

## ELEC 372 LECTURE NOTES, WEEK 9

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*Parts of these notes are adapted from the materials in the following references:*

- Modern Control Systems by Richard C. Dorf and Robert H. Bishop, Prentice Hall.
- Feedback Control of Dynamic Systems by Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, Prentice Hall.
- Automatic Control Systems by Farid Golnaraghi and Benjamin C. Kuo, John Wiley & Sons, Inc., 2010.

### Controller design using the root locus method (cont'd)

- The lead controller improves the transient response by adding positive phase to the open loop transfer function.
- One can improve the steady-state performance of the system by increasing the error constants ( $K_p$ ,  $K_v$ ,  $K_a$ ) using a proper controller.
- Consider a unity feedback control system with a forward-path transfer function

$$G(s) = K_0 \frac{\prod_{i=1}^m (s + z_i)}{s \prod_{j=1}^n (s + p_j)}. \text{ The velocity error constant will be:}$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} sK_0 \frac{\prod_{i=1}^m (s + z_i)}{s \prod_{j=1}^n (s + p_j)} = K_0 \frac{\prod_{i=1}^m (z_i)}{\prod_{j=1}^n (p_j)}$$

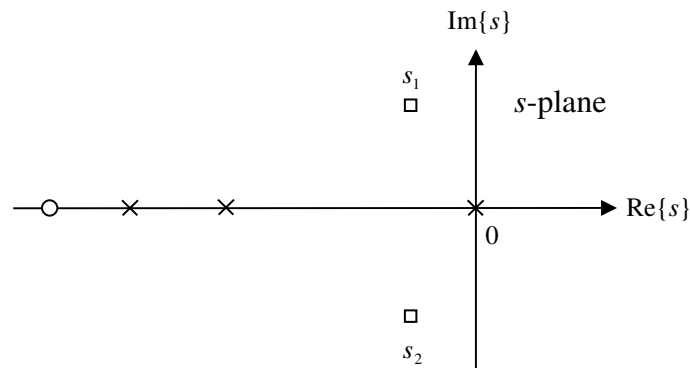
- Consider now a controller  $G_C(s) = \frac{(s+z)}{(s+p)}$  for this system. We will have:

$$K_{v,new} = \lim_{s \rightarrow 0} sG(s)G_C(s) = \frac{z}{p} \times K_{v,old}$$

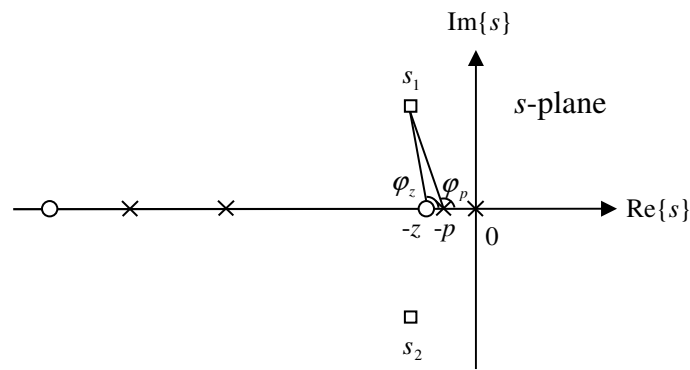
- This means that by choosing a sufficiently large ratio  $\frac{z}{p}$ , one can increase the velocity error constant arbitrarily. In other words, one can improve the steady-state response by using a controller with a dominant pole.

In general, a controller of the form  $G_c(s) = K \frac{(s+z)}{(s+p)}$  whose pole and zero are both in the LHP with its pole closer to the imaginary axis compared to its zero ( $-z < -p < 0$ ) is called phase-lag network or phase-lag compensator or simply lag controller.

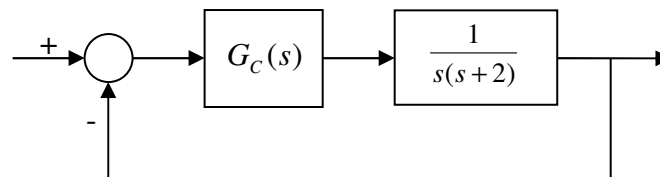
- Note that the location of the zero and pole of the lag controller must be chosen such that the dominant poles of the closed loop system remain unchanged (approximately). This means that the zero and pole of the lag controller must be close to each other so that the angle contributed by them in the condition on angles is negligible.
- For example, let  $s_1$  and  $s_2$  be the dominant poles of the closed loop system corresponding to the following pole zero configuration:



- Assume now that it is desired to improve the steady-state response of the system by using a lag controller as shown in the following figure.



- By adding the zero and pole shown in this figure, the angle introduced by the new open loop zero and pole  $\varphi_z - \varphi_p$  will be negligible, which means that  $s_1$  and  $s_2$  will still remain on the RL (or CRL) trajectories. In other words, since the new open loop zero ( $s = -z$ ) and pole ( $s = -p$ ) are very close, they will cancel out the effect of each other in the transient response to a great extent, and hence the dominance of  $s_1$  and  $s_2$  will not be affected by the controller. However, the DC gain introduced by the zero and the pole of the controller will improve the steady-state response by increasing the error constants.
- In order to have a large  $\frac{z}{p}$  ratio while  $z$  and  $p$  are very close, the zero and pole of the lag controller must be located close to the imaginary axis.
- Note that the main difference between the RL (or CRL) with and without the lag controller will be the section between the new open loop zero ( $s = -z$ ) and pole ( $s = -p$ ) on the real axis. If this section was part of the RL (or CRL) without the controller, it will be excluded from the new RL (or CRL) and if it was not part of the RL (or CRL), it will be included in the new RL (or CRL).
- **Example 9.1:** Design the controller  $G_C(s)$  for the following servomotor (which was given in Example 8.3) so that the settling time is less than or equal to 2 sec and the percentage overshoot of the step response is less than or equal to 5% and  $K_v \geq 16$ .



- **Solution:** From Example 8.3, we know that by using the lead controller

$$G_{C,\text{lead}}(s) = 8 \frac{(s+2)}{(s+4)}$$

the settling time and percentage overshoot specifications will

be met. With this controller, the velocity error constant will be:

$$K_v = \lim_{s \rightarrow 0} sG(s)G_C(s) = \lim_{s \rightarrow 0} 8s \frac{1}{s(s+4)} = 2.$$

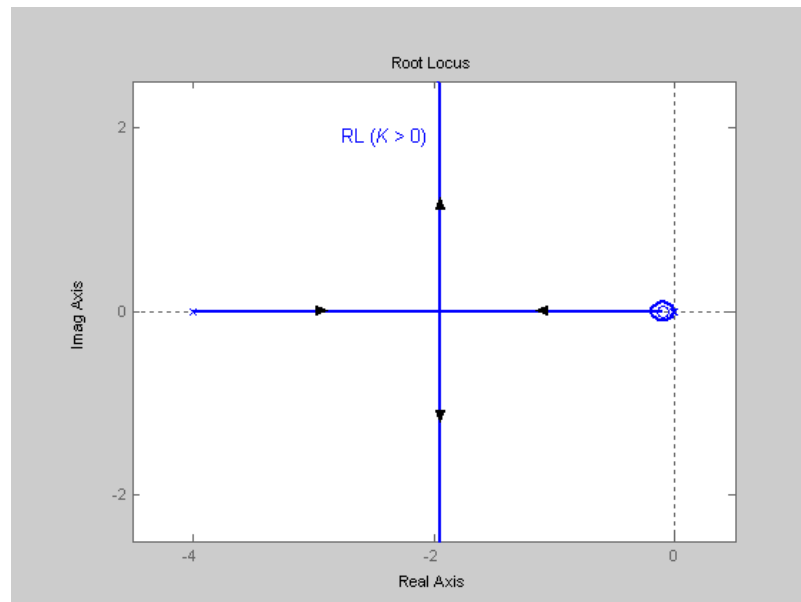
In order to increase  $K_v$  from 2 to 16, we need a lag controller  $G_{C,\text{lag}}(s) = \frac{(s+z)}{(s+p)}$

with  $\frac{z}{p} = 16/2 = 8$ . Let us choose  $z = 0.1$  and  $p = 0.0125$ . Therefore, the overall

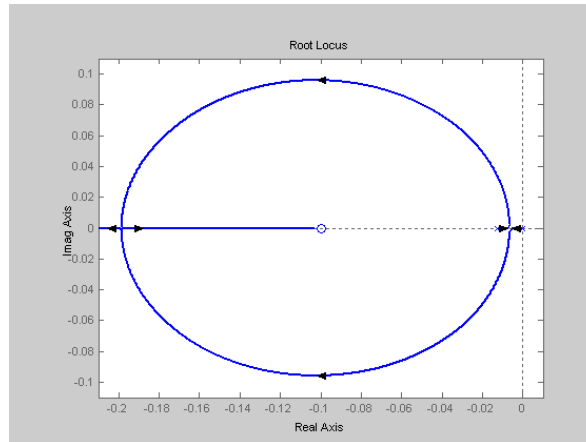
controller will be:

$$G_C(s) = 8 \frac{(s+2)(s+0.1)}{(s+4)(s+0.0125)}.$$

This controller consists of a lead and a lag controller, and is called a lead-lag (or lag-lead) controller. The RL with the above controller is as follows:



The new branch in the RL due to the lag compensator is shown in an extended scale in the following figure:

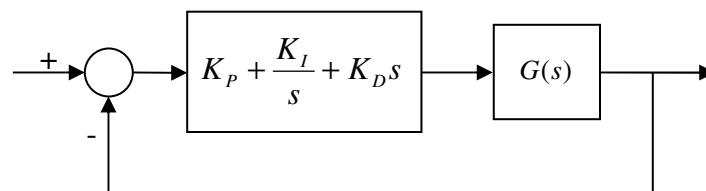


With the designed controller, the roots of the characteristic equation will be:

$$s_1, s_2 = -1.9539 \pm j1.9553, \quad s_3 = -0.1047.$$

This means that adding the lag compensator has resulted in a very small change in the location of the desired dominant closed loop poles. It has also resulted in a new closed loop pole whose effect is insignificant due to the zero of the system very close to that.

- In general, when a lead-lag controller is to be used, it is better to tentatively place the dominant poles inside the desired region (not on its boundary) so that in the presence of the new closed loop pole (introduced by the lag compensator) the system still meets the desired specifications. One may need to use trial and error to find the proper controller parameters for a lead-lag compensator.
- Three-term (PID) controllers: Proportional (P) integral (I) derivative (D) controllers are very popular in industry.



- The transfer function of this type of controller is:

$$G_C(s) = K_p + \frac{K_I}{s} + K_D s = \frac{K_D s^2 + K_p s + K_I}{s}.$$

- As it can be seen from the above equation, a PID controller has one pole in the origin and two zeros which can be located arbitrarily in the  $s$ -plane by adjusting the parameters  $K_P$ ,  $K_I$ , and  $K_D$ .
- The integral term acts like a lag compensator (improves the steady-state response) and the derivative term acts like a lead compensator (improves the transient response).
- One may use the proportional and integral terms only. In that case, we will have a PI controller with the following transfer function:

$$G_C(s) = K_P + \frac{K_I}{s} = K_P \frac{(s + \frac{K_I}{K_P})}{s}.$$

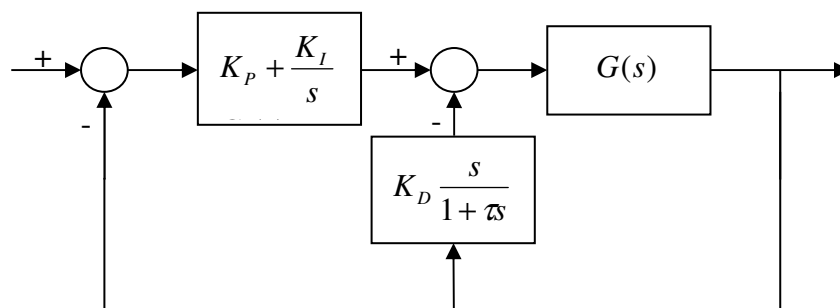
- A PI controller is usually used to improve the steady-state performance of the system (like a lag controller).
- One may use the proportional and derivative terms only. In that case, we will have a PD controller with the following transfer function:

$$G_C(s) = K_P + K_D s = K_D (s + \frac{K_P}{K_D}).$$

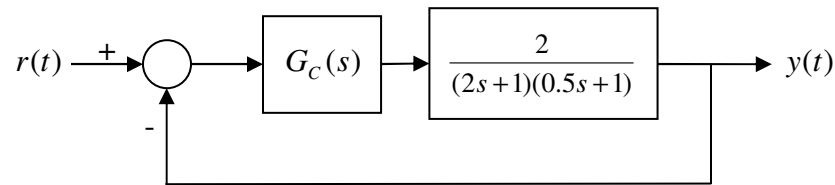
- A PD controller is usually used to improve the transient performance of the system (like a lead controller).
- Since the derivative term  $K_D s$  in a PID controller amplifies high-frequency noise, usually a combination of a differentiator and a low-pass filter is used instead of a derivative term as follows:

$$K_D \frac{s}{1 + \tau s}.$$

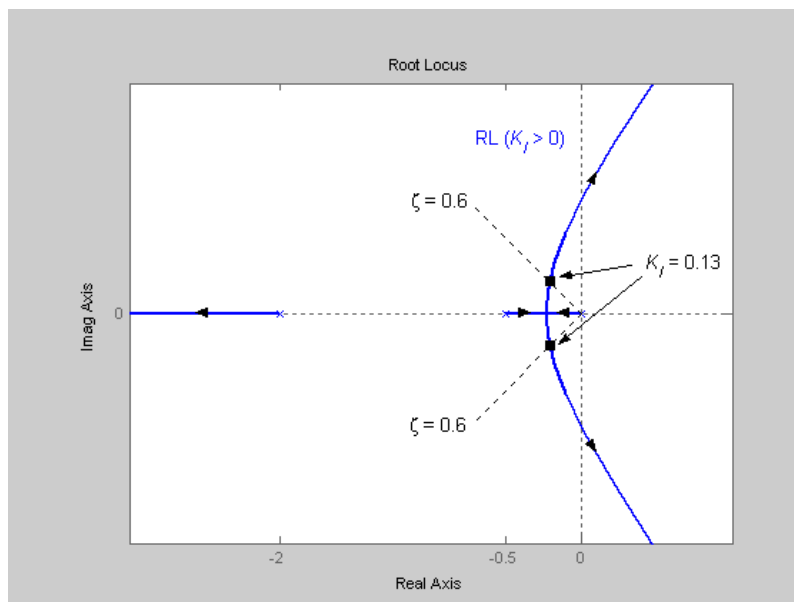
- In practice, usually the following structure is used to build the PID controller:



- Note that the PID controller is not stable (because of the pole in the origin). This can cause the saturation problem if the loop is opened for any reason. By moving the pole from the origin to the LHP, this problem can be fixed and the resultant controller will be a lag controller, in fact.
- **Example 9.2:** Design the controller  $G_C(s)$  for the following system so that the steady-state error due to a step reference input is zero and the percentage overshoot of the step response is less than or equal to 10%.



- **Solution:** First, we will choose an integrator  $G_C(s) = \frac{K_I}{s}$  to meet the steady-state specification. With this controller, the RL will be as follows:



In order to have a 10% overshoot for the step response, we must have  $\zeta = 0.6$  (or equivalently  $\theta = 53^\circ$ ). From the figure, it can be seen that this damping ratio can be achieved by choosing  $K_I = 0.13$ . For this controller, we will meet the desired specifications. The resultant closed loop poles will be:

$$s_1, s_2 = -0.2104 \pm j0.2842, \quad s_3 = -2.0792$$

and the settling time will be  $t_s = \frac{4}{0.2104} \cong 19$  sec.

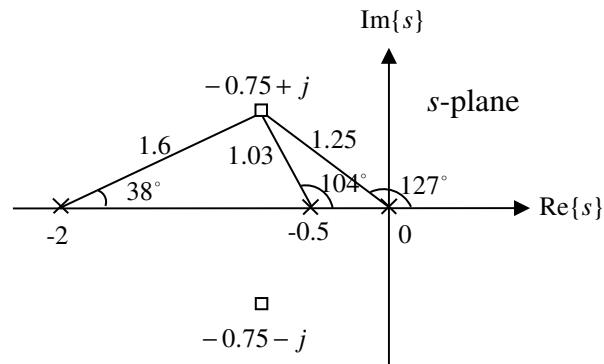
Assume now that we want to reduce the settling time to  $t_s = \frac{16}{3}$  sec. Therefore, we

must have  $\zeta\omega_n = \frac{4}{t_s} = 0.75$  and this condition cannot be satisfied by using an integrator only. In order to meet the desired percentage overshoot and settling time, we can choose the following dominant poles for the closed loop system:

$$s_1, s_2 = -0.75 \pm j$$

From the following figure it can be seen that the algebraic summation of the angles of the vectors from the open loop poles to the desired point is:

$$-127^\circ - 104^\circ - 38^\circ = -269^\circ$$



This means that we need a zero in the open loop transfer function to add  $89^\circ$  positive phase in order to meet the condition on angles at  $s_1, s_2 = -0.75 \pm j$ . This can be met by placing a zero at  $s = -0.75$ . Now, from the condition on magnitude we can find the corresponding controller gain as follows:



$$G_c(s)G(s) = \frac{K(s+0.75)}{s} \times \frac{2}{(2s+1)(0.5s+1)} = \frac{K(s+0.75)}{s} \times \frac{2}{(s+0.5)(s+2)}$$

$$|G_c(s)G(s)|_{s=-0.75+j} = 1 \Rightarrow \frac{K \times 1}{1.25} \times \frac{2}{1.03 \times 1.6} = 1 \Rightarrow K = 1.03$$

Therefore, the overall controller will be a PI controller as follows:

$$G_c(s) = \frac{1.03(s+0.75)}{s} = 1.03 + \frac{0.7725}{s}.$$

One should check in the end to see if the closed loop poles  $s_1, s_2$  are dominant poles. If they are not, then the controller parameters must be obtained through trial and error.

### Frequency domain representation

- We use frequency domain representation for stability analysis and control system design.
- There are different ways to represent the frequency response of a system  $G(j\omega)$ :
  1. The imaginary part versus the real part (Nyquist plot).
  2. Magnitude versus frequency and phase versus frequency (Bode plot).
  3. Magnitude versus phase (Nichols chart).

### Nyquist plot

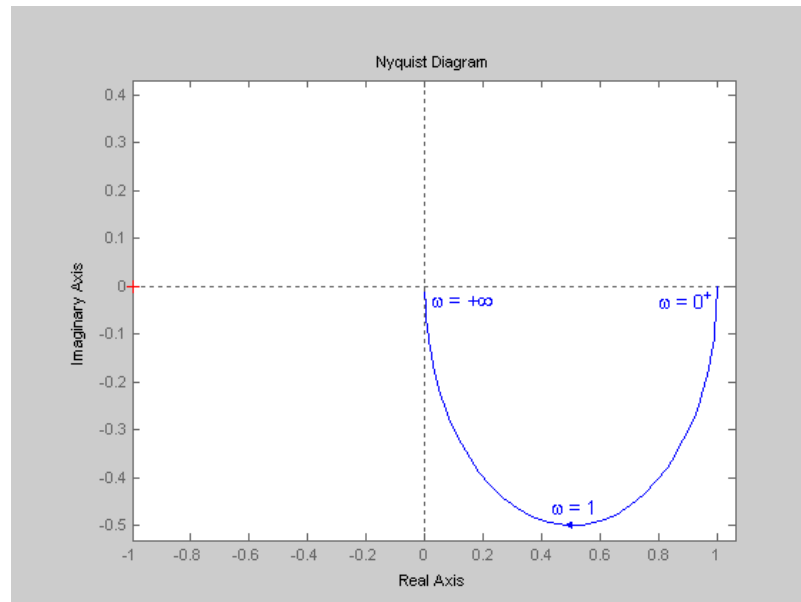
- The Nyquist plot is a map from the imaginary axis in the  $s$ -plane to the  $G(j\omega)$ -plane. In other words, the Nyquist plot gives the imaginary part of  $G(j\omega)$  versus the real part of  $G(j\omega)$  for different values of  $\omega \in (-\infty, +\infty)$ .
- **Example 9.3:** Sketch the Nyquist plot of the transfer function  $G(s) = \frac{1}{s+1}$ .

- **Solution:** We have  $G(j\omega) = \frac{1}{j\omega+1}$ .

$$\omega = 0 \Rightarrow G(j\omega) = 1$$

$$\omega = 1 \Rightarrow G(j\omega) = \frac{1}{j+1} = \frac{1}{2}(1-j)$$

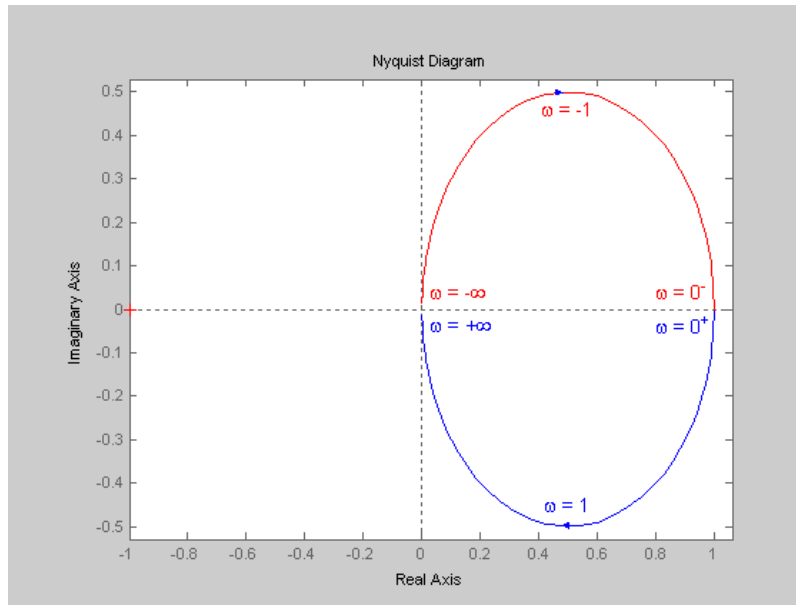
$$\omega \gg 1 \Rightarrow G(j\omega) \cong \frac{1}{j\omega} \rightarrow 0 \angle -\frac{\pi}{2}$$



It turns out that for this example the Nyquist plot is a circle of radius 0.5 centered at (0.5, 0). In other words, it can be verified that:

$$(\operatorname{Re}\{G(j\omega)\} - 0.5)^2 + (\operatorname{Im}\{G(j\omega)\})^2 = \frac{1}{4}.$$

- We know that the frequency response of any real system has the property  $G(-j\omega) = G^*(j\omega)$ , which means that the real part of the frequency response is always an even function of  $\omega$  and the imaginary part is an odd function of  $\omega$ . This implies that the Nyquist plot for  $-\infty < \omega < +\infty$  is symmetrical with respect to the real axis. In other words, one can find the plot for  $0 \leq \omega < +\infty$  and find the mirror image of the resulted trajectory with respect to the real axis. From this property, it can be concluded that for Example 9.3 we will have:



- In practice, the exact shape of the Nyquist plot is usually of no importance. We will see that for stability analysis only the shape of the Nyquist plot around the point  $s = -1$  (critical point) is very important.

- **Example 9.4:** Sketch the Nyquist plot of the transfer function  $G(s) = \frac{K}{s(\tau s + 1)}$ , where  $K$  and  $\tau$  are positive constants.

- **Solution:** We have  $G(j\omega) = \frac{K}{j\omega - \tau\omega^2}$ .

$$\omega \ll 1 \Rightarrow G(j\omega) \cong \frac{K}{j\omega} \rightarrow \infty \angle -90^\circ$$

$$\omega \gg 1 \Rightarrow G(j\omega) \cong -\frac{K}{\tau\omega^2} \rightarrow 0 \angle 180^\circ$$

$$G(j\omega) = \frac{K}{j\omega - \tau\omega^2} = \frac{K(-\tau\omega^2 - j\omega)}{\omega^2 + \tau^2\omega^4}$$

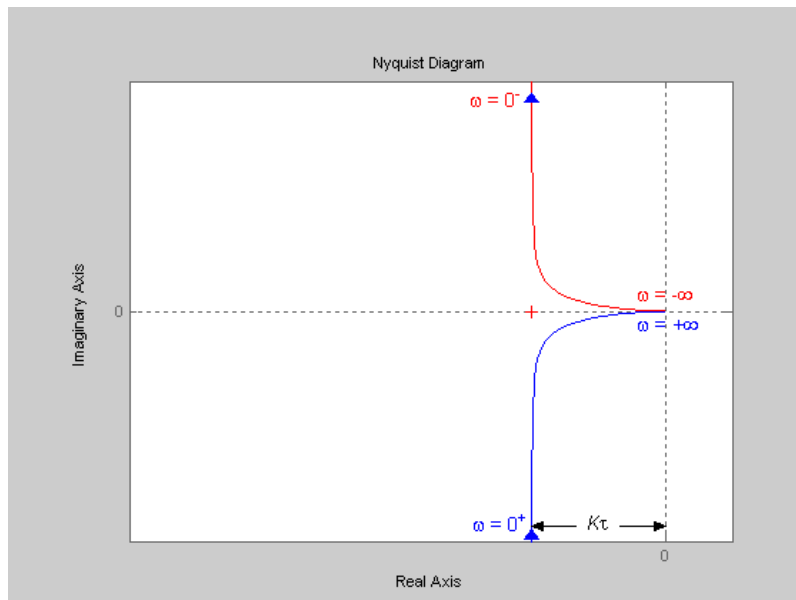
$$\operatorname{Re}\{G(j\omega)\} = \frac{-\tau K\omega^2}{\omega^2 + \tau^2\omega^4} < 0 \text{ for } \omega > 0$$

$$\operatorname{Im}\{G(j\omega)\} = \frac{-K\omega}{\omega^2 + \tau^2\omega^4} < 0 \text{ for } \omega > 0$$

One can easily verify that for  $\omega \ll 1$  where  $G(j\omega) \rightarrow \infty \angle -90^\circ$ , the real part of the frequency response is equal to:

$$\lim_{\omega \rightarrow 0} \operatorname{Re}\{G(j\omega)\} = \lim_{\omega \rightarrow 0} \frac{-\tau K \omega^2}{\omega^2 + \tau^2 \omega^4} = -K\tau$$

However, as pointed out before, the exact distance of the Nyquist plot from the imaginary axis when  $G(j\omega) \rightarrow \infty$  is not important.



- Useful tips to draw the Nyquist plot:
  1. Calculate  $G(j\omega)$  for  $\omega \ll 1$  and  $\omega \gg 1$ .
  2. Find the intersection with the real and imaginary axes.
  3. Sometimes you need to find the plot at some auxiliary points.
  4. The plot for  $\omega < 0$  is the mirror image of that for  $\omega > 0$ .
  
- **Example 9.5:** Sketch the Nyquist plot for a system with the transfer function

$$G(s) = \frac{6}{(s+1)(s+2)(s+3)}.$$

- **Solution:** We have  $G(j\omega) = \frac{6}{(j\omega+1)(j\omega+2)(j\omega+3)} = \frac{6}{6(1-\omega^2) + j\omega(11-\omega^2)}$ .

$$\omega = 0 \Rightarrow G(j\omega) = 1$$

$$\omega \gg 1 \Rightarrow G(j\omega) \cong \frac{6}{(j\omega)^3} \rightarrow 0 \angle -270^\circ$$

Intersection with the imaginary axis:

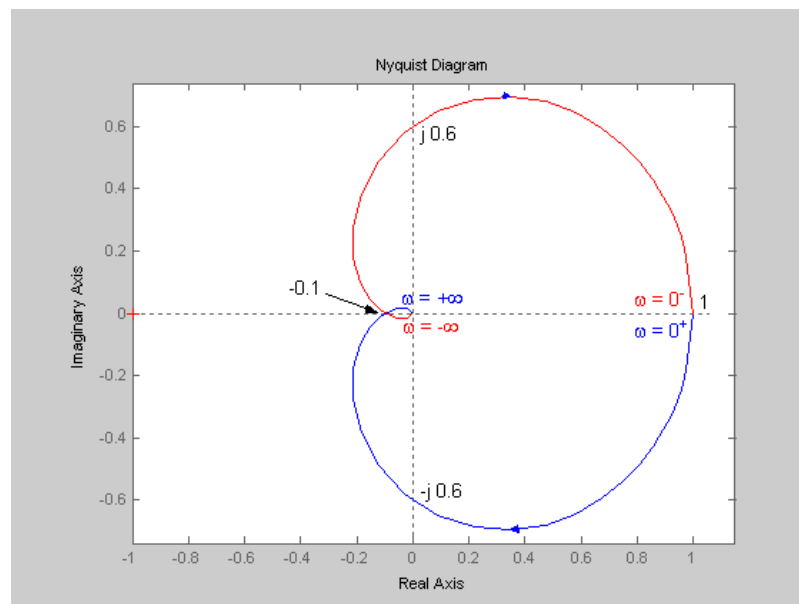
$$\operatorname{Re}\{G(j\omega)\} = 0 \Rightarrow \omega = 1$$

$$G(j1) = -0.6j$$

Intersection with the real axis:

$$\operatorname{Im}\{G(j\omega)\} = 0 \Rightarrow \omega = 0, \omega = \sqrt{11}$$

$$G(j\sqrt{11}) = -0.1$$



### Bode plot

- The Bode plot consists of two plots: the magnitude of the frequency response in dB ( $20\log_{10}|G(j\omega)|$ ) versus frequency in rad/sec and the phase of the frequency response (in degrees or radians) versus frequency in rad/sec. The frequency scale

in both graphs is logarithmic ( $\log_{10} \omega$ ). We will use the notation  $\log$  instead of  $\log_{10}$  to denote the logarithm to the base 10.

- We will find the Bode plot of some basic transfer functions. Any transfer function can be considered as a combination of these basic functions and the corresponding Bode plots can be obtained very easily.