Decentralized Control Systems

- When control theory is applied to a system that consists of geographically separated components, or a system consisting of a large number of input-output stations, it is often desired to have some form of decentralization. At each station, the controller observes only local system outputs and controls only local inputs.

Consider a LTI system with \( \nu \) local control stations given by:

\[
\begin{bmatrix}
\begin{array}{c}
\vdots \\
y_1(t) \\
\vdots \\
y_{\nu}(t)
\end{array}
\end{bmatrix} =
\begin{bmatrix}
\begin{array}{c}
\vdots \\
C_1 \\
\vdots \\
C_{\nu}
\end{array}
\end{bmatrix} x(t) +
\begin{bmatrix}
\begin{array}{c}
\vdots \\
D_{11} & \cdots & D_{1\nu} \\
\vdots & \ddots & \vdots \\
D_{\nu 1} & \cdots & D_{\nu \nu}
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
\vdots \\
u_1(t) \\
\vdots \\
u_{\nu}(t)
\end{array}
\end{bmatrix}
\]  

(3.1)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u_i(t) \in \mathbb{R}^{n_i} \) and \( y_i(t) \in \mathbb{R}^{r_i} \) are the input and output, respectively, of the \( i^{th} \) control station \( (i = 1, \ldots, \nu) \). The matrices \( A \in \mathbb{R}^{nxn} \), \( B_i \in \mathbb{R}^{nxn_i} \), \( C_i \in \mathbb{R}^{rxn} \), and \( D_{ij} \in \mathbb{R}^{rxn_j} \) \( (i, j = 1, \ldots, \nu) \) are real, constant matrices. The system (3.1) is often written in the following form:

\[
\begin{bmatrix}
\begin{array}{c}
\vdots \\
x(t)
\end{array}
\end{bmatrix} = Ax(t) + \sum_{i=1}^{\nu} B_i u_i(t)
\]  

(3.2)

\[
y_i(t) = C_i x(t) + \sum_{j=1}^{\nu} D_{ij} u_j(t), \quad i = 1, \ldots, \nu
\]

- The set of local dynamic LTI feedback controllers for (3.2) are given by:

\[
\begin{align*}
\dot{z}_i(t) &= S_i z_i(t) + R_i y_i(t) \\
u_i(t) &= Q_i z_i(t) + K_i y_i(t) + v_i(t), \quad i = 1, \ldots, \nu
\end{align*}
\]  

(3.3)
where $z_i(t) \in \mathbb{R}^{n_i}$ is the state vector of the $i^{th}$ feedback controller, $v_i(t) \in \mathbb{R}^{m_i}$ is the $i^{th}$ local external input. The matrices $S_i \in \mathbb{R}^{n_i \times n_i}$, $R_i \in \mathbb{R}^{n_i \times m_i}$, $Q_i \in \mathbb{R}^{n_i \times n_i}$, and $K_i \in \mathbb{R}^{n_i \times m_i}$ ($i = 1, \ldots, \nu$) are real, constant matrices. The controller (3.3) can be written in the following form:

$$
\dot{z}(t) = S z(t) + R v(t)
$$

$$
u(t) = Q z(t) + K y(t) + v(t)
$$

(3.4)

where $S$, $R$, $Q$, and $K$ are block diagonal matrices as follows:

$$
S := \begin{bmatrix}
S_1 & 0 & \cdots & 0 \\
0 & S_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & S_\nu
\end{bmatrix},

R := \begin{bmatrix}
R_1 & 0 & \cdots & 0 \\
0 & R_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & R_\nu
\end{bmatrix},

Q := \begin{bmatrix}
Q_1 & 0 & \cdots & 0 \\
0 & Q_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & Q_\nu
\end{bmatrix},

K := \begin{bmatrix}
K_1 & 0 & \cdots & 0 \\
0 & K_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & K_\nu
\end{bmatrix},

(3.5)

and $z(t)$, $y(t)$, $u(t)$, and $v(t)$ are given by:

$$
z(t) := \begin{bmatrix}
z_1(t) \\
\vdots \\
z_\nu(t)
\end{bmatrix},

y(t) := \begin{bmatrix}
y_1(t) \\
\vdots \\
y_\nu(t)
\end{bmatrix},

u(t) := \begin{bmatrix}
u_1(t) \\
\vdots \\
u_\nu(t)
\end{bmatrix},

v(t) := \begin{bmatrix}
v_1(t) \\
\vdots \\
v_\nu(t)
\end{bmatrix}.

$$

- Note that in the $s$-domain, the controller (3.4) will have the following form:

$$
\begin{bmatrix}
U_1(s) \\
\vdots \\
U_\nu(s)
\end{bmatrix} = \begin{bmatrix}
G_{c1}(s) & 0 & Y_1(s) \\
0 & \ddots & \vdots \\
0 & G_{c\nu} & Y_\nu(s)
\end{bmatrix} + V(s)
$$

- Using the augmented state vector $\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$, one can write the equations for the closed-loop system as follows:

$$
\begin{bmatrix}
\dot{x}(t) \\
\dot{z}(t)
\end{bmatrix} = \begin{bmatrix}
A & 0 \\
0 & \bar{A}
\end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix}
B & 0 \\
0 & \bar{B}
\end{bmatrix} v(t)
$$

$$
y(t) = \begin{bmatrix}
\bar{C}
\end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix}
\bar{D}
\end{bmatrix} v(t)
$$

(3.6)

For a strictly proper system ($D = 0$), the matrices $\bar{A}$, $\bar{B}$, and $\bar{C}$ are given by:

$$
\bar{A} := \begin{bmatrix}
A + BKC & BQ \\
RC & S
\end{bmatrix},

\bar{B} := \begin{bmatrix}
B \\
0
\end{bmatrix},

\bar{C} := \begin{bmatrix}
C & 0
\end{bmatrix},

\bar{D} := 0.
$$
where:

\[
B := \begin{bmatrix} B_1 & \cdots & B_v \end{bmatrix}, \quad C := \begin{bmatrix} C_1 \\ \vdots \\ C_v \end{bmatrix}.
\]  

(3.7)

- Note that the equations for a centralized controller are also similar to (3.4). However, for a decentralized controller, the matrices \( S, R, Q, \) and \( K \) are block diagonal as given by (3.5) whereas in the centralized case there is no such restriction in the structure of these matrices.

- **Decentralized fixed modes (DFM)** [9], [10]: Consider the \( m \)-input, \( r \)-output system (3.1), where \( m = \sum_{i=1}^{v} m_i \), \( r = \sum_{i=1}^{v} r_i \), and assume that the decentralized flow constraint \( K \) is defined as follows:

\[
K := \{ K \in \mathbb{R}^{m \times r} \mid K = \begin{bmatrix} K_1 & 0 \\ \vdots \\ 0 & K_v \end{bmatrix}, \quad K_i \in \mathbb{R}^{m_i, r_i}, \quad i = 1, \ldots, v, \quad \det(I - DK) \neq 0 \}\)

(3.8)

Then \( \lambda \in \mathcal{G} \) is a decentralized fixed mode (DFM) of (3.1) with respect to \( K \), if:

\[
\lambda \in \bigcap_{K \in \mathcal{K}} \text{sp}(A + BK(I - DK)^{-1}C),
\]

(3.9)

where \( \text{sp}(A + BK(I - DK)^{-1}C) \) denotes the set of eigenvalues of \( (A + BK(I - DK)^{-1}C) \). In other words, \( \lambda \in \mathcal{G} \) is a DFM of (3.1) with respect to \( K \), if:

\[
\text{rank}(A - \lambda I + BK(I - DK)^{-1}C) < n, \quad \forall K \in \mathcal{K}
\]

- For strictly proper systems, equation (3.9) can be simplified as:

\[
\lambda \in \bigcap_{K \in \mathcal{K}} \text{sp}(A + BKC).
\]

(3.10)

- This is, in fact, equivalent to the following:

\[
\lambda \in \bigcap_{K, \in \mathbb{R}^{m_i \times r_i}} \text{sp}(A + \sum_{i=1}^{v} B_i K_i C_i).
\]

(3.11)

- (3.10) and (3.11) can be used to find the DFM of a system numerically.

- Note that the set of CFMs of (3.1) is a subset of the set of DFMs of (3.1).
- **Invariance of DFM [11]:** Consider the system (3.1) with the decentralized dynamic controller (3.4). A mode \( \lambda \in \text{sp}(A) \) is a DFM of system (3.1) iff for all LTI controllers of the form (3.4), \( \lambda \) is an eigenvalue of the closed-loop system matrix of (3.6).

- The following numerical algorithm can be used to determine DFM of (3.1):
  1) Find the eigenvalues of \( A \).
  2) Select an arbitrary block diagonal feedback gain matrix \( K \in \mathbf{K} \) such that the matrix \( (I - DK) \) is nonsingular. This can be accomplished by use of a pseudorandom number generator (numerically it is better to properly scale the gain matrix \( K \) such that \( \left\| A \right\| \approx \left\| BK(I - DK)^{-1}C \right\| \)).
  3) Find the eigenvalues of the matrix \( A_c := A + BK(I - DK)^{-1}C \).
  4) For almost all \( K \in \mathbf{K} \), the set of DFMs with respect to \( \mathbf{K} \) is equal to the intersection set \( \text{sp}(A) \cap \text{sp}(A_c) \).

- **Theorem 3.1 [9], [10]:** Consider the system \( (C, A, B, D) \) given by (3.1), with \( B \) and \( C \) defined in (3.7). Let \( \mathbf{K} \) be the set of block diagonal matrices defined in (3.8). Then a necessary and sufficient condition for the existence of a decentralized LTI controller given by (3.4) such that the closed-loop system is asymptotically stable is that the system has no DFM in the closed right-half complex plane.

- **Example 3.1:** Consider the system (3.1) with \( \nu = 2 \) and the following parameters:

\[
A = \begin{bmatrix}
-1 & 0 & -3 \\
0 & \alpha & 0 \\
0 & 0 & -3
\end{bmatrix},
B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},
B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},
C_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},
C_2 = \begin{bmatrix} -1.1 & 0 & 0.1 \end{bmatrix},
D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

It can be easily seen that this system is controllable and observable and so, it has no CFMs. However, we have:
\[
A + B \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} C = \begin{bmatrix} -1 & K_1 & -3 \\ -1.1K_2 & \alpha & 0.1K_2 \\ -1.1K_2 & K_1 & -3 + 0.1K_2 \end{bmatrix}
\]

One can verify that for \( \alpha = 0.1 \) this system has a DFM at \( \lambda = 0.1 \) with respect to the diagonal information flow \[
\begin{bmatrix} \times & 0 \\ 0 & \times \end{bmatrix}.
\]

- The following analytical methods can be used to determine DFM of (3.1):

- **Theorem 3.2** [10]: Consider the system given by (3.1). Then \( \lambda \in \text{sp}(A) \) is a decentralized fixed mode of (3.1) with respect to \( K \) iff any one of the following conditions hold:

  1) \[
  \text{rank} \begin{bmatrix} A - \lambda I \\ C_1 \\ \vdots \\ C_v \end{bmatrix} < n
  \]

  2) \[
  \text{rank} \begin{bmatrix} A - \lambda I & B_1 & \cdots & B_v \end{bmatrix} < n
  \]

  3) \[
  \text{rank} \begin{bmatrix} A - \lambda I & B_{i_k} \\ C_{i_k} & D_{i_{k},i_{k}} \\ \vdots & \vdots \\ C_{i_v} & D_{i_{v},i_{v}} \end{bmatrix} < n
  \]

  for some \( i_k \in \{1,2,\ldots,v\} \), \( k = 1,2,\ldots,v \) such that \( \{i_1,i_2,\ldots,i_v\} = \{1,2,\ldots,v\} \).

  4) \[
  \text{rank} \begin{bmatrix} A - \lambda I & B_{i_1} & B_{i_2} \\ C_{i_1} & D_{i_{1},i_{1}} & D_{i_{1},i_{2}} \\ \vdots & \vdots & \vdots \\ C_{i_v} & D_{i_{v},i_{1}} & D_{i_{v},i_{2}} \end{bmatrix} < n
  \]

  for some \( i_k \in \{1,2,\ldots,v\} \), \( k = 1,2,\ldots,v \) such that \( \{i_1,i_2,\ldots,i_v\} = \{1,2,\ldots,v\} \).

  \vdots

  \[v + 1\) \[
  \text{rank} \begin{bmatrix} A - \lambda I & B_{i_1} & B_{i_2} & \cdots & B_{i_{v-1}} \\ C_{i_1} & D_{i_{1},i_{1}} & D_{i_{1},i_{2}} & \cdots & D_{i_{1},i_{v-1}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ C_{i_{v-1}} & D_{i_{v-1},i_{1}} & D_{i_{v-1},i_{2}} & \cdots & D_{i_{v-1},i_{v-1}} \end{bmatrix} < n
  \]
for some \( i_k \in \{1, 2, \ldots, \nu\}, k = 1, 2, \ldots, \nu \) such that \( \{i_1, i_2, \ldots, i_{\nu}\} = \{1, 2, \ldots, \nu\} \).

**Remark 3.1:** It is to be noted that the condition in Step 1 of Theorem 3.2 is in fact met iff the system is unobservable. Similarly, the condition in Step 2 of Theorem 3.2 is in fact met iff the system is uncontrollable. In other words, if Step 1 or Step 2 of Theorem 3.2 are satisfied, the corresponding DFM is also a CFM of the system (3.1).

- **Example 3.2:** Consider the system given in Example 3.1. We want to use Theorem 3.2 to find the value of \( \alpha \) for which the system has a DFM. We will check the rank of matrices given in Steps 1 to \( \nu + 1 = 3 \) in Theorem 3.2.

The matrix corresponding to Step 1 of Theorem 3.2:

\[
M_1 = \begin{bmatrix}
A - \lambda I \\
C_1 \\
C_2
\end{bmatrix}
\]

The matrix corresponding to Step 2 of Theorem 3.2:

\[
M_2 = \begin{bmatrix}
A - \lambda I & B_1 & B_2
\end{bmatrix}
\]

The matrices corresponding to Step 3 of Theorem 3.2:

\[
M_3 = \begin{bmatrix}
A - \lambda I & B_2 \\
C_1 & 0
\end{bmatrix},
M_4 = \begin{bmatrix}
A - \lambda I & B_1 \\
C_2 & 0
\end{bmatrix}.
\]

It can be verified that for all eigenvalues of \( A \ (\lambda \in \{-1, \alpha, -3\}) \) the matrices \( M_1 \) and \( M_2 \) are full-rank. In other words, the system is observable and controllable. However, the rank of the matrix \( M_4 \) for \( \lambda = \alpha = 0.1 \) will be less than 3, as follows:

\[
\text{rank}(M_4) = \text{rank}\left(\begin{bmatrix}
-1.1 & 0 & -3 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & -3.1 & 1 \\
-1.1 & 0 & 0.1 & 0
\end{bmatrix}\right) = 2 < n = 3
\]

From Theorem 3.2 it will be concluded that the system has a DFM at \( \lambda = 0.1 \) for \( \alpha = 0.1 \).
References:

