Permutations and Combinations

**Definition 1**: Permutation of a set of distinct objects is an ordered arrangement of these objects. (Order matters, no repetition).

Example: $S = \{1, 2, 3\}$, a permutation is $(3, 1, 2)$.

**Definition 2**: $P(n, r)$ is a number of $r$-permutations of a set with $n$ objects.

**Theorem**: Given $S$ with $n$ distinct elements, then:

$$P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$

Proof: given in class.
**Definition 3:** Combinations.  
An $r$-combination of $n$ elements is an unordered selection of $r$ elements from $n$ elements.

**Example:** $S = \{1, 2, 3, 4\}$ then $\{1, 2, 3\}$ is a 3-combination of $S$.

\[
C(n, r) = \frac{n!}{r!(n-r)!}; 0 \leq r \leq n
\]

\[
P(n, r) = C(n, r) \cdot P(r, r)
\]

\[
C(n, r) = C(n, n-r)
\]

**Binomial coefficient:** $C(n, r) = \binom{n}{r}$

**Theorem:** Pascal identity.

\[
C(n + 1, k) = C(n, k - 1) + C(n, k)
\]
Theorem: \( \sum_{k=0}^{n} C(n, k) = 2^n \)

Proof: given in class.

Vandermonde’s Identity
Let \( m, n, \) and \( r \) be nonnegative integers, and \( r \leq m, n. \) Then:
\[
C(m + n, r) = \sum_{k=0}^{r} C(m, r - k)C(n, k)
\]

Proof: given in class.
The Binomial Theorem
Let $x$ and $y$ be variables, and $n$ a positive integer. Then:

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

$$= \sum_{j=0}^{n} C(n, j)x^{n-j}y^j.$$

Proof: by induction.

Example: $(x + y)^4 = \sum_{j=0}^{4} C(4, j)x^{4-j}y^j$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$
Solution given in class.

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$
Solution given in class.

**Theorem:** Let $n$ be a positive integer. Then:

$$\sum_{k=0}^{n} (-1)^k C(n,k) = 0$$

Proof given in class.
Permutations with Repetition

Example: For \{1, 2, 3, 4\}, (2,2,1) and (2,2,3) are permutations with repetition allowed.

The number of r-permutations of n objects with repetition is:

\[ n^r = n \cdot n \cdot \ldots \cdot n \]

\(r\) times.
Combinations with Repetition

**Theorem:** There are $C(n + r - 1, r)$ $r$-combinations from a set with $n$ elements when repetition is allowed.

**Example:** How many ways to select 5 bills from a bag containing 1$, 2$, 5$, 10$, 20$, 50$, and 100$ bills? Assuming that the order in which the bills are chosen does not matter.

**Example:** How many solutions does the equation: $x_1 + x_2 + x_3 + x_4 = 11$ have, where $x_1, x_2, x_3, x_4$ are nonnegative integers?
## Combinations and Permutations with or without Repetition

<table>
<thead>
<tr>
<th>Type</th>
<th>Repetition Allowed?</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>r-permutations</td>
<td>No</td>
<td>(\frac{n!}{(n-r)!})</td>
</tr>
<tr>
<td>r-combination</td>
<td>No</td>
<td>(\frac{n!}{r!(n-r)!})</td>
</tr>
<tr>
<td>r-permutations</td>
<td>Yes</td>
<td>(n^r)</td>
</tr>
<tr>
<td>r-combination</td>
<td>Yes</td>
<td>(\frac{(n+r-1)!}{r!(n-1)!})</td>
</tr>
</tbody>
</table>

**Note:** \(C(n + r - 1, r) = \frac{(n+r-1)!}{r!(n-1)!}\)
Permutations of Sets with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*?

**Theorem:** The number of different permutations of $n$ objects, where there are $n_1$ indistinguishable objects of type 1, $n_2$ indistinguishable objects of type 2, ..., and $n_k$ indistinguishable objects of type $k$, is:

$$\frac{n!}{n_1!n_2! \cdots n_k!}$$
Distributing Objects into Boxes

Example: How many ways are there to distribute 5 cards to each of four players from a deck of 52 cards?

Theorem: The number of ways to distribute $n$ distinguishable objects into $k$ distinguishable boxes so that $n_i$ objects are placed into box $i$, $i = 1, 2, ..., k$ equals:

$$\frac{n!}{n_1!n_2! \cdots n_k!}$$