Stabilization of Decentralized Control Systems (Cont’d)

- **Structured matrix** [14], [15]: A matrix \( \overline{M} \) with a number of fixed zeros at certain locations and arbitrary entries elsewhere is called a structured matrix. Let the number of arbitrary entries be denoted by \( \kappa \). Then, a parameter space \( R^\kappa \) is associated with these entries such that a matrix \( M = \overline{M}(h) \) corresponds to every data point \( h \in R^\kappa \), where the elements of \( M \) are obtained by replacing the arbitrary entries of \( \overline{M} \) with the corresponding elements of \( h \).

- Conversely, to any arbitrary matrix \( M \) with \( \kappa \) nonzero entries, there corresponds a structured matrix \( \overline{M} \) such that \( M = \overline{M}(h) \) for some \( h \in R^\kappa \).

- Two matrices \( M_1 \) and \( M_2 \) are called structurally equivalent if there is a one-to-one correspondence between the locations of their zero and nonzero entries, that is, if there corresponds the same structured matrix \( \overline{M} \) to both \( M_1 \) and \( M_2 \).

- **Structured systems** [14], [15]: To any arbitrary system represented by \( (C, A, B) \), there corresponds a structured system \( (\overline{C}, \overline{A}, \overline{B}) \), where \( \overline{A} \), \( \overline{B} \), and \( \overline{C} \) are the structured matrices associated with \( A \), \( B \), and \( C \), respectively. Similarly, two systems \( (C_1, A_1, B_1) \) and \( (C_2, A_2, B_2) \) are called structurally equivalent if there is a one-to-one correspondence between the location of zero and nonzero entries of \( A_1 \) and \( A_2 \), \( B_1 \) and \( B_2 \), and \( C_1 \) and \( C_2 \).

- **Structurally fixed modes** [15]: The system represented by \( (C, A, B) \) is said to have structurally fixed modes with respect to a decentralized flow constraint \( K \), if every system structurally equivalent to \( (C, A, B) \) has fixed modes w.r.t. \( K \).

- **Example 8.1**: Consider a decentralized system with the following matrices:
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
1 & 3 & 2 \\
0 & 4 & 0 \\
\end{bmatrix},
B_1 = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix},
B_2 = \begin{bmatrix}
0 \\
1 \\
0 \\
\end{bmatrix},
C_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix},
C_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix},
D = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

- It can be easily verified by using MATLAB and replacing nonzero values of the above matrices with random values, that this system has structurally fixed modes. In other words, all structurally equivalent systems represented by:

\[
\tilde{A} = \begin{bmatrix}
0 & \times & 0 \\
\times & \times & \times \\
0 & \times & 0 \\
\end{bmatrix},
\tilde{B}_1 = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix},
\tilde{B}_2 = \begin{bmatrix}
0 \\
\times \\
0 \\
\end{bmatrix},
\tilde{C}_1 = \begin{bmatrix}
\times & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix},
\tilde{C}_2 = \begin{bmatrix}
\times & 0 \\
0 & \times \\
\end{bmatrix},
\tilde{D} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

have a DFM.

- The notion of structurally fixed modes is introduced to identify the modes that cannot be shifted by decentralized feedback regardless of the numerical values of the system’s nonzero parameters. It is to be noted that structurally fixed modes cannot be referred to in terms of individual DFMs (except for zero DFMs or systems with a single DFM). This is due to the fact that the system resulted by perturbing the nonzero entries of the original system matrices will, in general, have different modes compared to the original system.

**Structured DFMs [16]:** Consider the following system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + \sum_{i=1}^{\nu} B_i u_i(t) \\
y_i(t) &= C_i x(t), \quad i = 1, \ldots, \nu
\end{align*}
\]  
(8.1)

Assume that the decoupled model corresponding to (8.1) is as follows

\[
\begin{align*}
\dot{\tilde{x}}(t) &= \Lambda \tilde{x}(t) + \sum_{i=1}^{\nu} \tilde{B}_i \tilde{u}_i(t) \\
y_i(t) &= \tilde{C}_i \tilde{x}(t), \quad i = 1, \ldots, \nu
\end{align*}
\]  
(8.2)

where \( \Lambda \) is a diagonal matrix consisting of the eigenvalues of the system. Structured DFMs of (8.1) are those modes (if any) that continue to be DFMs after perturbing the nonzero values of matrices \( \tilde{B}_i \) and \( \tilde{C}_i \), \( i = 1, \ldots, \nu \) in the decoupled model. If a DFM is not structured, it is called an unstructured DFM.
- This definition was originally made for systems with distinct eigenvalues. In general case, to determine the structured DFM of a system, one should perturb the nonzero values of \( \hat{B}_i \) and \( \hat{C}_j \), \( i=1,\ldots,\nu \), as well as the diagonal elements of \( \Lambda \), corresponding to the repeated eigenvalue, to verify if any of the modes of the perturbed system continues to be a DFM.

- DFMs can be referred to as being either structured or unstructured, because unlike structurally fixed modes, the elements of matrix \( \Lambda \) are not perturbed.

- **Lemma 8.1** [16]: Consider the controllable and observable system (8.1) with \( \nu = 2 \) and the decoupled model given by (8.2). Assume that both control stations are SISO \( (m_i = r_i = 1, \ i = 1, 2) \). Assume also, that the entries of \( \hat{B}_i \) and \( \hat{C}_i \), \( i = 1, 2 \) are given in the following form:

\[
\hat{B}_i = \begin{bmatrix}
\hat{b}_1^i \\
\vdots \\
\hat{b}_n^i
\end{bmatrix}, \quad \hat{C}_i = \begin{bmatrix}
\hat{c}_1^i \\
\hat{c}_2^i \\
\vdots \\
\hat{c}_n^i
\end{bmatrix},
\]

- Then \( \lambda_i \) is a structured DFM of (8.1) iff either condition (i) or condition (ii) given below holds:

  i) \( \hat{b}_1^i = 0 \) and \( \hat{c}_1^i = 0 \) and \( \hat{b}_j^i \hat{c}_2^j = 0, \ j = 2, 3, \ldots, n \).

  ii) \( \hat{b}_2^i = 0 \) and \( \hat{c}_1^i = 0 \) and \( \hat{b}_j^i \hat{c}_1^j = 0, \ j = 2, 3, \ldots, n \).

and \( \lambda_i \) is an unstructured DFM of (3.1) iff conditions (i), (ii) do not hold, and either condition (iii) or condition (iv) given below holds:

  iii) \( \hat{b}_1^i = 0 \) and \( \hat{c}_1^i = 0 \) and

\[
\begin{bmatrix}
\hat{c}_2^1 & \hat{c}_3^1 & \cdots & \hat{c}_n^1 \\
\hat{c}_2^2 & \hat{c}_3^2 & \cdots & \hat{c}_n^2 \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
0 & & & \ddots \\
0 & 1 & & \\
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\lambda_2 - \lambda_i} \\
\frac{1}{\lambda_3 - \lambda_i} \\
\vdots \\
\frac{1}{\lambda_n - \lambda_i}
\end{bmatrix}
= 0
\]

Then \( \lambda_i \) is a structured DFM of (8.1) iff either condition (i) or condition (ii) given below holds:
iv) \( \hat{b}_2^1 = 0 \) and \( \hat{c}_1^1 = 0 \) and

\[
\begin{bmatrix}
\frac{1}{\lambda_2 - \lambda_1} & 0 \\
\frac{1}{\lambda_3 - \lambda_1} & \ldots \\
0 & \frac{1}{\lambda_n - \lambda_1}
\end{bmatrix}
\begin{bmatrix}
\hat{c}_1^1 \\
\hat{c}_1^2 \\
\vdots \\
\hat{c}_1^n
\end{bmatrix}
= 0.
\]

- The effect of sampling on DFMs will be investigated now. The equivalent discrete-time system corresponding to (8.1) with a constant sampling interval \( T > 0 \) and a zero-order hold is as follows:

\[
x[k + 1] = A_d x[k] + \sum_{i=1}^{\nu} B_{d_i} u_i[k] \\
y_i[k] = C_{d_i} x[k], \quad i = 1, \ldots, \nu
\]

where \( A_d = e^{\Lambda T} \) and \( C_{d_i} = \hat{C}_i \), and assuming all modes are nonzero, \( B_{d_i} = \Lambda^{-1}(A_d - I)B_i \). If the system (8.1) is given in decoupled form, then:

\[
A_d =
\begin{bmatrix}
e^{\lambda_1 T} & 0 \\
e^{\lambda_2 T} & \vdots \\
0 & e^{\lambda_n T}
\end{bmatrix},
B_{d_i} :=
\begin{bmatrix}
e^{\lambda_1 T} - 1 \\
\frac{e^{\lambda_2 T} - 1}{\lambda_2} \\
\frac{e^{\lambda_n T} - 1}{\lambda_n}
\end{bmatrix}
\]

- **Theorem 8.1 [16]:** Given the system (8.1), assume that it is controllable and observable, that it can be written in the decoupled form (8.2), and that it contains \( P_u \geq 0 \) unstructured DFMs and \( P_s \geq 0 \) structured DFMs \( \lambda_j, \ j = 1, 2, \ldots, P_s \). Suppose that all unstructured DFMs are nonzero; then the sampled system (8.3) contains \( P_s \) structured DFMs \( e^{\lambda_j T}, \ j = 1, 2, \ldots, P_s \), \( \forall T > 0 \), and no unstructured DFMs for almost all \( T > 0 \).
- It is to be noted that sampling cannot eliminate the zero unstructured DFMs and non-cyclic repeated unstructured DFMs. However, sampling can eliminate cyclic nonzero repeated unstructured DFMs (which means the Jordan blocks corresponding to the repeated unstructured DFMs have maximal dimension).

- As a result of Theorem 8.1, one can design a discrete-time decentralized controller for a system that has a nonzero RHP distinct unstructured DFM to stabilize the system, although a decentralized LTI controller cannot stabilize the system. A discrete-time controller is, in fact, a time-varying controller for the original continuous-time system.

- **Example 8.2:** Consider the system (8.1) with \( \nu = 2 \) and the following system matrices:

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 1 & -4 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}
\]

It can be easily verified that this system has a structured DFM at \( \lambda_2 = 2 \) and the sampled system has a DFM at \( e^{2\nu} \) for any sampling period \( T > 0 \). This means that this system cannot be stabilized by neither a decentralized LTI controller, nor a discrete-time decentralized controller.

- **Example 8.3:** Consider the system (8.1) with \( \nu = 2 \) and the following system matrices:

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 & -4 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}
\]

It can be easily verified that this system has an unstructured DFM at \( \lambda_1 = 1 \) and the sampled system has no DFM. This means that this system cannot be stabilized by applying a decentralized LTI controller but one can use a discrete-time decentralized controller to stabilize this system.
References:

