ELEC361: Signals And Systems

Topic 5:
Discrete-Time Fourier Transform (DTFT)

- DT Fourier Transform
  - Overview of Fourier methods
  - DT Fourier Transform of Periodic Signals
  - Properties of DT Fourier Transform
  - Relations among Fourier Methods
  - Summary
  - Appendix:
    - Transition from DT Fourier Series to DT Fourier Transform

Figures and examples in these course slides are taken from the following sources:

DT Fourier Transform

- (Note: a Fourier transform is unique, i.e., no two same signals in time give the same function in frequency)
- The DT Fourier Series is a good analysis tool for systems with periodic excitation but cannot represent an aperiodic DT signal for all time
- The DT Fourier Transform can represent an aperiodic discrete-time signal for all time
- Its development follows exactly the same as that of the Fourier transform for continuous-time aperiodic signals
DT Fourier Transform

- Let $x[n]$ be the aperiodic DT signal
- We construct a periodic signal $\tilde{x}[n]$ for which $x[n]$ is one period
  - $\tilde{x}[n]$ is comprised of infinite number of replicas of $x[n]$
  - Each replica is centered at an integer multiple of $N$
  - $N$ is the period of $\tilde{x}[n]$
- Consider the following figure which illustrates an example of $x[n]$ and the construction of $\tilde{x}[n]$
- Clearly, $x[n]$ is defined between $-N_1$ and $N_2$
- Consequently, $N$ has to be chosen such that $N > N_1 + N_2 + 1$ so that adjacent replicas do not overlap
- Clearly, as we let
  $$N \to \infty, \quad \tilde{x}[n] = x[n]$$
  as desired
DT Fourier Transform

Let us now examine the FS representation of \( \tilde{x}[n] \)

\[
\tilde{x}[n] = \sum_{<N>} a_k e^{jk(2\pi/N)n}
\]

where

\[
a_k = \frac{1}{N} \sum_{<N>} \tilde{x}[n] e^{-jk(2\pi/N)n}
\]

Since \( x[n] \) is defined between \(-N_1\) and \(N_2\)

\[ a_k \text{ in the above expression simplifies to } \]

\[
\begin{align*}
a_k &= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk(2\pi/N)n} \\
&= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n} \\
\end{align*}
\]

\( \omega = 2\pi/N \)
DT Fourier Transform

- Now defining the function \( X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \)
- We can see that the coefficients \( a_k \) are related to \( X(e^{j\omega}) \) as \( a_k = \frac{1}{N}X(e^{jk\omega_0}) \)
- where \( \omega_0 = \frac{2\pi}{N} \) is the spacing of the samples in the frequency domain
- Therefore \( \tilde{x}[n] = \sum_{<N>} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n} \)
  \[ = \frac{1}{2\pi} \sum_{<N>} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0 \]
- As \( N \) increases \( \omega_0 \) decreases, and as \( N \to \infty \) the above equation becomes an integral
One important observation here is that the function $X(e^{j\omega})$ is periodic in $\omega$ with period $2\pi$.

Therefore, as $N \to \infty$, $\tilde{x}[n] = x[n]$.

(Note: the function $e^{j\omega}$ is periodic with $N=2\pi$).

This leads us to the DT-FT pair of equations:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \text{Synthesis equation}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \text{Analysis equation}$$
DT Fourier Transform: Forms

Inverse $F$ Form Forward

$x[n] = \int X(F)e^{j2\pi Fn}dF \overset{F}{\Rightarrow} X(F) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi Fn}$

Inverse $\Omega$ Form Forward

$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\Omega)e^{j\Omega n}d\Omega \overset{F}{\Rightarrow} X(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$

$\otimes$ DT Fourier Transform: $\text{DT} \Rightarrow \text{CT} + P_{2\pi}$

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$\otimes$ Inverse DT Fourier Transform: $\text{CT} + P_{2\pi} \Rightarrow \text{DT}$

$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
DT Fourier Transform: Examples

- Let $x[n] = \delta[n] \rightarrow X(e^{j\omega}) = 1$

- Let $x[n] = \begin{cases} 1, & |n| \leq 5 \\ 0, & \text{e.w.} \end{cases} \rightarrow X(e^{j\omega}) = \sum_{n=-5}^{5} e^{-j\omega n} = \frac{\sin[\omega(11/2)]}{\sin[\omega/2]}$
Consider the sequence \( x(n) = \alpha^n u(n), \ |\alpha| < 1 \).

For this sequence, \( X(\Omega) = \sum_{n=0}^{\infty} \alpha^n \exp[-j\Omega n] = \frac{1}{1 - \alpha \exp[-j\Omega]} \).

The magnitude is given by \( |X(\Omega)| = \frac{1}{\sqrt{1 + \alpha^2 - 2\alpha \cos \Omega}} \).

And the phase by \( \text{Arg} X(\Omega) = -\tan^{-1}\frac{\alpha \sin \Omega}{1 - \alpha \cos \Omega} \).
Outline

- DT Fourier Transform
- **Overview of Fourier methods**
  - DT Fourier Transform of Periodic Signals
  - Properties of DT Fourier Transform
  - Relations among Fourier Methods
  - DTFT: Summary
- Appendix:
  - Transition from DT Fourier Series to DT Fourier Transform
Overview of Fourier Analysis Methods: Types of signals

- Analog:
  - Continuous-Time Signal
  - Discrete-Time Signal
  - Continuous-Value Signal

- Digital:
  - Discrete-Time Signal
  - Continuous-Time Signal
  - Continuous-Value Signal

Periodic → Non-periodic

- Noise

Number of cars crossing an intersection between red lights
Overview of Fourier Analysis Methods: Types of signals

Continuous (analog)
- Periodic
  - Fourier Series
- Non-periodic
  - Fourier Transform

Discrete (digital)
- Periodic
  - Discrete Fourier Series
- Non-periodic
  - Discrete-time Fourier Transform & Z-transform
Overview of Fourier Analysis

Methods: Continuous-Value and Continuous-Time Signals

- All continuous signals are CT but not all CT signals are continuous

[Diagrams showing continuous and continuous-time signals, including points of discontinuity]
# Overview of Fourier Analysis Methods

<table>
<thead>
<tr>
<th></th>
<th>Periodic in Time</th>
<th>Discrete in Frequency</th>
<th>Aperiodic in Time</th>
<th>Continuous in Frequency</th>
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</thead>
<tbody>
<tr>
<td>Continuous</td>
<td>CT Fourier Series: CT - P_T (\Rightarrow) DT</td>
<td>(a_k = \frac{1}{T} \int_{0}^{T} x(t)e^{-jk\omega t} dt)</td>
<td>CT Fourier Transform: CT (\Rightarrow) CT</td>
<td>(X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt)</td>
</tr>
<tr>
<td></td>
<td>CT Inverse Fourier Series: DT (\Rightarrow) CT - P_T</td>
<td>(x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t})</td>
<td>Inverse CT Fourier Transform: CT (\Rightarrow) CT</td>
<td>(x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega)</td>
</tr>
<tr>
<td>Aperiodic</td>
<td>DT Fourier Series: DT - P_N (\Rightarrow) DT - P_N</td>
<td>(X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\omega kn})</td>
<td>DT Fourier Transform: DT (\Rightarrow) CT + P (2\pi)</td>
<td>(X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n})</td>
</tr>
<tr>
<td>Discrete</td>
<td>Inverse DT Fourier Series: DT - P_N (\Rightarrow) DT - P_N</td>
<td>(x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j\omega kn})</td>
<td>Inverse DT Fourier Transform: CT + P (2\pi) (\Rightarrow) DT</td>
<td>(x[n] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(e^{j\omega})e^{j\omega n} d\omega)</td>
</tr>
<tr>
<td>Time</td>
<td>Periodic in Frequency</td>
<td>Continuous in Frequency</td>
<td></td>
<td></td>
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\(\omega = \frac{2\pi}{T}\)
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- Overview of Fourier methods
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- Relations among Fourier Methods
- DTFT: Summary
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Fourier Transform of Periodic DT Signals

- Consider the continuous time signal $x(t) = e^{j\omega_0 t}$
- This signal is periodic
- Furthermore, the Fourier series representation of this signal is just an impulse of weight one centered at $\omega = \omega_0$
- Now consider this signal $x[n] = e^{j\omega_0 n}$
- It is also periodic and there is one impulse per period
  However, the separation between adjacent impulses is $2\pi$, which agrees with the properties of DT Fourier Transform
- In particular, the DT Fourier Transform for this signal is

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi l)$$
Fourier Transform of Periodic DT Signals: Example

- Let \( x[n] = \cos w_0 n \) with \( w_0 = \frac{2\pi}{5} \)

- The signal can be expressed as \( x[n] = \frac{1}{2} (e^{jw_0 n} + e^{-jw_0 n}) \)

- We can immediately write
  \[
  X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \pi \delta(\omega - \frac{2\pi}{5} - 2\pi l) + \sum_{l=-\infty}^{\infty} \pi \delta(\omega + \frac{2\pi}{5} - 2\pi l)
  \]

- Or equivalently
  \[
  X(e^{j\omega}) = \pi \delta(\omega - \frac{2\pi}{5}) + \pi \delta(\omega + \frac{2\pi}{5}) \quad -\pi \leq \omega < \pi
  \]

where \( X(e^{j\omega}) \) is periodic in \( \omega \) with period \( 2\pi \)
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Properties of the DT Fourier Transform

- Periodicity

\[ X(e^{j(\omega+2\pi)}) = X(e^{j\omega}) \]

- Note: the function \( e^{j\omega} \) is periodic with \( N=2\pi \)
Properties of the DT Fourier Transform

- Linearity: If \( x_1[n] \xrightarrow{F} X_1(e^{j\omega}) \) and \( x_2[n] \xrightarrow{F} X_2(e^{j\omega}) \)

Then \( \Rightarrow \alpha x_1[n] + \beta x_2[n] \overset{F}{\longrightarrow} \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega}) \)

\( \alpha x[n] + \beta y[n] \overset{F}{\longrightarrow} \alpha X(F) + \beta Y(F) \)

\( \alpha x[n] + \beta y[n] \overset{F}{\longrightarrow} \alpha X(j\Omega) + \beta Y(j\Omega) \)
Properties of the DT Fourier Transform

- **Time-Shifting:** If \( x[n] \xrightarrow{F} X(e^{j\omega}) \)
  Then \( x[n - n_0] \xrightarrow{F} e^{-j\omega n_0} X(e^{j\omega}) \)

\[
x[n - n_0] \xrightarrow{F} e^{-j2\pi F n_0} X(F)
\]

\[
x[n - n_0] \xrightarrow{F} e^{-j\Omega n_0} X(j\Omega)
\]
Properties of the DT Fourier Transform

- **Frequency Shifting:** If $x[n] \xrightarrow{F} X(e^{j\omega})$

  Then

  $e^{-j\omega_0 n} x[n] \xrightarrow{F} X(e^{j(\omega-\omega_0)})$

  $e^{j2\pi F_0 n} x[n] \xrightarrow{F} X(F - F_0)$

  $e^{j\Omega_0 n} x[n] \xrightarrow{F} X(j(\Omega - \Omega_0))$
Properties of the DT Fourier Transform

- Conjugation and Conjugate Symmetry

\[ x[n] \overset{F}{\rightarrow} X(e^{j\omega}) \]
\[ x^*[n] \overset{F}{\rightarrow} X^*(e^{-j\omega}) \]

- For real-valued signals,

\[ x^*[n] = x[n] \Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega}) \]

- For real-valued and even signals, the Fourier transform is real and even

- For real-valued and odd signals, the Fourier transform is purely imaginary and odd
Properties of the DT Fourier Transform

- Differencing

\[ x[n] - x[n-1] \xrightarrow{F} (1 - e^{-j\omega}) X(e^{j\omega}) \]

\[ x[n] - x[n-1] \xrightarrow{F} (1 - e^{-j2\pi F}) X(F) \]

\[ x[n] - x[n-1] \xrightarrow{F} (1 - e^{-j\Omega}) X(j\Omega) \]
Properties of the DT Fourier Transform

- Accumulation

\[
\sum_{m=-\infty}^{n} x[m] \overset{F}{\longleftrightarrow} \frac{1}{1 - e^{-jw}} X(e^{jw}) + \pi X(e^{j0}) \sum_{m=-\infty}^{\infty} \delta(w - 2\pi k)
\]

\[
\sum_{m=-\infty}^{n} x[m] \overset{F}{\longleftrightarrow} \frac{X(F)}{1 - e^{-j2\pi F}} + \frac{1}{2} X(0) \text{comb}(F)
\]

\[
\sum_{m=-\infty}^{n} x[m] \overset{F}{\longleftrightarrow} \frac{X(j\Omega)}{1 - e^{-j\Omega}} + \frac{1}{2} X(0) \text{comb}\left(\frac{\Omega}{2\pi}\right)
\]

where the impulse train on the right-hand side of the above equation reflects the average value (or dc component) that may result from the summation.

- Accumulation Definition of a Comb Function

\[
\sum_{n=-\infty}^{\infty} e^{j2\pi Fn} = \text{comb}(F)
\]
Properties of the DT Fourier Transform

- Accumulation
Properties of the DT Fourier Transform

- Time Reversal: If
  \[ x[n] \xrightarrow{F} X(e^{j\omega}) \]
  Then
  \[ x[-n] \xrightarrow{F} X(e^{-j\omega}) \]
Properties of the DT Fourier Transform

- **Time Expansion:**
  Let \( k \) be a positive integer.
  
  Define
  
  \[
  x_{(k)}[n] = \begin{cases} 
  x[n/k], & \text{if } n \text{ is a multiple of } k \\
  0, & \text{otherwise}
  \end{cases}
  \]
  
  Now if \( x[n] \xrightarrow{F} X(e^{jw}) \)
  
  then \( x_{(k)}[n] \xrightarrow{F} X(e^{jkw}) \)
Properties of the DT Fourier Transform

- Differentiation in Frequency: If

\[ x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega}) \]

then

\[ nx[n] \overset{F}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega} \]
Properties of the DT Fourier Transform

- Parseval’s Relation

\[ \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega \]

\[ \sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{-\infty}^{\infty} |X(F)|^2 dF \]

\[ \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega \]

The signal energy is proportional to the integral of the squared magnitude of the DTFT of the signal over one period.
Properties of the DT Fourier Transform

**Multiplication-Convolution Duality**

- $x[n] \ast y[n] \xrightarrow{F} X(F)Y(F)$
- $x[n] \ast y[n] \xrightarrow{F} X(j\Omega)Y(j\Omega)$
- $x[n]y[n] \xrightarrow{F} X(F) \odot Y(F)$
- $x[n]y[n] \xrightarrow{F} \frac{1}{2\pi} X(j\Omega) \odot Y(j\Omega)$

As in other transforms, convolution in the time domain is equivalent to multiplication in the frequency domain:

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = h[n] \ast x[n]$  
$X(F) \rightarrow \boxed{H(F)} \rightarrow Y(F) = H(F)X(F)$
Properties of the DT Fourier Transform
Properties of the DT Fourier Transform: Example

- Let $y[n] = x[n] * h[n]$ Then $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

Example: Consider the following system

$x[n] = b^n u[n]$

$h[n] = a^n u[n]$

$y[n] = ??$

From the convolution property, we have $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

$$Y(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})(1 - be^{-j\omega})} = \frac{A}{1 - ae^{-j\omega}} + \frac{B}{1 - be^{-j\omega}}$$

using partial fraction $A = \frac{a}{a-b}$ and $B = -\frac{b}{a-b}$

Therefore,

$$y[n] = \frac{1}{a-b} \left[ a^{n+1} u[n] - b^{n+1} u[n] \right] = \frac{a^{n+1} - b^{n+1}}{a - b} u[n]$$
Properties of the DT Fourier Transform

- Multiplication: Let

\[ y[n] = x_1[n] \cdot x_2[n] \]

then

\[ Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \]
Properties of the DT Fourier Transform: Difference equation

- DT LTI Systems are characterized by Linear Constant-Coefficient Difference Equations
- A general linear constant-coefficient difference equation for an LTI system with input $x[n]$ and output $y[n]$ is of the form
  \[ \sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k] \]

- Now applying the Fourier transform to both sides of the above equation, we have
  \[ \sum_{k=0}^{N} a_k e^{-jkw} Y(e^{jw}) = \sum_{k=0}^{M} b_k e^{-jkw} X(e^{jw}) \]

- But we know that the input and the output are related to each other through the impulse response of the system, denoted by $h[n]$, i.e.,
  \[ y[n] = x[n] * h[n] \]
Properties of the DT Fourier Transform: Difference equation

- Applying the convolution property

\[ Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \quad \text{or} \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jkw}}{\sum_{k=0}^{N} a_k e^{-jkw}} \]

=> if one is given a difference equation corresponding to some system, the Fourier transform of the impulse response of the system can be found directly from the difference equation by applying the Fourier transform.

- Fourier transform of the impulse response = Frequency response
- Inverse Fourier transform of the frequency response = Impulse response
Properties of the DT Fourier Transform: Example

- With $|a| < 1$, consider the causal LTI system that is characterized by the difference equation
  \[ y[n] - ay[n - 1] = x[n] \]

- From the discussion, it is easy to see that the frequency response of the system is
  \[ H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \]

- From tables (or by applying inverse Fourier transform), one can easily find that
  \[ h[n] = a^n u[n] \]
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# Relations Among Fourier Methods

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<th>Aperiodic in Time</th>
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<td>CT Fourier Transform</td>
<td>CT - P&lt;sub&gt;T&lt;/sub&gt; ⇒ DT</td>
<td>$a_k = \frac{1}{T} \int_{0}^{T} x(t)e^{-j\omega_k t} , dt$</td>
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<td>CT Inverse Fourier Series</td>
<td>CT Inverse Fourier Transform</td>
<td>$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} , d\omega$</td>
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<td>DT Fourier Series</td>
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<td>DT - P&lt;sub&gt;N&lt;/sub&gt; ⇒ DT - P&lt;sub&gt;N&lt;/sub&gt;</td>
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<td>$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$</td>
<td>DT ⇒ CT + P&lt;sub&gt;2\pi&lt;/sub&gt;</td>
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Relations Among Fourier Methods

DT Fourier Series

CT Fourier Series

DT Fourier Transform

CT Fourier Transform
\[ X(f) = \sum_{k=-\infty}^{\infty} X[k] \delta(f - kf_0) \]
CT Fourier Transform - CT Fourier Series

\[
X_p[k] = f_p X(kf_p)
\]
CT Fourier Transform - DT Fourier Transform

Let \( x_\delta(t) = x(t) \frac{1}{T_s} \text{comb} \left( \frac{t}{T_s} \right) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s) \)

and let \( x[n] = x(nT_s) \)

There is an “information equivalence” between \( x_\delta(t) \) and \( x[n] \). They are both completely described by the same set of numbers.

\[
X_{DTFT}(F) = X_\delta(f_sF) \quad X_\delta(f) = X_{DTFT} \left( \frac{f}{f_s} \right)
\]

\[
X_{DTFT}(F) = f_s \sum_{k=-\infty}^{\infty} X_{CTFT} \left( f_s(F - k) \right)
\]
CT Fourier Transform - DT Fourier Transform

\[ x[n] \xrightarrow{\mathcal{F}} X(F) \]

\[ x_s(t) \xrightarrow{\mathcal{F}} X_s(f) \]

\[ x(t) \xrightarrow{\mathcal{F}} X(f) \]
DT Fourier Series - DT Fourier Transform

\[ X(F) = \sum_{k=-\infty}^{\infty} X[k] \delta(F - kF_0) \]
DT Fourier Series - DT Fourier Transform

\[ X_p[k] = \frac{1}{N_p} X(kF_p) \]
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DTFT: Summary

- DT Fourier Transform represents a discrete time aperiodic signal as a sum of infinitely many complex exponentials, with the frequency varying continuously in (-π, π)

\[
x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega t} \, d\omega, \quad X(e^{j\omega}) = \sum_{n} x[n] e^{-jn\omega}
\]

- DTFT is periodic
  - only need to determine it for \( \omega \in (-\pi, \pi) \)
DTFT: Summary

- Know how to calculate the DTFT of simple functions
  - Know the geometric sum:

  \[ \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad \text{if} \quad |a|<1 \]

- Know Fourier transforms of special functions, e.g. \( \delta[n] \), exponential
- Know how to calculate the inverse transform of rational functions using partial fraction expansion
- Properties of DT Fourier transform
  - Linearity, Time-shift, Frequency-shift, …
A discrete-time LTI system has impulse response \( h[n] = \left(\frac{1}{2}\right)^n u[n] \)

Find the output \( y[n] \) due to input \( x[n] = \left(\frac{1}{7}\right)^n u[n] \)

*(Suggestion: work with and using the convolution property)*

**Solution**

This can be solved using convolution of \( h[n] \) and \( x[n] \).

However, the point was to use the convolution in time \( \rightarrow \) multiplication in frequency property.

Therefore,
\[
y[n] = h[n] * x[n] \Rightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})
\]

It can be readily shown that
\[
m[n] = (a)^n u[n] \Rightarrow M(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}, \quad a < 1
\]

Therefore, 
\[
H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \quad \text{and} \quad X(e^{j\omega}) = \frac{1}{1 - \frac{1}{7}e^{-j\omega}}
\]
DT-FT Summary: a quiz

- Exploiting the convolution in time $\rightarrow$ multiplication in frequency property gives:

$$Y(e^{j\omega}) = \left(\frac{1}{1 - \frac{1}{7}e^{-j\omega}}\right)\left(\frac{1}{1 - \frac{1}{2}e^{-j\omega}}\right)$$

- Using partial fraction expansion method of finding inverse Fourier transform gives:

$$Y(e^{j\omega}) = \frac{-2/5}{1 - \frac{1}{7}e^{-j\omega}} + \frac{7/5}{1 - \frac{1}{2}e^{-j\omega}}$$

- Therefore,
  - since a Fourier transform is unique, (i.e. no two same signals in time give the same function in frequency) and
  - since $m[n] = (a)^n u[n] \Rightarrow M(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$

  $\Rightarrow$ It can be seen that a Fourier transform of the type $\frac{1}{1 - ae^{-j\omega}}$
  should correspond to a signal $a^n u[n]$.

- Therefore,
  - the inverse Fourier transform of $\frac{-2/5}{1 - \frac{1}{7}e^{-j\omega}}$ is $-\frac{2}{5}\left(\frac{1}{7}\right)^n u[n]$
  - the inverse transform of $\frac{7/5}{1 - \frac{1}{2}e^{-j\omega}}$ is $\frac{7}{5}\left(\frac{1}{2}\right)^n u[n]$

- Thus the complete output

$$y[n] = -\frac{2}{5}\left(\frac{1}{7}\right)^n u[n] + \frac{7}{5}\left(\frac{1}{2}\right)^n u[n]$$
Outline

- DT Fourier Transform
- Overview of Fourier methods
- DT Fourier Transform of Periodic Signals
- Properties of DT Fourier Transform
- Relations among Fourier Methods
- DTFT: Summary

**Appendix:**
- Transition from DT Fourier Series to DT Fourier Transform
Transition: DT Fourier Series to DT Fourier Transform

- DT Pulse Train Signal: \( x(n) = \text{rect}_{N_w}[n] \ast \text{comb}_{N_0}[n] \)

- This DT periodic rectangular-wave signal is analogous to the CT periodic rectangular-wave signal used to illustrate the transition from the CT Fourier Series to the CT Fourier Transform.
Transition: DT Fourier Series to DT Fourier Transform

- DTFS of DT Pulse Train
- As the period of the rectangular wave increases, the period of the DT Fourier Series increases and the amplitude of the DT Fourier Series decreases
Transition: DT Fourier Series to DT Fourier Transform

- Normalized DT Fourier Series of DT Pulse Train
- By multiplying the DT Fourier Series by its period and plotting versus instead of $k$, the amplitude of the DT Fourier Series stays the same as the period increases and the period of the normalized DT Fourier Series stays at one
Transition: DT Fourier Series to DT Fourier Transform

- The normalized DT Fourier Series approaches this limit as the DT period approaches infinity.