Advanced Counting Techniques

Many counting problems cannot be solved by the previous counting techniques.

Example: How many bit strings of length n do not contain 2 consecutive 0's?

Answer: $a_n = a_{n-1} + a_{n-2}$, $a_1 = 2$, $a_2 = 3$.

The answer is a recurrence relation.

Example: Compound interest at 7%.

$$P_n = P_{n-1} + 0.07P_{n-1} = 1.07P_{n-1}$$
$$P_n = (1.07)^n P_0$$

The Tower of Hanoi: The problem of moving n disks from one peg to another peg, one at a time, via a third peg in such a way that no disk is on top of a smaller one. Let H_n be a minimum number of moves needed to solve the problem.
We can summarize the solution as follows:

move $n-1$ top disks from peg 1 to 2
move the largest disk from peg 1 to peg 3
move $n-1$ disks from peg 2 to 3

Thus we have

$$
\begin{aligned}
&H_n = H_{n-1} + 1 + H_{n-1} \\
&H_n = 2H_{n-1} + 1 \\
&H_n = 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&\quad \ldots\ldots\ldots\ldots \\
&H_n = 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \ldots + 1 \\
&H_n = \sum_{i=0}^{n-1} 2^i = 2^n - 1
\end{aligned}
$$

Example: How many bit strings of length n do not contain 2 consecutive 0's?

Answer: Denote by a_n the number of such strings. We will try to relate a_n with a_{n-1} and a_{n-2}.

Case 1: If the string begins with a 1 then it can be followed by a_{n-1} strings that do not contain 2 consecutive 0's.

Case 2: If the string begins with a 0 then the next bit must be 1, and it can be followed by a_{n-2} strings that do not contain 2 consecutive 0's.
Therefore: $a_n = a_{n-1} + a_{n-2}$, the initial conditions are easily found: $a_1 = 2$ and $a_2 = 3$.

Example: How many strings of n decimal digits (0-9) contain an even number of 0's?

Answer: Let a_n denote the number of such strings. Hence

$a_1 = 9$.

$a_2 = 9 \cdot 9 + 1 \;=\; 82$

Case 1: Take a valid string length $n-1$ and append a digit $\neq 0$ (there are 9):

There are: $9a_{n-1}$ such strings.

Case 2: Take a non-valid string length $n-1$ and append a 0:

There are: $(10^{n-1} - a_{n-1})$ such strings.

The total is: $a_n = 10^{n-1} - a_{n-1} + 9a_{n-1} = 8a_{n-1} + 10^{n-1}$.

Problems:

1. How many bit strings of length n do not contain 00?

2. How many bit strings of length n contain 00?

3. How many bit strings of length 7 either begin with 00 or (inclusive or) end with 111?

4. How many bit strings of length 10 either have 5 consecutive $0's$ or 5 consecutive $1's$?

Solving Linear Recurrences

Linear homogeneous recurrences

Linear homogeneous recurrence relation of degree k with constant coefficients:

$$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}$$

where $c_1, \ldots, c_k$ are real numbers with $c_k \neq 0$.

Characteristic equation:

$$r^k - c_1 r^{k-1} - \ldots - c_{k-1} r - c_k = 0.$$

For example, solve $a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 0$ with proper initial conditions.

Method:

1. Find the characteristic equation for the homogeneous recurrence.

2. Solve the characteristic equation for the roots $r_1, r_2, ..., r_k$.

There are two cases:

Case 1 If all the roots are distinct, then the solution is of the form:

$$C_1 r_1^n + C_2 r_2^n + ... + C_k r_k^n$$

Case 2 If some roots are the same,
for example, three roots with $r_1 = r_2 = r_3 = r$
then the solution is $C_1 r^n + C_2 n r^n + C_3 n^2 r^n$.
If the three roots are $r_1, r_2 = r_3 = r$
then the solution is $C_1 r_1^n + C_2 r^n + C_3 n r^n$.
If the three roots are $r_1 = r_2 = r, r_3$
then the solution is $C_1 r^n + C_2 n r^n + C_3 r_3^n$.

Examples

Example 1. Solve $a_n - 3a_{n-1} - 4a_{n-2} = 0, n \geq 2$,
$a_0 = 0, a_1 = 1$

Solution. The characteristic equation is $r^2 - 3r - 4 = 0$ which has roots $r_1 = -1, r_2 = 4$ which are distinct, so use Case 1:

$a_n = C_1(-1)^n + C_2(4)^n$.

Using the initial conditions we obtain

$$a_0 = 0 = C_1 + C_2$$
$$a_1 = 1 = -C_1 + 4C_2.$$

Solving the system above yields $C_1 = -1/5$ and $C_2 = 1/5$, thus

$$a_n = \frac{(-1)^{n+1} + 4^n}{5}.$$

Example 2. Solve
$a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 0$,
$n \geq 3, a_0 = 0, a_1 = 1, a_2 = 2$.
Solution. Characteristic equation: $r^3 - 5r^2 + 8r - 4 = 0$.
We find one root $r = 1$. We factor the equation $r^3 - 5r^2 + 8r - 4 = 0$ by dividing the left side by $r - 1$ $\Rightarrow (r-1)(r-2)^2 = 0 \Rightarrow r_1 = 1(multiplicity = 1), r_2 = 2(multiplicity = 2)$. It's Case 2.
Therefore, $a_n = C_1 1^n + C_2 2^n + C_3 n 2^n$.
With initial conditions $a_0 = 0, a_1 = 1, a_2 = 2$
we find $C_1 = -2, C_2 = 2, C_3 = -1/2$
Therefore

$$a_n = -2 + 2(2^n) - \frac{1}{2}n(2^n).$$

Linear nonhomogeneous recurrences

Linear nonhomogeneous recurrence relation of degree k with constant coefficients:

$$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} + F(n)$$

where $c_1, \ldots, c_k$ are real numbers with $c_k \neq 0$ and $F(n)$ is the function not identically zero depending only on n.

For example, solve $a_n - 5a_{n-1} + 6a_{n-2} = 7^n$ with proper initial conditions. Denote $F(n)$ = the nonhomogeneous part, i.e. $F(n) = 7^n$ in this example.

1. Find the solution $a_n^{(h)}$ of the associated homogeneous recurrence (see above).

2. Find a particular solution $a_n^{(p)}$ of the nonhomogeneous equation. A particular solution can be found by using a *trial solution*. The trial solution depends on the nonhomogeneous term $F(n)$.

Case 1 If $F(n)$ has the form $p(n)s^n$ where $p(n)$ is a polynomial in n of degree k, s is a constant, and if s **is not** a root of the characteristic equation, then the trial solution has the same form as $q(n)s^n$, where $q(n)$ is a polynomial of degree k (see example below).

Case 2 If $F(n)$ has the form $p(n)s^n$ where $p(n)$ is a polynomial in n of degree k and s is a root of the characteristic equation with multiplicity m, then the trial solution has the form $n^m q(n)s^n$, where $q(n)$ has the same degree as $p(n)$.

3. The solution to the nonhomogenous equation is $a_n = a_n^{(h)} + a_n^{(p)}$.

Examples

Example. Solve $a_n - 5a_{n-1} + 6a_{n-2} = 7^n$
Solution. The characteristic equation: $r^2 - 5r + 6 = 0$
Therefore $r_1 = 3, r_2 = 2 \Rightarrow a_n^{(h)} = C_1 3^n + C_2 2^n$.

Since 7 is not a root of the characteristic equation (the roots are 3 and 2) hence the trial solution is $C7^n$ (Case 1).

Substituting the trial solution into the nonhomogeneous equation we have:

$$C7^n - 5C7^{n-1} + 6C7^{n-2} = 7^n$$
$$C7^2 - 5 \cdot 7C + 6C = 7^2$$
$$49C - 35C + 6C = 49$$

or $C = \frac{49}{20} \Rightarrow a_n^{(p)} = \frac{49}{20}7^n$.

Combining $a_n^{(h)}$ and $a_n^{(p)}$ to get

$a_n = C_1 3^n + C_2 2^n + \frac{49}{20}7^n$.

You can find C_1 and C_2 from the initial conditions (whatever they may be).

Big-O notation: Let $f(x)$ and $g(x)$ be two functions from the set of integers, we say
$f(x) = O(g(x))$ if there are constants C and k such that:
$|f(x)| \leq C|g(x)|$, for $x > k$.

Example: $f(x) = 2x^2 + 3x - 12$, and $g(x) = x^2$ we say $f(x) = O(x^2)$.

Example: $logn! = O(nlogn)$.

Divide-and-Conquer relations

$$T(n) = aT(\frac{n}{b}) + c.$$

Assume: $n = b^k$ or $k = log_b n$.

$$T(\frac{n}{b}) = aT(\frac{n}{b^2}) + c.$$

$$T(n) = a[aT(\frac{n}{b^2}) + c] + c$$

$$T(n) = a^2 T(\frac{n}{b^2}) + ac + c.$$

$$T(n) = a^k T(\frac{n}{b^k}) + (a^{k-1} + ... + a + 1)c.$$

$$T(n) = a^k T(1) + c \sum_{i=0}^{k-1} a^i.$$

Assume $T(1) = 1$.
If $a = 1$ then

$$T(n) = T(1) + c \cdot k = 1 + c \cdot log_b n = O(logn)$$

If $a > 1$ then

$$T(n) = a^k + c \sum_{i=0}^{k-1} a^i.$$

Since

$$\sum_{i=0}^{k-1} a^i = \frac{a^k - 1}{a - 1}.$$

We have

$$T(n) = a^k[1 + \frac{c}{a-1}] - \frac{c}{a-1}.$$

Therefore

$$T(n) = O(a^k) = O(a^{log_b n}) = O(n^{log_b a}).$$

More details on Divide-and-Conquer relations
We first derive solution in general case and then infer particular cases.

Theorem 1 *Let $f(n)$ be an increasing function satisfying*

$$f(n) = af(n/b) + g(n)$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and $g(n)$ is the positive sequence of real numbers. Then

$$f(n) = a^k f\left(\frac{n}{b^k}\right) + \sum_{i=0}^{k-1} a^i g\left(\frac{n}{b^i}\right)$$

when $n = b^k$ and

$$f(n) \leq f(b^{k+1}) = a^{k+1} f(1) + \sum_{i=0}^{k} a^i g\left(\frac{n}{b^i}\right)$$

when $n \neq b^k$.

Proof. Assume $n = b^k$. We have by iterative approach

$$\begin{aligned}
&f(n) = af(n/b) + g(n) \\
&= a[af\left(\frac{n}{b^2}\right) + g\left(\frac{n}{b}\right)] + g(n) \\
&= a^2 f\left(\frac{n}{b^2}\right) + ag\left(\frac{n}{b}\right) + g(n) \\
&= a^2[af\left(\frac{n}{b^3}\right) + g\left(\frac{n}{b^2}\right)] + ag\left(\frac{n}{b}\right) + g(n) \\
&= a^3 f\left(\frac{n}{b^3}\right) + a^2 g\left(\frac{n}{b^2}\right) n + ag\left(\frac{n}{b}\right) + g(n) \\
&\ldots \\
&= a^k f(1) + \sum_{i=0}^{k-1} a^i g\left(\frac{n}{b^i}\right).
\end{aligned}$$

When $n \neq b^k$ then $b^k < n < b^{k+1}$ for some positive integer k. Since f is an increasing function

$$f(n) \leq f(b^{k+1}) = a^{k+1} f(1) + \sum_{i=0}^{k} a^i g\left(\frac{n}{b^i}\right).$$

Special cases.

1. $g(n) = c$.

   i) $a = 1$. Let $n = b^k$. Then

$$f(n) = a^k f(1) + c \sum_{i=0}^{k-1} a^i$$

   thus

$$f(n) = f(1) + ck.$$

   But $n = b^k \implies k = \log_b n$. Hence

$$f(n) = f(1) + c \log_b n = O(\log n).$$

   When $n \neq b^k$, we have

$$\begin{aligned} f(n) &= f(1) + c(k+1) \\ &= f(1) + c + c \log_b n = O(\log n). \end{aligned}$$

   ii) $a > 1$. Let $n = b^k$. Using formula for the sum of geometric progression

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1}, \quad r \neq 0$$

   we have

$$
\begin{aligned}
f(n) &= f(1) + c\sum_{i=0}^{k-1} a^i \\
&= a^k f(1) + c\frac{a^k - 1}{a - 1} \\
&= a^k[f(1) + \frac{a}{a-1}] + (-\frac{c}{a-1}) \\
&= \left(f(1) + \frac{a}{a-1}\right) n^{\log_b a} + (-\frac{c}{a-1}) \\
&= C_1 n^{\log_b a} + C_2 = O(n^{\log_b a})
\end{aligned}
$$

where we have used the identity $a^{\log_b n} = n^{\log_b a}$.

When $n \neq b^k$ then

$$f(n) \leq (C_1 a) n^{\log_b a} + C_2 = O(n^{\log_b a}).$$

We have thus proved

Theorem 2 *Let $f(n)$ be an increasing function satisfying*

$$f(n) = af(n/b) + c$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) = \begin{cases} O(\log n) & \textit{if} \quad a = 1 \\ O(n^{\log_b a}) & \textit{if} \quad a > 1. \end{cases}$$

2. $g(n) = cn^d$, where d is a positive real number.

Let $n = b^k$. By Theorem 1 we have

$$f(n) = a^k f(n/b^k) + \sum_{i=0}^{k-1} a^i c \left(\frac{n}{b^i}\right)^d$$
$$= a^k f(1) + cn^d \sum_{i=0}^{k-1} \left(\frac{a}{b^d}\right)^i.$$

When $n \neq b^k$

$$f(n) \leq f(b^{k+1}) = a^{k+1} f(1) + cn^d \sum_{i=0}^{k} \left(\frac{a}{b^d}\right)^i.$$

(i) $a = b^d$.

$$
\begin{aligned}
f(n) &= a^k f(1) + cn^d \sum_{i=0}^{k-1} 1^i \\
&= a^k f(1) + cn^d k \\
&= a^k f(1) + cn^d \log_b n \\
&= a^{\log_b n} f(1) + cn^d \log_b n \\
&= n^{\log_b a} f(1) + cn^d \log_b n \\
&= n^d f(1) + cn^d \log_b n \\
&= O(n^d \log n).
\end{aligned}
$$

(ii) $a \neq b^d$. Then

$$
\begin{aligned}
& f(n) \\
&= a^k f(1) + cn^d \sum_{i=0}^{k-1} \left(\frac{a}{b^d}\right)^i \\
&= a^k f(1) + cn^d \frac{\left(\frac{a}{b^d}\right)^k - 1}{\frac{a}{b^d} - 1} \\
&= a^k f(1) + cn^d \frac{\frac{a^k}{b^{kd}} b^d - b^d}{a - b^d} \\
&= a^k [f(1) + c\frac{n^d \frac{b^d}{b^{kd}}}{a - b^d}] - cn^d \frac{b^d}{a - b^d} \\
&= a^{\log_b n} [f(1) + c\frac{\left(\frac{n}{b^k}\right)^d b^d}{a - b^d}] - cn^d \frac{b^d}{a - b^d} \\
&= n^{\log_b a} [f(1) + c\frac{b^d}{a - b^d}] - cn^d \frac{b^d}{a - b^d} \\
&= C_1 n^{\log_b a} + C_2 n^d
\end{aligned}
$$

where $C_1 = [f(1) + c\frac{b^d}{a-b^d}]$ and $C_2 = c\frac{b^d}{b^d-a}$. If $a > b^d$ then $\log_b a > d$ and the last equation implies

$$f(n) = O(n^{\log_b a})$$

otherwise if $a < b^d$ then $\log_b a < d$ and we have

$$f(n) = O(n^d).$$

If $n \neq b^k$ then

$$f(n) \le f(b^{k+1}) = a^{k+1} f(1) + cn^d \sum_{i=0}^{k} \left(\frac{a}{b^d}\right)^i$$

$$= \begin{cases} O(n^{\log_b a}) & \text{if} \quad a > b^d \\ O(n^d) & \text{if} \quad a < b^d \end{cases}$$

whenever $a \neq b^d$ and

$$\begin{aligned} & f(n) \le f(b^{k+1}) \\ & = a^{k+1} f(1) + cn^d (k+1) \\ & = O(n^d \log n) \end{aligned}$$

when $a = b^d$. We have thus proven the master theorem

Theorem 3 *Let* $f(n)$ *be an increasing function satisfying*

$$f(n) = af(n/b) + cn^d$$

whenever n *is divisible by* b*, where* $a \geq 1$*,* b *is an integer greater than 1, and* c, d *are positive real numbers. Then*

$$f(n) = \begin{cases} O(n^{\log_b a}) & \text{if} \quad a > b^d \\ O(n^d \log n) & \text{if} \quad a = b^d \\ O(n^d) & \text{if} \quad a < b^d. \end{cases}$$

Example. Solve

$$f(n) = 8f(n/2) + n^2$$

where f is an increasing function, $f(1) = 1$ and show that $f(n) = O(n^{\log 3})$.

Solution. Following the proof Theorem 3 we have $a = 8, b = 2, c = 1, d = 2$, thus $a > b^d$ and

$$\begin{aligned} f(n) \quad &= [1 + \frac{1 \cdot 2^2}{8 - 2^2}]n^{\log_2 8} + \frac{1 \cdot 2^2}{2^2 - 8}n^2 \\ &= 2n^3 - n^2 \\ &= O(n^3) \end{aligned}$$

which agrees with Theorem 3.

Here is the complete solution.

$$\begin{aligned}
f(n) &= 8f(n/2) + n^2 \\
&= 8[8f(n/2^2) + (n/2)^2] + n^2 \\
&= 8^2 f(n/2^2) + 8(n/2)^2 + n^2 \\
&= 8^2[8f(n/2^3) + (n/2^2)^2] + 8(n/2)^2 + n^2 \\
&= 8^3 f(n/2^3) + 8^2(n/2^2)^2 + 8(n/2)^2 + n^2 \\
&= 8^k f(1) + n^2 \sum_{i=0}^{k-1} 8^i \left(\frac{1}{2^2}\right)^i \\
&= 8^k f(1) + n^2 \sum_{i=0}^{k-1} \left(\frac{8}{4}\right)^i \\
&= 8^k f(1) + n^2 \frac{2^k - 1}{2 - 1} \\
&= 8^k + n^2(2^k - 1).
\end{aligned}$$

Next, $k = \log_2 n$, $\quad 8^k = 8^{\log_2 n} = n^{\log_2 8} = n^3$. Thus

$$\begin{aligned}
f(n) &= 8^{\log_2 n} + n^2(2^{\log_2 n} - 1) \\
&= n^{\log_2 8} + n^2(n^{\log_2 2} - 1) \\
&= n^3 + n^2(n - 1) \\
&= 2n^3 - n^2.
\end{aligned}$$

Hence

$$f(n) = 2n^3 - n^2 = O(n^3).$$

Inclusion-Exclusion

Example: How many solutions are there to the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 30$$

where x_i, $i = 1, 2, 3, 4, 5, 6$, is nonnegative integer such that

1. $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?

2. $x_1 \leq 5$?

3. $x_1 \leq 7$ and $x_2 > 8$?

Solution.

1. We require each $x_i \geq 2$. This uses up 12 of the 30 total required, so the problem is the same as finding the number of solutions to $x'_1 + x'_2 + x'_3 + x'_4 + x'_5 + x'_6 = 18$ with each $x'_i s = x_i - 1$ a nonnegative integer. The number of solutions is therefore $C(6 + 18 - 1, 18) = C(23, 18) = 33649$.

2. The number of solutions without restriction is $C(6 + 30 - 1, 30) = C(35, 30) = 324632$. The number of solutions violating the restriction by having $x_1 \geq 6$ is $C(6 + 24 - 1, 24) = C(29, 24) = 118755$. Therefore the answer is 324632 - 118755 = 205877.

3. The number of solutions with $x_2 \geq 9$ (as required) but without the restriction on x_1 is $C(6+21-1, 21) = C(26, 21) = 65780$. The number of solution violating the additional restriction by having $x_1 \geq 8$ is $C(6+13-1, 13) = C(18, 13) = 8568$. Therefore the answer is 65780 - 8568 = 57212.

Example: How many solutions are there to the equation:

$$x_1 + x_2 + x_3 = 13$$

where x_i, $i = 1, 2, 3$ are nonnegative integer such that

$$1 \le x_1 \le 4, x_2 \le 6, x_3 \le 9?$$

Solution. First we take care of the double constraint on x_1 by substitution $x_1 = p + 1$, where $0 \le p \le 3$. The original problem is equivalent to finding solutions to

$$p + d + h = 12$$

where p, d, h are nonnegative integers such that

$$p \le 3, d \le 6, h \le 9?$$

Let S denote the set of all nonnegative integer solutions (p, d, h) of $p + d + h = 12$;

let P denote the set of all (p, d, h) in S such that $p \ge 4$;

let D denote the set of all (p, d, h) in S such that $d \ge 7$;

let H denote the set of all (p, d, h) in S such that $h \ge 10$.

By the principle of inclusion-exclusion, we have

$$\begin{aligned}&|S-(P\cup D\cup H)|\\&=|S|-(|P|+|D|+|H|)\\&+(|P\cap D|+|P\cap H|+|D\cap H|)\\&-(|P\cap D\cap H|)\end{aligned} \tag{1}$$

we also have $|S| = \binom{14}{2} = 91$ and
$|P| = \binom{10}{2} = 45$ (the number of nonnegative integer solutions of $p' + d + h = 8$),
$|D| = \binom{7}{2} = 21$ (the number of nonnegative integer solutions of $p + d' + h = 5$),
$|H| = \binom{4}{2} = 6$ (the number of nonnegative integer solutions of $p + d + h' = 2$),
$|P \cap D| = \binom{3}{2} = 3$ (the number of nonnegative integer solutions of $p' + d' + h = 1$),
and $|P \cap H| = |D \cap H| = |P \cap D \cap H| = 0$.
Substituting these partial results to (1) we get the answer 22.

Example: A well-known result implies that a composite integer is divisible by a prime not exceeding its square root. Find a number of primes not exceeding 100.

Solution.

The only primes less than 10 are 2,3,5,7, so the primes not exceeding 100 are these four primes and all positive integers $1 < n \leq 100$ not divisible by 2,3,5,7. Let A_i be a subset of elements that have property P_i that an integer is divisible by $i, i = 2, 3, 5, 7$ and let $|A_i| = N(P_i)$. By the principle of inclusion-exclusion the answer is

$$
\begin{aligned}
&4 + N(P_2'P_3'P_5'P_7') \\
&= 4 + (99 - N(P_2) - N(P_3) - N(P_5) - N(P_7) \\
&+N(P_2P_3) + N(P_2P_5) + N(P_2P_7) \\
&+N(P_3P_5) + N(P_3P_7) + N(P_5P_7) \\
&-N(P_2P_3P_5) - N(P_2P_3P_7) - N(P_2P_5P_7) - N(P_3P_3P_7) \\
&+N(P_2P_3P_5P_7)) \\
&= 4 + (99 - \left\lfloor \frac{100}{2} \right\rfloor - \left\lfloor \frac{100}{3} \right\rfloor - \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{7} \right\rfloor \\
&+ \left\lfloor \frac{100}{2 \cdot 3} \right\rfloor + \left\lfloor \frac{100}{2 \cdot 5} \right\rfloor + \left\lfloor \frac{100}{2 \cdot 7} \right\rfloor + \left\lfloor \frac{100}{3 \cdot 5} \right\rfloor + \left\lfloor \frac{100}{3 \cdot 7} \right\rfloor \\
&- \left\lfloor \frac{100}{2 \cdot 3 \cdot 5} \right\rfloor - \left\lfloor \frac{100}{2 \cdot 3 \cdot 7} \right\rfloor - \left\lfloor \frac{100}{2 \cdot 5 \cdot 7} \right\rfloor - \left\lfloor \frac{100}{3 \cdot 5 \cdot 7} \right\rfloor \\
&+ \left\lfloor \frac{100}{2 \cdot 3 \cdot 5 \cdot 7} \right\rfloor) \\
&= 4 + (99 - 50 - 33 - 20 - 14 \\
&+16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0) \\
&= 25.
\end{aligned}
$$