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NON-RECIPROCAL WAVE TRANSMISSION IN 1D MECHANICAL METAMATERIALS

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Reciprocity is often treated as a fundamental principle in elastodynamics: if the locations of the source and receiver are interchanged, the recorded signal remains unchanged. Despite their intriguing power, reciprocity relations hold only when certain conditions are satisfied in the elastic medium. For example, breakdown of reciprocity may take place in nonlinear, active or rotating systems. Non-reciprocity leads to asymmetric wave propagation for opposite directions, which can be used to develop devices for rectification and control of elastic waves. Recent experiments have shown the feasibility of realizing such devices in mechanical metamaterials. In this light, we review the scenarios in which asymmetric wave propagation is possible in the context of 1D phononic systems and mechanical/acoustic metamaterials.

Keywords: Reciprocity, asymmetric waves, nonreciprocal wave propagation, 1D phononics

Introduction

Reciprocity theorems provide relations between two elastodynamics states of a medium [1]. Most famously, reciprocity describes the symmetry of wave propagation: if the locations of the source and receiver are interchanged, the recorded signal remains unchanged. This property, known for well over a century [2, 3], has been instrumental in developing certain acoustic and vibration measurement techniques [4, 5, 6], such as measurement of transfer functions and microphone calibration.

Despite their incredible usefulness in acoustics and vibration, reciprocity relations hold only when certain conditions are satisfied in the elastic medium. Most notably [1, 4, 7], a reciprocity theorem relies on the linearity and time invariance of the medium (material properties that do not change over time). Deviations from these conditions are encountered in nonlinear, active or rotating systems, and may lead to asymmetric wave propagation for opposite directions. This *nonreciprocal* property of wave propagation can be employed for developing biased elastic waveguides.

In the light of the recent theoretical and experimental developments in the mechanics community regarding nonreciprocal wave propagation, we review the scenarios in which asymmetric wave propagation is possible in the context of 1D periodic media. We also distinguish between asymmetric and nonreciprocal wave propagation, a distinction that is prone to being overlooked. There exist scenarios for asymmetric wave propagation that do not involve breaking reciprocity; rather, the asymmetric transmission follows directly from reciprocity considerations in a linear, time-invariant medium. We will discuss these scenarios, along with examples of nonreciprocal systems.

Nonreciprocity in 1D periodic media

Elastodynamic nonreciprocity is perhaps most familiarly associated with nonlinear media. Reciprocity theorems do not generally hold for nonlinear systems, but not every nonlinear system is nonreciprocal. It has been shown in 1D elastodynamics [8] that nonreciprocal propagation could depend on the boundary conditions, the symmetries of the governing equations, and the choices of the locations where nonreciprocity is being tested. In the context of periodic media (phononic crystals and mechanical metamaterials with repeating unit cells and symmetric boundary conditions), breaking the symmetries of the nonlinear medium, either in gradually changing the geometry of the unit cell or in the functional form of nonlinearities, is often a good predictor of nonreciprocal response. For example, defective granular chains were used for switching and rectification of signals based on the post-bifurcation behavior of the medium [9]: the onset of bifurcation depends on the location of the input force. Nonreciprocal propagation is also possible in granular chains operating in the pre-bifurcation regime; in this case, the tonal frequency of the signal is mostly preserved [10]. Both bifurcation-based and frequency-preserving nonreciprocity may also be realized in lattice materials with bilinear elasticity [11]. In this case, the asymmetry is within the functional form of nonlinearity: different stiffness in compression and extension. Whether nonreciprocity is based on weak nonlinearity or bifurcation, asymmetry is an indispensable ingredient: if the system is symmetric, then the onset of instability and the post-bifurcation behavior of the medium remain invariant upon exchanging the source and receiver.

Reciprocity theorems do not generally hold in rotating systems due to the existence of gyroscopic forces, a property that has been understood since the nineteenth century [4, 3]. Gyroscopic forces play a significant role in realization of a class of (two-dimensional) topological metamaterials [12]. However, gyroscopic forces are of limited practical relevance in the context of one-dimensional mechanical systems and will not be emphasized in this review.

Another well known scenario where reciprocity does not hold is for acoustic waves in the presence of flow (e.g. sound waves in certain windy environments [2, 3]). This case is not emphasized here either.

Mechanical systems with time-dependent properties are another class of media in which reciprocity does not hold. Because of the power required for maintaining the modulations, such mechanical systems are often active media. In particular, periodic composites in which the elastic/inertial properties change in time and space in a wave-like fashion have been known, theoretically, to violate reciprocity since the twentieth century [13]. However, it was only recently [14] that this phenomenon was demonstrated experimentally. In mechanical systems subject to spatiotemporal modulations, there are frequency ranges in which unidirectional wave propagation is possible: waves can only propagate freely in one direction. Nonreciprocity in modulated phononic crystals and metamaterials occurs in the linear operating range.

Although reciprocity is a very familiar theorem in elastodynamics, it can sometimes be misinterpreted or applied in a setting that is not appropriate. For example, some misconceptions regarding acoustic and thermal diodes were addressed recently [15]. The choice of the input(s) and output(s) is another important consideration. If one's goal is simply to extract different outputs by interchanging the locations of the source and receiver, there is no need to break reciprocity. We discuss an example in which asymmetric wave transmission takes place in a linear time-invariant system in compliance with reciprocity. Our main goal is to highlight the importance of choosing the correct inputs and outputs when checking for reciprocity.

Reciprocal asymmetric wave propagation

Consider the array of N coupled oscillators shown in Fig. 1. The equations governing the motion of this mechanical system may be written in matrix form as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{1}$$

where $[M]$ is a diagonal mass matrix, $[C]$ is a proportional viscous damping matrix that is included to avoid infinities at resonance frequencies, $[K]$ is a tridiagonal stiffness matrix, $\{x\} = [x_1(t), ..., x_N(t)]^T$ is the state vector of displacements, $\{f(t)\}$ is the vector of external forces applied at each degree of freedom, and an overdot represents time derivative. Because the governing equations are linear, we can focus on the steady-state response of the system to harmonic excitation without loss of generality. Assuming a time dependence of $\exp(i\omega t)$ for both the input and output, we can rewrite Eq. (1) as

$$[D]\{X\} = \{F\} \Rightarrow \{X\} = [H]\{F\}, \qquad [H] \equiv [D]^{-1} \equiv \left(-\omega^2[M] + i\omega[C] + [K]\right)^{-1} \tag{2}$$

where $\{X\} = [X_1, ..., X_N]^T$ and $\{F\} = [F_1, ..., F_N]^T$ are vectors of complex-valued displacement and force amplitudes. Without loss of generality, we assume $\{F\}$ to be real-valued throughout this work. $[D]$ and $[H]$ are known, respectively, as the dynamic stiffness and receptance matrices in structural dynamics [7]. Note that all the components of $\{X\}$ and $\{F\}$ may be functions of the forcing frequency; e.g. $X_n = X_n(\omega)$ for $1 \leq n \leq N$. For brevity, we will not explicitly write the dependence on ω hereafter.

We consider two configurations for the input (source) and output (receiver) locations: (i) *forward* configuration: where the input is applied to the left boundary and the output is recorded at the right boundary; (ii) *backward* configuration: the same source is applied to the right boundary and the output is recorded at the left boundary. The recorded output is the steady-state displacement amplitude for both configurations. The input is either a prescribed force amplitude or a prescribed displacement amplitude. These two cases are treated separately.

Force input

We first choose a prescribed force amplitude as the input. For the forward configuration, we have $\{F\} = \{P^F, 0, ..., 0\}^T$ where P^F is the prescribed input. Thus, using the components of the receptance matrix $[H]$, we have $X_n^F = H_{n1}P^F$ for $1 \leq n \leq N$. For the backward configuration, we have $\{F\} = [0, ..., 0, P^B]^T$ where P^B is the prescribed input. In this case, we have $X_n^B = H_{nN}P^B$ for $1 \leq n \leq N$. Denoting the output of the forward and backward configurations by U^F and U^B, we have

$$\left.\begin{aligned} U^F = X_N^F = H_{N1}P^F \\ U^B = X_1^B = H_{1N}P^B \end{aligned}\right\} \xrightarrow{H_{1N}=H_{N1}} \frac{U^F}{P^F} = \frac{U^B}{P^B} \tag{3}$$

where we have used the symmetry property of the receptance matrix [7]; i.e. $[H]^T = [H]$. If the input amplitude is the same for both configurations, $P^F = P^B$, then Eq. (3) reduces to $U^F = U^B$, which is

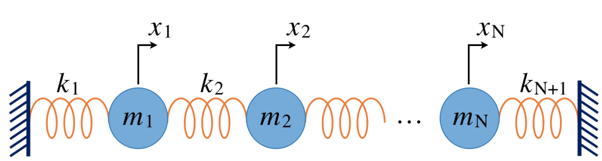

Figure 1: A one-dimensioal array of N coupled oscillators.

the most familiar statement of reciprocity: interchanging the locations of the input and output leaves the recorded output unchanged.

We emphasize that the appropriate output in this context is a displacement amplitude, not a ratio of displacement amplitudes. To clarify this point, we introduce the displacement ratio parameter, r. For the forward configuration, the input is located at $n = 1$ (left end) and $r^F = X_N^F/X_1^F$. For the backward configuration, the input is located at $n = N$ and $r^B = X_1^B/X_N^B$. Assuming $P^F = P^B$ for simplicity, we have

$$\left.\begin{aligned} r^F &= H_{N1}/H_{11} \\ r^B &= H_{1N}/H_{NN} \end{aligned}\right\} \xrightarrow[H_{11} \neq H_{NN}]{H_{1N}=H_{N1}} r^F \neq r^B \tag{4}$$

Thus, if a displacement ratio is monitored as the output, then the forward and backward configurations do not have the same output. This might appear in contrast with reciprocity in the first glance (same output expected for the same input), but in fact there is no contrast. The reciprocity theorem in this context [7] requires the symmetry of the receptance matrix $[H]$ and has no bearing on the relation between H_{11} and H_{NN}.

Displacement input

In the second scenario, we choose a prescribed displacement as the input. For the forward configuration, we apply the displacement input amplitude $Y_{in}^F = Y_{in}^F(\omega)$ to the first unit ($X_1^F = Y_{in}^F$) and monitor the output at the other end ($U^F = X_N$). Because the governing equations are linear and there is only one input and one output, without loss of generality, we can replace the prescribed input displacement at $n = 1$ with an unknown input force at $n = 1$, denoted by Q^F for the forward configuration. Considering that $X_n^F = H_{n1}Q^F$, we can write $Y_{in}^F = X_1^F = H_{11}Q^F$ to obtain $Q^F = Y_{in}^F/H_{11}$, which is the external force that ensures the prescribed displacement amplitude Y_{in}^F at $n = 1$. Similarly, for the backward configuration, we apply the prescribed displacement input $Y_{in}^B = Y_{in}^B(\omega)$ to the last unit ($X_N^B = Y_{in}^B$) and monitor the output at the first unit, $U^B = X_1^B$. Following the same reasoning as in the forward configuration, we can write $X_N^B = Y_{in}^B = H_{NN}Q^B$ and obtain $Q^B = Y_{in}^B/H_{NN}$. Here, Q^B is the external force that is applied to the N-th uint to ensure the prescribed displacement amplitude Y_{in}^B at $n = N$.

Now we are equipped to compare the outputs of the forward and backward configurations for this scenario:

$$\left.\begin{aligned} U^F &= X_N^F = H_{N1}Q^F = \frac{H_{N1}}{H_{11}} Y_{in}^F \\ U^B &= X_1^B = H_{1N}Q^B = \frac{H_{1N}}{H_{NN}} Y_{in}^B \end{aligned}\right\} \xrightarrow[H_{11} \neq H_{NN}]{H_{1N}=H_{N1}} \frac{U^F}{Y_{in}^F} \neq \frac{U^B}{Y_{in}^B} \tag{5}$$

In contrast to the force-input scenario of Section 3.1, we can see that prescribing the same displacement as the input ($Y_{in}^F = Y_{in}^B$) does not result in the same output – compare Eq. (5) to Eq. (3).

There are of course situations in which we can force the outputs to be equal in this scenario. This is done by arranging the mass and stiffness matrices such that $H_{11} = H_{NN}$. Most notably, this can be achieved by having a spatially symmetric system; e.g. for the system shown in Fig. 1 this condition can be written as $k_n = k_{N+1-n}$ and $m_n = m_{N+1-n}$ for $1 \leq n \leq N$. Of course, this result is trivial for a spatially symmetric system because in this case the forward and backward configurations are identical.

Numerical example

To illustrate the results in Eq. (3) and Eq. (5), we use the mechanical system in Fig. 1 with $N = 13$ units. We use the following non-dimensional parameters: $m_n = 1$ for all n and $k_n = 1 + \delta_n$ for $1 \leq n \leq N + 1$ with δ_n increasing linearly from -0.33 to 0.67. We consider viscous damping forces

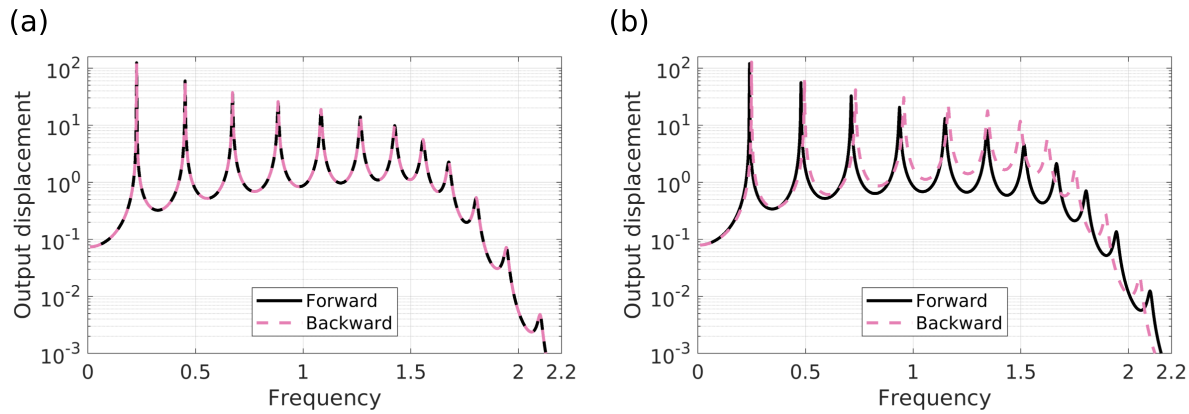

Figure 2: The response of the mechanical system to (a) unit force input, Eq. (3), (b) unit displacement input, Eq. (5). Reciprocity holds in both cases. Displacement input leads to asymmetric wave propagation.

to act between adjacent units with uniform coefficients 0.005 – a stiffness-proportional damping matrix. The steady-state response is calculated based on Eq. (2). We note that a linear viscous damping, even if it is non-proportional, does not alter the outcome.

For the force-input scenario (Section 3.1), Fig. 2(a) shows the output for the forward and backward configuration. As expected from Eq. (3), the propagation is symmetric and the two responses are indistinguishable. Fig. 2(b) shows the outputs for the case of displacement input (Section 3.2). We can see that the outputs for the forward and backward configurations are not generally the same, as predicted by Eq. (5). We conclude that choosing a prescribed displacement input results in asymmetric propagation of waves.

Concluding Remarks

It is important and helpful to distinguish between asymmetric and nonreciprocal wave propagation because they are not always interchangeable. A main focus in systems with asymmetric wave propagation is to show that if the locations of the input and output are interchanged, the recorded output does not remain unchanged. Nonreciprocity can lead to asymmetric propagation, but the inverse is not necessarily true. We provided an example in which asymmetric propagation occurs in a one-dimensional elastic medium in compliance with reciprocity. See [16] for an example in a two-dimensional elastic medium involving surface waves.

Reciprocity theorems in linear elastodynamics are powerful invariance relations that hold for scleronomic, holonomic systems [7]. These restrictions exclude, for example, systems with moving boundaries, time-dependent properties or rotating parts. Reciprocity in this context can be reduced in the frequency domain to the symmetry of the receptance matrix.

Care should also be taken in the choice of inputs and outputs when checking for reciprocity. As demonstrated, using a prescribed displacement as the input or using amplitude ratios as the output are both inappropriate. A good check is that the product of the variables to be interchanged under reciprocity yields the power or energy [17].

The matrix formulation in Section 3 is general and applies to any elastodynamic problem that can be described by Eq. (1), for example by discretizing the governing equations using the finite element method. The presence of extra features, such as internal resonances or non-proportional damping, does not alter the outcome.

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