on the location of the added pole. This means that the distance of the added pole from the imaginary axis will affect the location of the closed loop poles, which will also affect the transient response (mainly percentage overshoot and rise time).
- Effect of adding a LHP zero to the forward-path transfer function: Consider the controller $G_C(s) = K(s + z)$. The RL in this case will be as follows:



- By adding the zero, the trajectory parallel to the imaginary axis moves and bends toward the added LHP zero and a new section on the real axis is added to the RL.
- Note that as long as the zero is in the LHP, the closed loop system will remain stable for all positive values of K.
- The shape of the trajectory depends on the location of the added zero. This means that the distance of the added zero from the imaginary axis will affect the location of the closed loop poles, which will also affect the transient response (mainly percentage overshoot and rise time).
- Using a controller with both zeros and poles $K(s) = K\dfrac{(s+z)}{(s+p)}$, one may locate the roots of the characteristic equation in a desired region in the *s*-plane.

Controller design using the root locus method

- The root locus method can be used to design a controller by placing the closed-loop poles properly, based on the design specification.
- **Example 8.3:** Design the controller $G_C(s)$ for the following servomotor so that the settling time is less than 2 sec and the percentage overshoot is less than or equal to 5%.

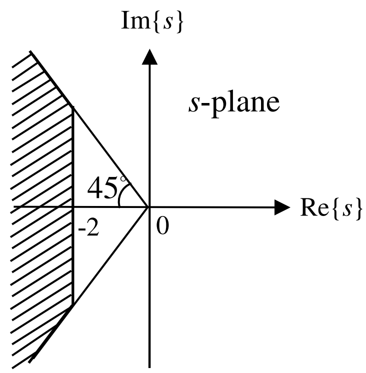


- **Solution:** We have:

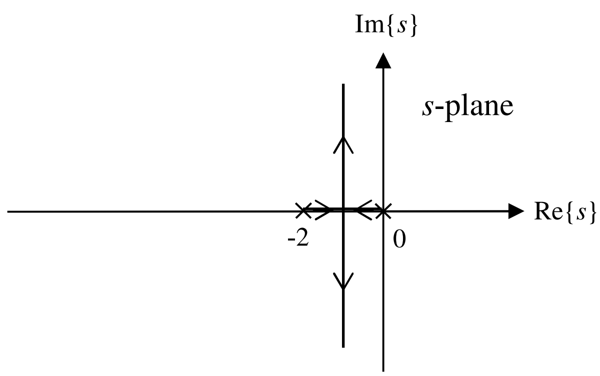
$$t_s \le 2 \text{ sec} \Rightarrow \frac{4}{\xi\omega_n} \le 2 \Rightarrow \xi\omega_n \ge 2$$

$$P.O. \le 5\% \Rightarrow \zeta \ge 0.7 \Rightarrow \theta = \cos^{-1}\zeta \le 45^\circ$$

The desired region for the closed loop poles is given in the following figure:



Using a constant gain $G_C(s) = K$, we will have the following root loci:



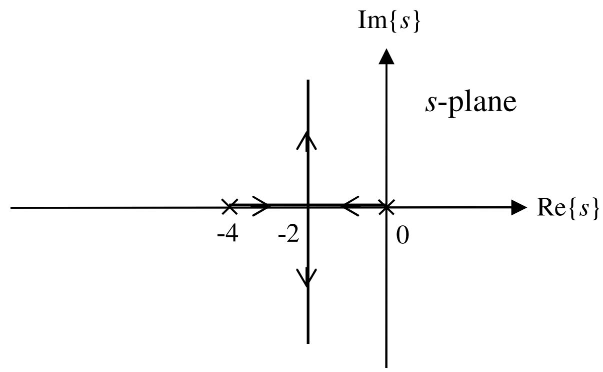
It can be easily seen that using a constant controller the desired specifications cannot be met (no complex poles on the trajectory will be inside the desired region).

Now, consider the following controller:

$$G_C(s) = K\frac{(s+2)}{(s+4)}$$

Note that using this controller, there will be a pole-zero cancellation in the loop transfer function, which results in:

$$G_C(s)G(s) = \frac{K}{s(s+4)}$$



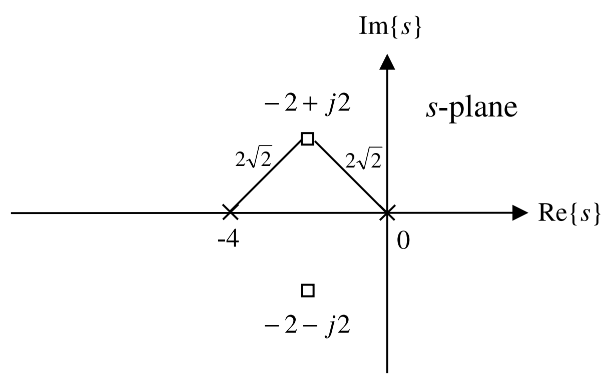
Set the following closed loop poles which are on the RL:

$$s_1,\, s_2 = -2 \pm j2$$

The condition on magnitude will now be used to find the value of K corresponding to the above poles.

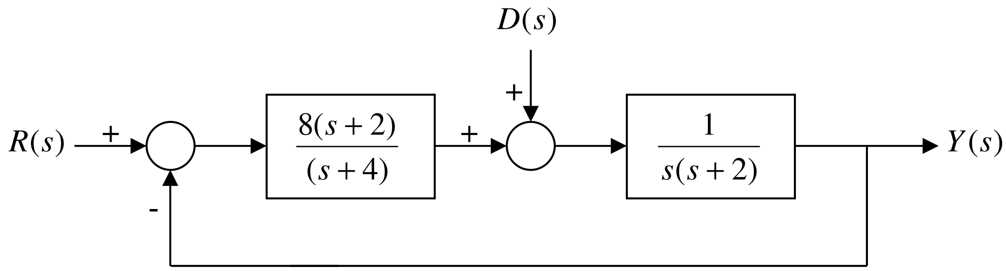
$$\left|\frac{K}{s(s+4)}\right|_{s=-2+j2} = 1 \Rightarrow K = 8$$

One can also use the length of the vectors in the following figure to find K.



$$\frac{K}{2\sqrt{2}\times 2\sqrt{2}} = 1 \Rightarrow K = 8$$

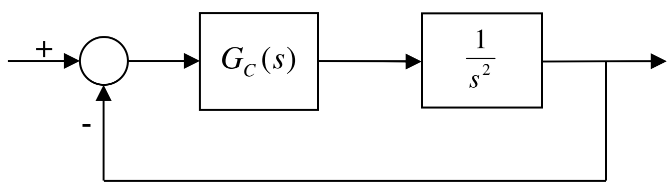
Note that in the presence of disturbance, the cancelled pole will appear in the transfer function from disturbance to the output. This is shown in the following block diagram:



$$\frac{Y(s)}{R(s)} = \frac{8}{s^2+4s+8}$$
$$\frac{Y(s)}{D(s)} = \frac{(s+4)}{(s+2)(s^2+4s+8)}$$

This implies that if there is any RHP pole-zero cancellation, that pole will appear in the transfer function from other inputs to the output, which makes the overall system unstable.

- **Example 8.4:** Design the controller $G_C(s)$ for the following servomotor so that the settling time is less than or equal to 2 sec and the percentage overshoot of the step response is less than or equal to 5%.

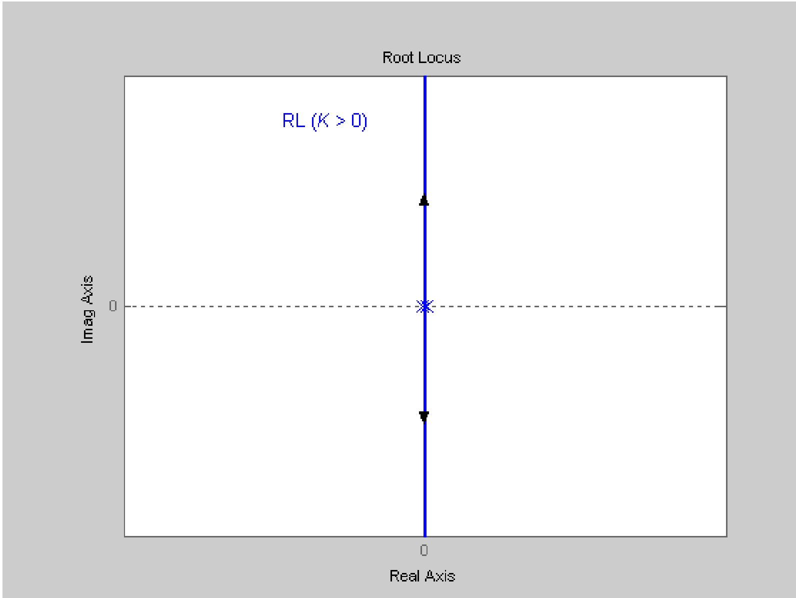


- **Solution:** From the design specifications we have:

$$t_s \leq 2 \text{ sec} \Rightarrow \frac{4}{\xi\omega_n} \leq 2 \Rightarrow \xi\omega_n \geq 2$$

$$P.O. \leq 5\% \Rightarrow \zeta \geq 0.7 \Rightarrow \theta = \cos^{-1}\zeta \leq 45^\circ$$

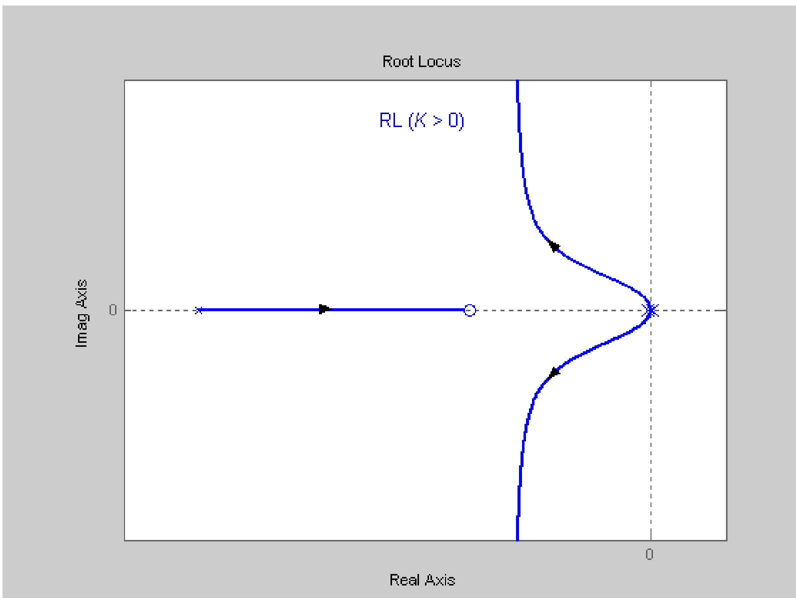
We will examine a simple constant controller first. From the following root locus trajectory it can be easily seen that the roots of the characteristic equation resulted from a constant gain will be on the imaginary axis, and hence will not do the job.



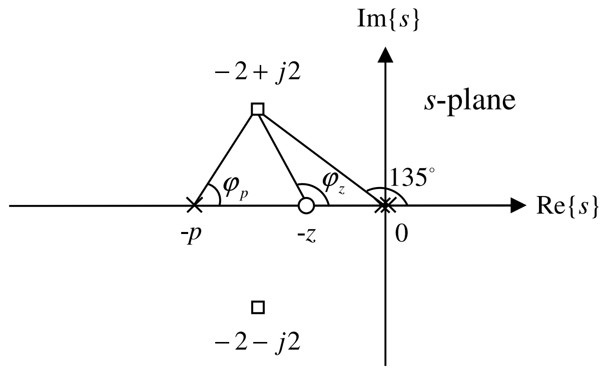
Consider now the following controller:

$$G_C(s) = K\frac{(s+z)}{(s+p)}.$$

For $-p < -z < 0$, the RL will be as follows:



It is desired to find K, z and p, so that $s_1, s_2 = -2 \pm j2$ be the dominant poles of the closed loop system.



$$\varphi_z - \varphi_p - 135^\circ - 135^\circ = 180^\circ \Rightarrow \varphi_z - \varphi_p = 90^\circ$$

In general, when the zero of the controller $G_C(s) = K\dfrac{(s+z)}{(s+p)}$ is closer to the imaginary axis compared to the pole ($-p < -z < 0$), the angle introduced by the controller will be positive and the controller is called phase-lead network or phase-lead compensator or simply lead controller. One can fix φ_z and find φ_p. Then z and p can be obtained accordingly. If we choose values so that p/z is greater than 15, the electronic elements for the design of the controller will be unreasonable. To find K, we should use condition on magnitude as follows:

$$\left.\frac{|K|\times|s+z|}{|s^2(s+p)|}\right|_{s=-2+j2} = 1\,.$$

It is to be noted that with this controller the characteristic equation will be third order and so the closed loop system will have three poles. One must check to see if for the chosen values of z and p, the complex poles are dominant. This means that the third root of the characteristic equation corresponding to the gain K obtained from the condition on magnitude must be at least 5 times farther from the imaginary axis in the LHP compared to the complex poles. Since the real part of the complex roots is equal to -2, for the third root s_3 we must have:

$$s_3 < -10\,.$$

One may find the proper values of z and p by trial and error to satisfy the above inequality.