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CUTTING-PLANE PROOFS AND THE STABILITY NUMBER OF A GRAPH

by

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ABSTRACT

Many claims in combinatorics can be stated by saying that every integer solution of a specified system of linear inequalities satisfies another specified linear inequality. Such claims can be proved in a certain canonical way involving the notion of cutting planes. We investigate the structure of these proofs in the particular case where the claim is that a specified graph contains at most a specified number of pairwise nonadjacent vertices.

1. INTRODUCTION

Many combinatorial results may be stated by saying that every integer solution of a specified system of linear inequalities,

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad (i = 1, 2, \ldots, m), \tag{1}$$

must satisfy a specified linear inequality

$$\sum_{j=1}^{n} a_{Mj}x_j \leq b_M \,. \tag{2}$$

One way of proving such results consists of exhibiting a sequence of inequalities

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad (i = 1, 2, \ldots M) \tag{3}$$

along with nonnegative numbers y_{ki} $(m < k \leq M,\ 1 \leq i < k)$ such that, for all $k = m+1, m+2, \ldots, M$, we have

$$\begin{aligned} a_{kj} &= \sum_{i=1}^{k-1} y_{ki}a_{ij} = \text{integer} \quad \text{for all } j = 1, 2, \ldots, n\,, \\ \lfloor \sum_{i=1}^{k-1} y_{ki}b_i \rfloor &\leq b_k \end{aligned} \tag{4}$$

(with $\lfloor t \rfloor$ standing for t rounded down to the nearest integer). We shall refer to every such sequence (3), presented along with the number y_{ki} , as a *cutting-plane proof* of (2) from (1). Obviously, if there is a cutting-plane proof of (2) from (1) then every integer solution of (1) must satisfy (2); the following two theorems show that the converse holds as soon as certain nonrestrictive assumptions are placed on (1).

Theorem 1. *Let the polyhedron defined by (1) be bounded. If every integer solution of (1) satisfies (2) then there is a cutting-plane proof of (2) from (1).*

Theorem 2. *Let all the numbers a_{ij} and b_i in (1) be rational, and let (1) have an integer solution. If every integer solution of (1) satisfies (2) then there is a cutting-plane proof of (2) from (1).*

Theorem 1 follows from Gomory's analysis of the cutting plane algorithms [5], [6], [7]; an alternative proof may be found in [1]. Theorem 2 is a corollary of a result of Schrijver [8], from which Theorem 1 may be also derived. The analogue of Theorems 1 and 2 is false if no assumption at all is placed on (1): for instance, as pointed out by Schrijver, if a is irrational then every integer solution of

$$\begin{aligned} x_1 - ax_2 &\le 0 \\ -x_1 + ax_2 &\le 0 \\ -x_1 - \ x_2 &\le 0 \end{aligned} \tag{5}$$

must satisfy $x_1 + x_2 \le 0$, and yet there is no cutting-plane proof of this inequality from (5). Similarly, every integer solution of

$$\begin{aligned} 3x_1 - 3x_2 &\le 2 \\ -3x_1 + 3x_2 &\le -1 \\ - x_1 - \ x_2 &\le 0 \end{aligned} \tag{6}$$

must satisfy $x_1 + x_2 \le 0$ (in fact, (6) has no integer solutions), and yet there is no cutting-plane proof of this inequality from (6).

We shall be concerned with finite undirected graphs. Given any such graph G , we shall associate a variable x_v with each vertex v of G , and consider the system

$$\begin{aligned} \sum_{v\in C} x_v \le 1 &\quad \text{for all cliques } C \text{ in } G\,, \\ -x_v \le 0 &\quad \text{for all vertices } v \text{ of } G\,. \end{aligned} \tag{7}$$

Clearly, the integer solutions of (7) are precisely the incidence vectors of sets S such that no two vertices in S are adjacent. Such sets are called independent or *stable*; the largest size of a stable set in G is called the *stability number* of G and denoted by $\alpha(G)$. Thus, with V standing for the set of all the vertices of G , every integer solution of (7) must satisfy

$$\sum_{v\in V} x_v \le \alpha(G)\,. \tag{8}$$

The subject of this note is the structure of cutting-plane proofs of (8) from (7).

2. DEPTH AND LENGTH OF CUTTING-PLANE PROOFS

In a way, the sequential order of the inequalities in (3) obscures the structure of the cutting-plane proof: this structure is revealed by a directed graph with vertices $1, 2, \ldots, M$, in which a directed edge goes from vertex i to vertex k if and only if $y_{ki} > 0$ in (4). Trivially, this graph is acyclic; we shall refer to the number of edges in a longest path terminating at a vertex k as the *depth* of the k-th inequality, and to the depth of the M-th inequality as the *depth* of the proof. Similarly, we shall refer to M in (3) as the *length* of the proof.

Theorem 3. *If some cutting-plane proof of a linear inequality from a system S of linear inequalities in n variables has depth d , then some cutting plane proof of this inequality from a subsystem of S has length at most*

$$1 + (n+1)(1 + n + \ldots + n^{d-1})\,.$$

Proof. From all the cutting-plane proofs of (2) from subsystems (1) of S , whose depth is at most d , choose one that minimizes M in (3). Note that

$$\text{the } k\text{-th inequality in (3) has depth less than } d \text{ whenever } k < M: \tag{9}$$

else the k-th inequality could be deleted from (3), contradicting minimality of M . Next, writing $y(k,i) = y_{ki}$, set

$$v_i = y(M,i) + y(M, M-1)y(M-1, i) \quad \text{for all } i = 1, 2, \ldots, M-2$$

and observe that

$$\sum_{i=1}^{M-2} v_i a_{ij} = a_{Mj} \quad \text{for all } j = 1, 2, \ldots, n\,.$$

Having made these observations, we propose to show that

$$\text{the system } \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad (i = 1, 2, \ldots, M-2) \text{ has a solution.} \tag{10}$$

To prove (10), let us assume the contrary. Now, by one of the fundamental results of linear programming (see, for instance, Theorem 9.2 in [2]), there are nonnegative numbers $w_1, w_2, \ldots, w_{M-2}$ such that

$$\sum_{i=1}^{M-2} w_i a_{ij} = 0 \quad \text{for all } j = 1, 2, \ldots, n$$

and

$$\sum_{i=1}^{M-2} w_i b_i = -1\,.$$

Choosing a sufficiently large positive t , we obtain

$$\sum_{i=1}^{M-2} (v_i + t w_i) a_{ij} = a_{Mj} \quad \text{for all } j = 1, 2, \ldots, n$$

and

$$\sum_{i=1}^{M-2}(v_i+tw_i)b_i=\sum_{i=1}^{M-2}v_ib_i-t<b_M\ .$$

Thus, numbers y_{Mi} in (4) could be replaced by v_i+tw_i , and the next-to-last inequality deleted from (3): by (9), the resulting cutting-plane proof would still have depth at most d . Since this contradicts minimality of M , claim (10) is established.

Next, we propose to establish the existence of nonnegative numbers $\tilde{y}_{ki}$ $(m<k\leq M$, $1\leq i<k)$, satisfying (4) in place of y_{ki} , such that

$$\tilde{y}_{ki}=0 \quad \text{whenever} \quad y_{ki}=0\ , \tag{11}$$

and

$$\text{at most } n \text{ of the numbers } \tilde{y}_{ki} \text{ are positive} \\ \text{for each} \quad k=m+1,m+2,\ldots,M-1\ , \tag{12}$$

and

$$\text{at most } n+1 \text{ of the numbers } \tilde{y}_{Mi} \text{ are positive.} \tag{13}$$

For each fixed $k=m+1,m+2,\ldots,M-1$, let I consist of all the subscripts i that have $y_{ki}>0$, and consider the linear programming problem

$$\begin{aligned} &\text{minimize} && \sum_{i\in I} b_iy_i \\ &\text{subject to} && \sum_{i\in I} a_{ij}y_i=a_{kj} \quad \text{for all} \quad j=1,2,\ldots,n\ , \\ &&& y_i\geq 0 \quad \text{whenever} \quad i\in I\ . \end{aligned}$$

This problem has a feasible solution $(y_i=y_{ki})$; by (10), its dual has a feasible solution; hence the problem has an optimal solution. A basic optimal solution, with at most n variables positive, yields the desired numbers $\tilde{y}_{ki}$. To find the numbers $\tilde{y}_{Mi}$, let I consist of all the subscripts i that have $y_{Mi}>0$, and consider the system

$$\begin{aligned} &\sum_{i\in I} a_{ij}y_i=a_{Mj} \quad (j=1,2,\ldots,n) \\ &\sum_{i\in I} b_iy_i=\sum_{i\in I} b_iy_{Mi}\ . \end{aligned}$$

This system has a nonnegative solution $(y_i=y_{Mi})$; hence, by another fundamental result of linear programming (see, for instance, Theorem 9.3 in [2]), it has a nonnegative solution with at most $n+1$ variables positive.

Finally, replace the numbers y_{ki} in (4) by $\tilde{y}_{ki}$. By (11), the resulting cutting-plane proof still has depth at most d . Consider the directed graph associated with this proof: by (12) and (13), at most n directed edges enter each vertex other than M , and at most $n+1$ directed edges enter vertex M . It follows that the component containing M has at most $1+(n+1)(1+n+\ldots+n^{d-1})$ vertices. □

The upper bound in Theorem 3 can be replaced by

$$1+n+\ldots+n^d$$

as soon as S has an integer solution: in this case, the entire system (3) is solvable, and so (13) can be established with n in place of $n+1$. However, this improved upper bound does not hold in general. For instance, the sequence

$$\begin{aligned} 2x&\leq 3 \\ -2x&\leq -3 \\ x&\leq 1 \\ x&\leq 0 \end{aligned}$$

with $y_{31}=\frac{1}{2}$, $y_{32}=0$, $y_{41}=0$, $y_{42}=1$, $y_{43}=3$ constitutes a cutting-plane proof of the last integrality from the first two, and the depth of this proof is two; at the same time, every cutting-plane proof of the last integrality from a subset of the first two must have length at least four.

3. SHORT AND SHALLOW PROOFS

Either of Theorems 1 and 2 guarantees that, for every graph G , there is a cutting-plane proof of (8) from (7). In this section, we shall present two easy results bounding from above the minimum length and the minimum depth of such proofs.

Throughout the remainder of this note, we shall reserve the letters n , α , and V for the number of vertices, the stability number, and the set of all the vertices, respectively, of a graph G . We shall say that an inequality

$$\sum_{j=1}^{n} a_jx_j\leq b$$

is a *linear combination* of inequalities

$$\sum_{j=1}^{n} a_{ij}x_j\leq b_i \quad (i=1,2,\ldots,m)$$

if there are nonnegative numbers $y_1,y_2,\ldots,y_m$ such that

$$\sum_{i=1}^{m} y_ia_{ij}=a_j \quad \text{for all} \quad j\ , \quad \text{and} \quad \sum_{i=1}^{m} y_ib_i=b\ .$$

The stability number of a graph can be always determined by examining all the sets of at most α vertices; the following theorem shows that there are always cutting-plane proofs of (8) from a subsystem of (7) of comparable complexity.

Theorem 4. *For every graph G , there is a cutting-plane proof of (8) from a subsystem of (7), whose length is at most $\binom{n+1}{2}+\binom{n}{\alpha}$.*

Proof. Since the case of $n \le 2$ is trivial, we may assume that $n \ge 3$. Now there is a family $\mathcal{C}$ of at most $\binom{n}{2}$ cliques in G such that each edge of G belongs to some clique in $\mathcal{C}$ and such that each vertex of G belongs to some clique in $\mathcal{C}$. For each subset R of V , let $\alpha(R)$ stand for the largest size of a stable subset of R . We claim that, for each nonempty R , there is a cutting-plane proof of

$$\sum_{v\in R} x_v \le \alpha(R) \tag{14}$$

from

$$\begin{aligned} \sum_{v\in C} x_v \le 1 \quad &\text{whenever } C \in \mathcal{C} , \\ -x_v \le 0 \quad &\text{whenever } v \in V , \end{aligned} \tag{15}$$

in which the number of inequalities not belonging to (15) is at most $\binom{|R|}{\alpha(R)}$.

This claim will be justified by induction on $|R|$. If $\alpha(R-w) < \alpha(R)$ for some vertex w in R then the conclusion is immediate, for the inequality $x_w \le 1$ is a linear combination of (15). Thus, we may assume that

$$\alpha(R-w) = \alpha(R)$$

whenever $w \in R$; in particular, each vertex in R has at least one neighbour in R . Choose a vertex w in R that belongs to some stable set of $\alpha(R)$ vertices in R , set $P = R - w$, and let Q stand for the subset of P obtained by deleting all the neighbours of w . The induction hypothesis guarantees that (15) extends into a cutting-plane proof of

$$\sum_{v\in P} x_v \le \alpha(P) \tag{16}$$

from (15) by a sequence P^* of at most $\binom{|P|}{\alpha(P)}$ inequalities, and that (15) extends into a cutting-plane proof of

$$\sum_{v\in Q} x_v \le \alpha(Q) \tag{17}$$

from (15) by a sequence P^* of at most $\binom{|Q|}{\alpha(Q)}$ inequalities. Since $|P| = |R|-1$, $|Q| \le |R|-2$, $\alpha(P) = \alpha(R)$, and $\alpha(Q) = \alpha(R) - 1$, we have

$$|P^*| \le \binom{|R|-1}{\alpha(R)} \quad \text{and} \quad |Q^*| \le \binom{|R|-2}{\alpha(R)-1} \le \binom{|R|-1}{\alpha(R)-1} - 1 ,$$

and so

$$|P^*| + |Q^*| \le \binom{|R|}{\alpha(R)} - 1 .$$

Now we only need exhibit a linear combination of (15), (16), (17) that reads

$$\sum_{v\in V} x_v \le b \tag{18}$$

with $b < \alpha(R) + 1$, for then the concatenation of (7), P^*, Q^* , and (14) will constitute the desired cutting-plane proof.

Since each vertex in $P - Q$ is a neighbour of w , and since the inequality $x_v + x_w \le 1$ is a linear combination of (15) for each neighbour v of w , the inequality

$$|P-Q|\, x_w + \sum_{v\in P-Q} x_v \le |P-Q| \tag{19}$$

is a linear combination of (15). Adding $|P-Q|-1$ times (16) to the sum of (17) and (19), we obtain the inequality

$$|P-Q| \cdot \sum_{v\in V} x_v \le (|P-Q|-1)\alpha(R) + (\alpha(R)-1) + |P-Q| .$$

Hence (18) with

$$b = \alpha(R) + 1 - \frac{1}{|P-Q|}$$

is a linear combination of (15), (16), (17). □

In the following theorem, as usual, $\lceil t \rceil$ stands for t rounded up to the nearest integer, and ln stands for the natural logarithm.

Theorem 5. *For every graph G with $n \le \alpha + 2$ there is a cutting-plane proof of (8) from (7) whose depth is at most $n - \alpha - 1$; if $n \ge 2\alpha + 1$ then the upper bound can be replaced by*

$$\alpha + \lceil (2\alpha+1)\, ln \frac{n}{2\alpha+1} \rceil .$$

Proof. Let us define a sequence $n_1,\ n_2,\ n_3, \ldots$ of integers by setting $n_1 = \alpha + 2$ and letting each n_k with $k > 1$ be the largest integer smaller than $(\alpha+1)n_{k-1}/\alpha$. We claim that, for every set A of at most k vertices, there is a cutting-plane proof of

$$\sum_{v\in A} x_v \le \alpha \tag{20}$$

from (7) of depth at most k . This claim is easy to justify by induction on k ; only two observations are required. First, each inequality (20) with $|A| \le \alpha + 1$ is a linear combination of (7); second, if $a < b$ then each inequality

$$\sum_{v\in B} x_v \le \frac{b}{a}\alpha$$

with $|B| = b$ is a linear combination of all the inequalities (20) with $|A| = a$.

Now we only need prove that the smallest k with $n_k \ge n$ satisfies $k \le n - \alpha - 1$ and, in case $n \ge 2\alpha + 1$,

$$k \le \alpha + \lceil (2\alpha+1)\, ln \frac{n}{2\alpha+1} \rceil .$$

For this purpose, we first use induction on k to show that $n_k \ge \alpha + k + 1$ for all k . Then, observing that

$$(2\alpha+1)(1+\frac{1}{2\alpha})^{k-\alpha} + 1 < \frac{\alpha+1}{\alpha}(2\alpha+1)(1+\frac{1}{2\alpha})^{k-1-\alpha}$$

whenever $k > \alpha$, we use further induction on k to show that

$$n_k \geq (2\alpha+1)(1+\frac{1}{2\alpha})^{(k-\alpha)} \quad \text{for all } k=\alpha,\ \alpha+1,\ \alpha+2,\ldots .$$

Since

$$1+\frac{1}{2\alpha} \geq exp\,\frac{1}{2\alpha+1} ,$$

the desired conclusion follows. □

4. LOWER BOUNDS ON DEPTH

This section contains our main results; a large part of the argument involved in their proofs may be summarized as follows.

Lemma *Let G be a graph and let k, s be positive integers such that $k < s$ and such that every subgraph of G with s vertices is k-colorable. Then every cutting-plane proof of (8) from (7) has depth at least*

$$\frac{s}{k}\,ln\,\frac{n}{\alpha k} .$$

Proof. We shall establish a stronger conclusion: if some cutting-plane proof of an inequality

$$\sum_{v\in V} a_v x_v \leq b \tag{21}$$

from (7) has depth at most d then

$$b \geq \frac{1}{k}(\frac{s}{s+k})^d \sum_{v\in V} a_v . \tag{22}$$

This will be done by induction on d .

The case of $d=0$ is trivial, since G has no clique with more than k vertices. Now assume that $d>0$ and note that each a_v in (21) is an integer. Let us show at once that

$$b \geq \frac{1}{k}\sum_{v\in W} a_v \quad \text{whenever } |W| \leq s : \tag{23}$$

since the subgraph of G induced by W is k-colorable, there are numbers x_{iv} $(1 \leq i \leq k\ ,\ v \in W)$ such that

$$\sum_{i=1}^{k} x_{iv} = 1 \quad \text{for all } v ,$$

$$\sum_{v\in W} a_v x_{iv} \leq b \quad \text{for all } i ,$$

and so

$$\sum_{v\in W} a_v = \sum_{i=1}^{k}\sum_{v\in W} a_v x_{iv} \leq kb .$$

Now, we shall distinguish between two cases.

Case 1: Fewer than s vertices v have $a_v > 0$. In this case, (23) implies

$$b \geq \frac{1}{k}\sum_{v\in V} a_v$$

which is stronger than (22).

Case 2: At least s vertices v have $a_v > 0$. In this case, (23) implies $b \geq s/k$, and so

$$b \geq \frac{s}{s+k}(b+1) . \tag{24}$$

By assumption, there are inequalities

$$\sum_{v\in V} a_{iv} x_v \leq b_i \quad (i=1,2,\ldots,m)$$

whose cutting-plane proofs from (7) have depth at most $d-1$, and there are nonnegative numbers $y_1,\ y_2,\ldots,\ y_m$ such that

$$\sum_{i=1}^{m} y_i a_{iv} = a_v \quad \text{whenever } v \in V ,$$

$$\sum_{i=1}^{m} y_i b_i < b+1 .$$

By the induction hypothesis, we have

$$b_i \geq \frac{1}{k}(\frac{s}{s+k})^{d-1}\sum_{v\in V} a_{iv} \quad \text{for all } i=1,2,\ldots,m ,$$

and so

$$b+1 > \sum_{i=1}^{m} y_i b_i \geq \frac{1}{k}(\frac{s}{s+k})^{d-1}\sum_{i=1}^{m}\sum_{v\in V} y_i a_{iv} = \frac{1}{k}(\frac{s}{s+k})^{d-1}\sum_{v\in V} a_v . \tag{25}$$

Now (22) follows from (24) and (25). □

It is an open question whether for every graph G there is a cutting-plane proof of (8) from (7) whose length is only $(1+o(1))^n$. The first of our two main results shows that an affirmative answer to this question cannot be justified by simply appealing to Theorem 3.

Theorem 6. *There are arbitrarily large graphs G and a positive constant ϵ such that the depth of every cutting-plane proof of (8) from (7) exceeds ϵn .*

Proof. Erdös [4] has proved that for every positive c there is a positive δ with the following property: there are arbitrarily large graphs G in which $\alpha < n/c$, and yet every subgraph of G

with at most δn vertices is 3-colorable. Any c greater than 3 will do for our purpose: we only need set $k = 3$ and $s = \lfloor \delta n \rfloor$ in the Lemma. □

Theorem 4 guarantees that for every graph G with $\alpha = 2$ there is a cutting-plane proof of (8) from (7) whose length is at most n^2. Yet, as we are about to show, no constant bounds from above the depth of such cutting-plane proofs.

Theorem 7. *There are arbitrarily large graphs G with $\alpha = 2$ such that the depth of every cutting-plane proof of (8) from (7) exceeds $\frac{1}{3} \ln n$.*

Proof. Erdös [3] has proved that, for some positive c, there are arbitrarily large graphs G in which $\alpha = 2$ and yet every clique has at most $cn^{\frac{1}{2}} \ln n$ vertices. Now we only need use the Lemma with k equal to the largest size of a clique in G and with $s = k + 1$. □

Note that, by Theorem 5, the lower bounds of Theorems 6 and 7 cannot be improved beyond a constant factor.

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