A self-centered look at cutting planes

Vašek Chvátal

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    if x^* belongs to $\mathcal{S}$
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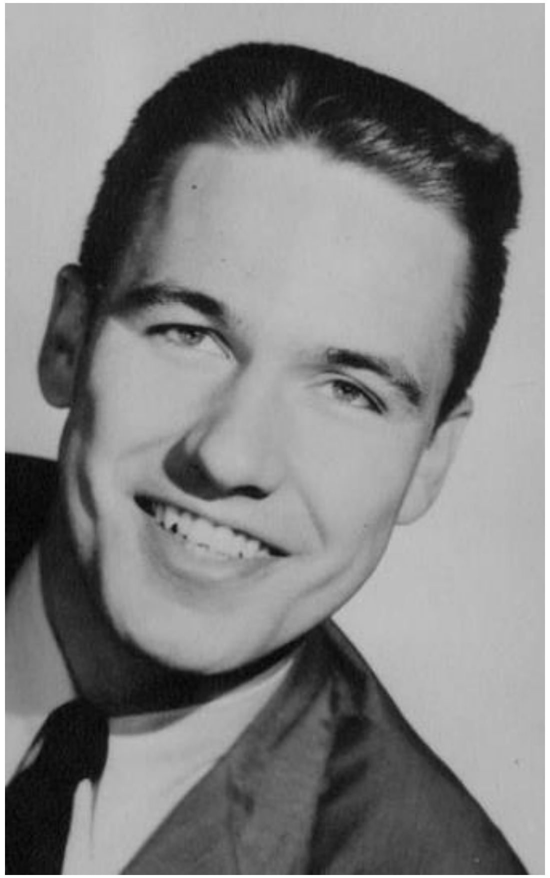

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Corollary (generalization of the Birkhoff – von Neumann Theorem) :

The convex hull of incidence vectors of matchings in a bipartite graph
is described by the set of linear inequalities

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Corollary:

The convex hull of incidence vectors of matchings in an arbitrary graph is described by the set of linear inequalities

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Corollary (The Matching Polyhedron Theorem):

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Because for every vector c in $\mathbf{R}^n$, a linear description $Ax \leq b$ of the convex hull of $\mathcal{S}$ reduces the discrete optimization problem maximize $c^T x$ subject to $x \in \mathcal{S}$
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Why are we interested in reducing discrete optimization problems to LP problems?
Because the LP Duality Theorem provides certificates of optimality:
If every solution x such that $Ax \leq b$ satisfies $c^T x \leq d$,
then there is a nonnegative vector y such that $y^T A = c$ and $y^T b \leq d$.

Incidence vectors of matchings in an arbitrary graph satisfy the inequalities

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then, for every set S of vertices, they satisfy the inequalities

$$-x_e \leq 0 \quad \text{with all } e \text{ such that } e \cap S \neq \emptyset, e - S \neq \emptyset$$

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Generalization: If a vector x satisfies a system $Ax \le b$ of inequalities, then, for every nonnegative vector y, it satisfies the inequality $\left(y^{\mathrm{T}}A\right)x \le y^T b$. Furthermore, If both x and $y^T A$ are integer-valued, then the left-hand side is an integer, and so the right-hand side can be rounded down to the nearest integer.

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Definition: An inequality $c^T x \leq d$ belongs to the *elementary closure* of a system $Ax \leq b$ if the vector c is integer-valued there is a nonnegative vector y such that $c = y^T A$ and $d \geq \lfloor y^T b \rfloor$.

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Given a graph G, let $\Sigma(G)$ denote the system

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Corollary of the Matching Polyhedron Theorem:
A linear description of the convex hull of all integer solutions of $\Sigma(G)$
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integer solutions of $\Sigma(G)$ are precisely the incidence vectors of matchings.

When H is a graph, $T(H)$ denotes the system

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If H is a line graph of G, then $T(H)$ subsumes $\Sigma(G)$, and so a linear description of the convex hull of all integer solutions of $T(H)$ is contained in $e\big(T(H)\big)$.

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Be wise: Generalize!

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How about arbitrary graphs ???

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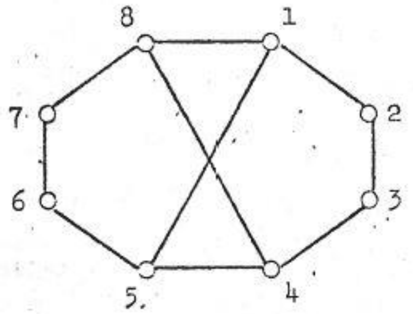

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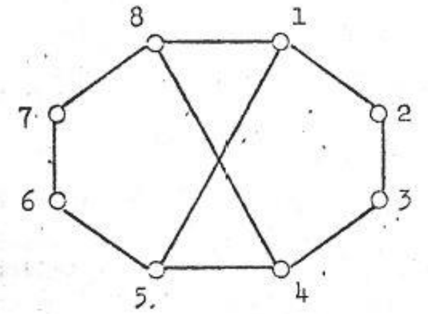

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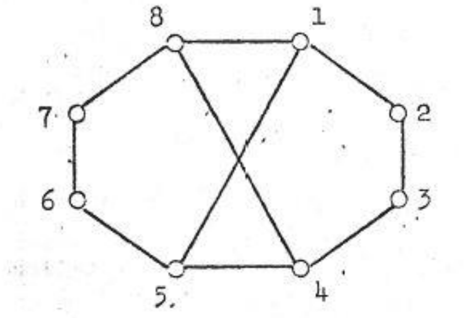
When G is this graph, $T(G)$ consists of the eight inequalities $-x_j \leq 0$, the eight inequalities $x_j \leq 1$ and the ten inequalities $x_j + x_k \leq 1$.



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$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 2,$$
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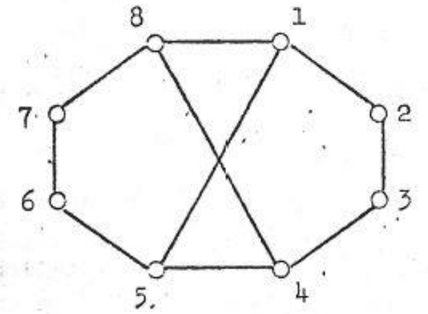

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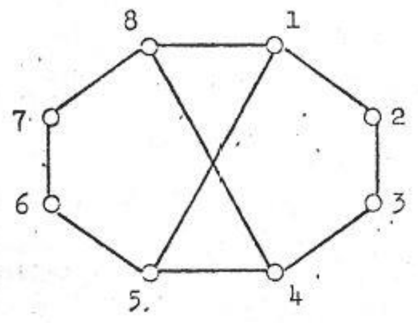
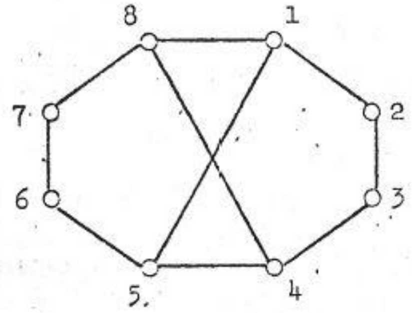
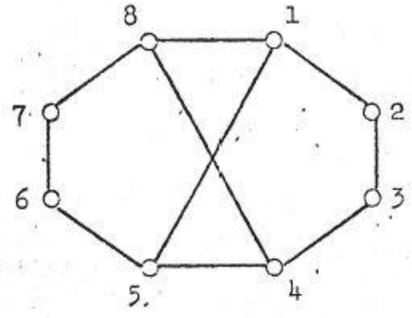
Actually, all inequalities in $e(T(G))$ are linear combinations of these 30 inequalities.



The largest stable set in this G has three vertices.



The largest stable set in this G has three vertices,
but the maximum of $x_1 + x_2 + \cdots x_8$ subject to $e(T(G))$ is bigger than 3:
$x_1 = x_4 = x_5 = x_8 = \frac{1}{3}$ and $x_2 = x_3 = x_6 = x_7 = \frac{1}{2}$ satisfy $e(T(G))$ and make $x_1 + x_2 + \cdots + x_8 = 3\frac{1}{3}$.



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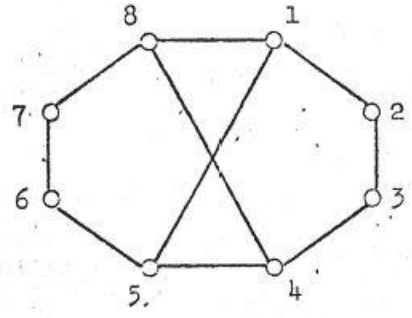
Hence the inequality $x_1 + x_2 + \cdots + x_8 \leq 3$ is not a linear combination of inequalities in $e(T(G))$.



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Nevertheless, this inequality belongs to $e(e(T(G)))$: the sum of

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$$x_2 + x_3 \leq 1,$$
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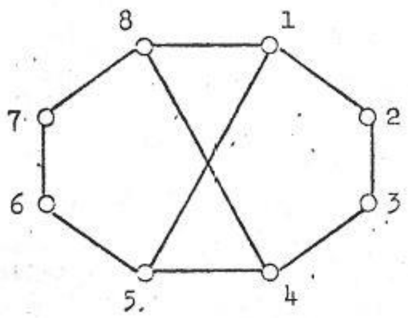
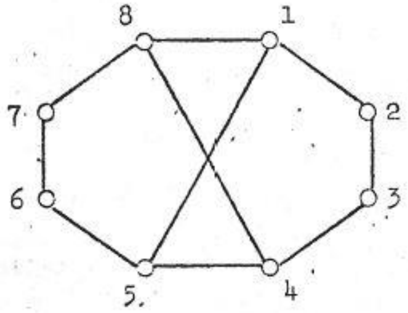
reads $3(x_1 + x_2 + \cdots + x_8) \leq 10$, which scales to $x_1 + x_2 + \cdots + x_8 \leq 3\frac{1}{3}$.



The inequality $x_1 + x_2 + \cdots + x_8 \leq 3$ is not a linear combination of inequalities in $e\big(T(G)\big)$, but it belongs to $e\big(e\big(T(G)\big)\big)$.



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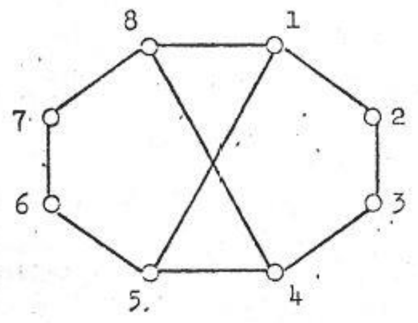
..... inequalities in $T(G)$



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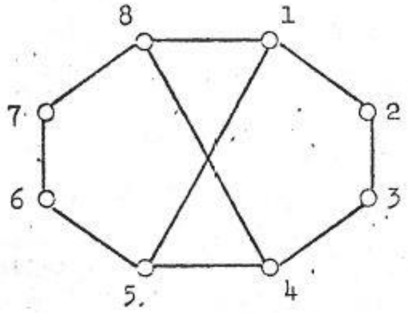

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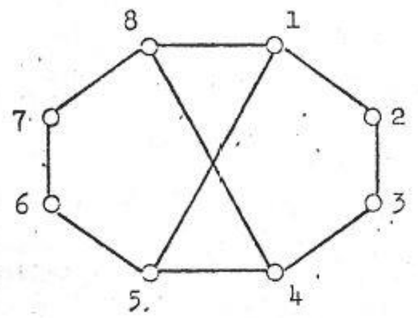

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Given a system T of linear inequalities, we set $e^0(T) = T$ and, for every positive integer k, $e^k(T) = e(e^{k-1}(T))$.

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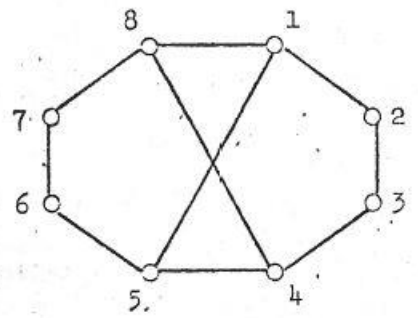
inequality in $e^2\big(T(G)\big)$



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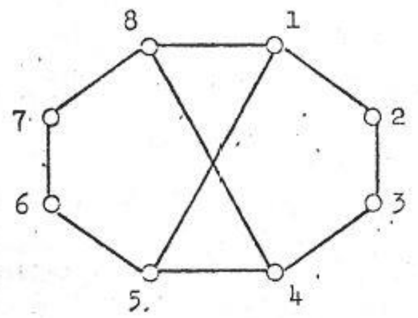

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Given a system T of linear inequalities, we set $e^0(T) = T$ and, for every positive integer k, $e^k(T) = e(T \cup e^{k-1}(T))$.

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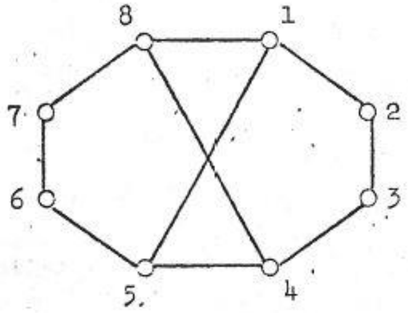

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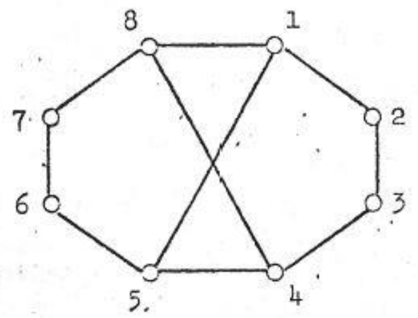

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$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leq 3$ inequality in $e^2\left(T(G)\right)$ = rank 2 inequality

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$x_3 + x_4 \leq 1$

$x_4 + x_5 \leq 1$

$x_5 + x_6 \leq 1$

$x_6 + x_7 \leq 1$

$x_7 + x_8 \leq 1$

$x_8 + x_1 \leq 1$

$x_1 + x_5 \leq 1$

$x_8 + x_4 \leq 1$ ..... inequalities in $e^0(T(G))$ = rank 0 inequalities

Given a system T of linear inequalities, we set $e^0(T) = T$ and, for every positive integer k, $e^k(T) = e(T \cup e^{k-1}(T))$.

The *rank* of an inequality is defined as the smallest k such that $e^k(T)$ includes this ineq.

$x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$

$x_1 + x_5 + x_6 + x_7 + x_8 \leq 2$

$x_1 + x_2 + x_3 + x_4 + x_8 \leq 2$

$x_4 + x_5 + x_6 + x_7 + x_8 \leq 2$ ..... inequalities in $e^1(T(G))$ = rank 1 inequalities

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leq 3$ inequality in $e^2\big(T(G)\big)$ = rank 2 inequality

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And he was right: Ralph E. Gomory, “An algorithm for integer solutions to linear programs, *Recent advances in mathematical programming* (R.L. Graves and P. Wolfe, eds.), McGraw-Hill, 1963, pp. 269-302.

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[5] Cutting planes in combinatorics, *European Journal of Combinatorics* **6** (1985), 217-226 under the name of *cutting-plane proofs* and in complete generality.

Definition: Δ-system is a family of sets $S_1, S_2, \dots S_m$ such that the intersection of distinct S_i and S_j does not depend on i, j.

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The conjecture can be formulated in integer linear programming terms and this formulation leads to a way of proving it.

Integer linear programming formulation of the Erdős-Lovász conjecture:

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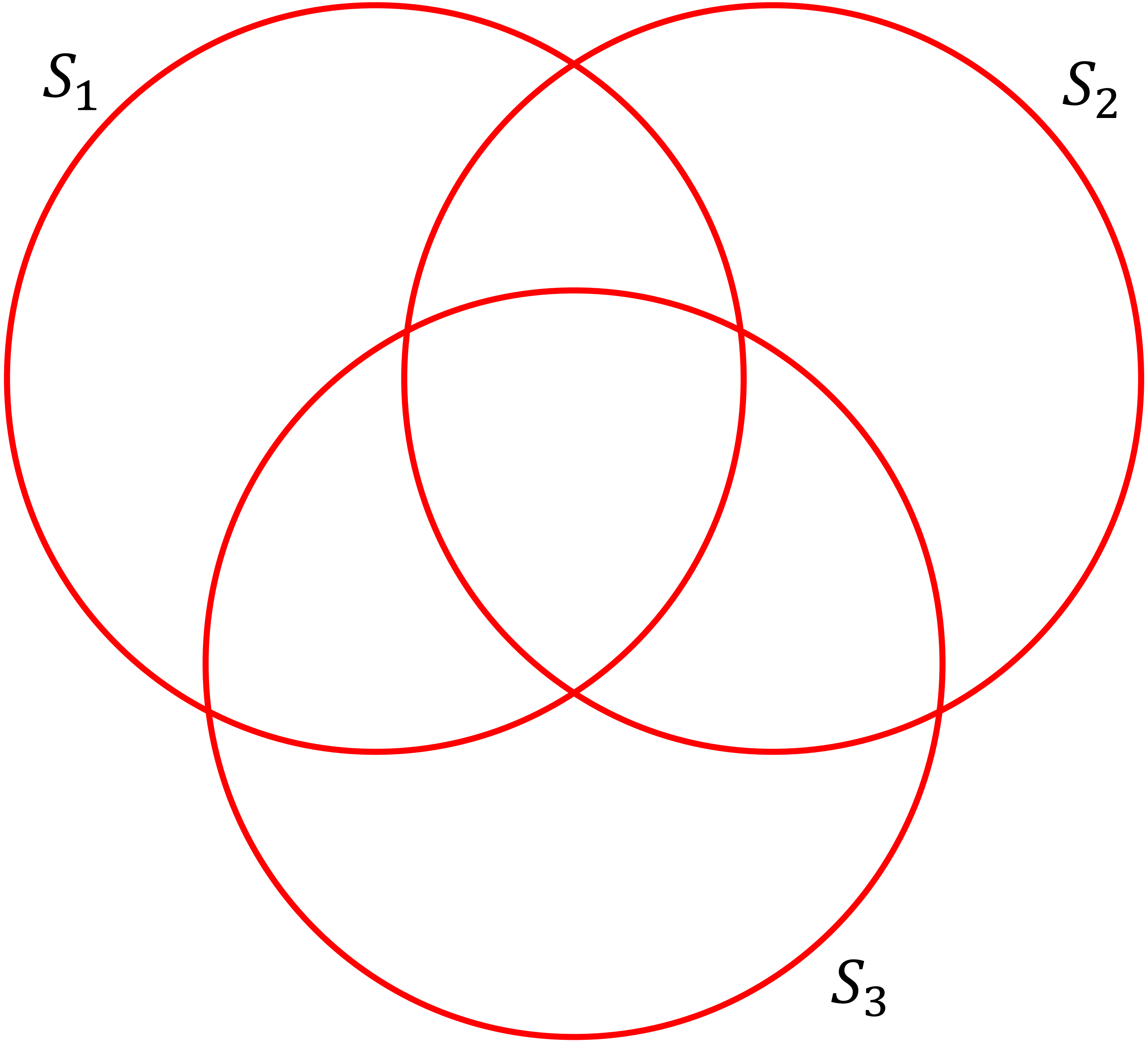
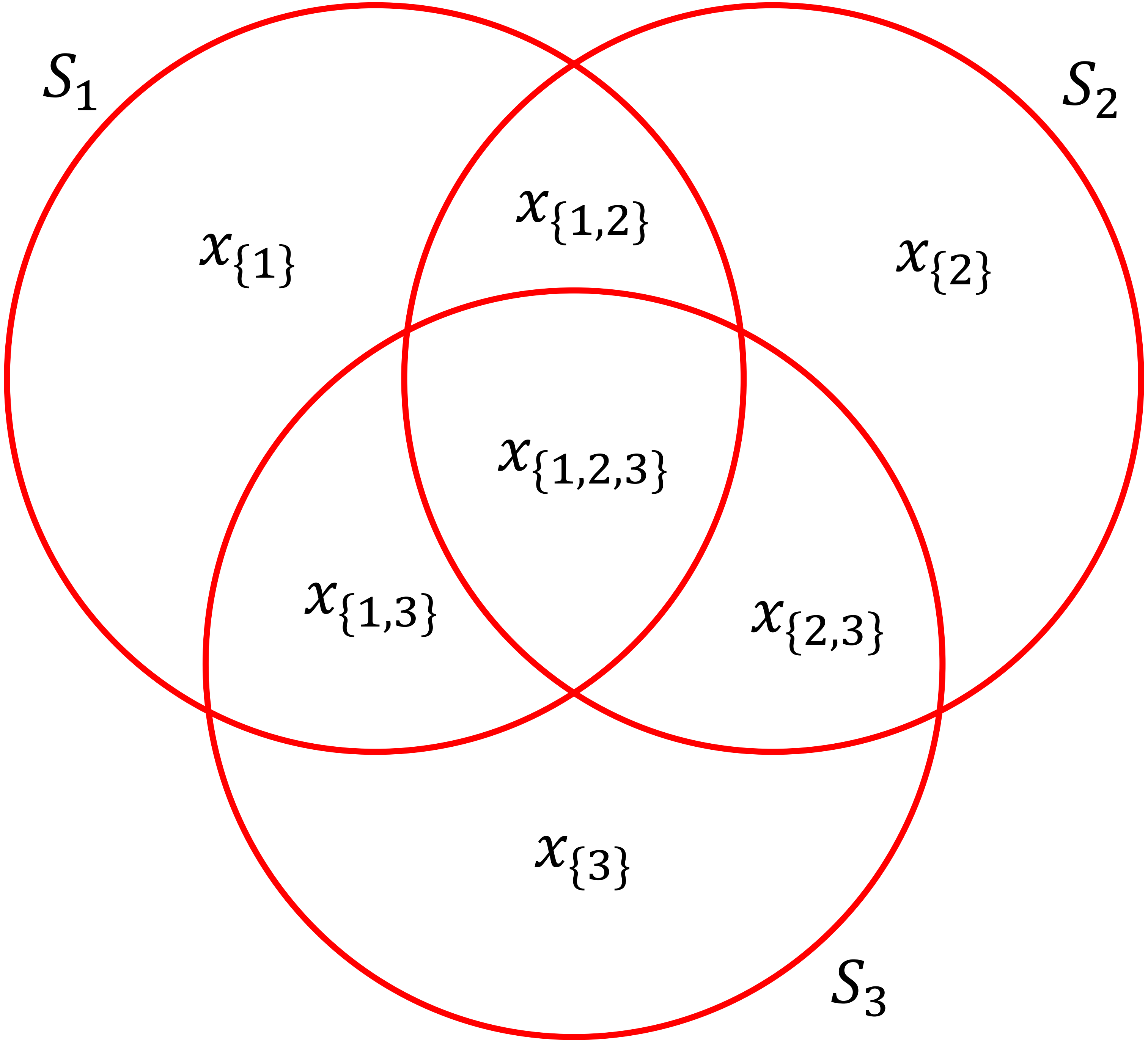
False start. Back to the drawing board.


S_1
S_2
S_3


S_1
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S_3
$x_{\{1\}}$
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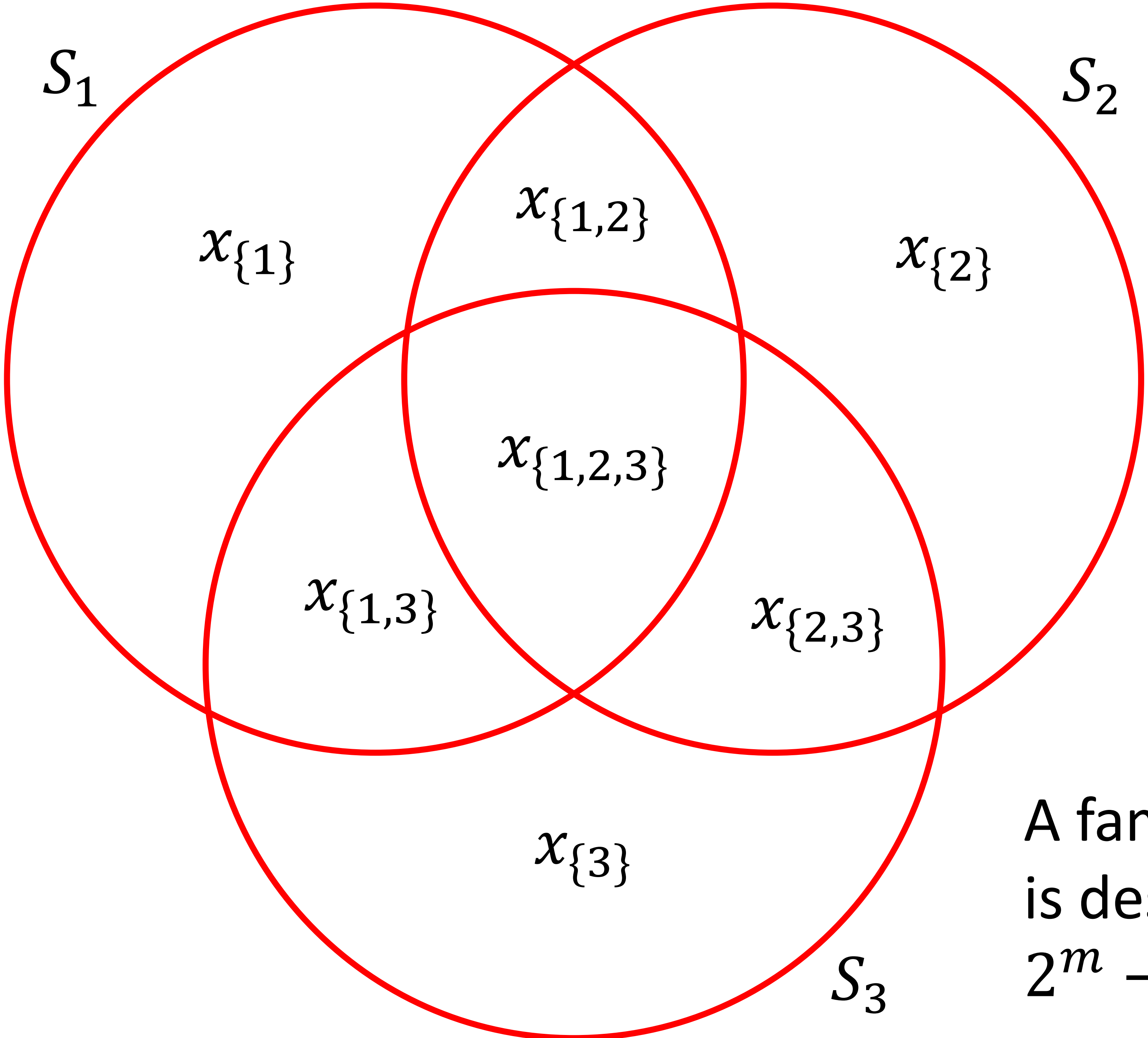
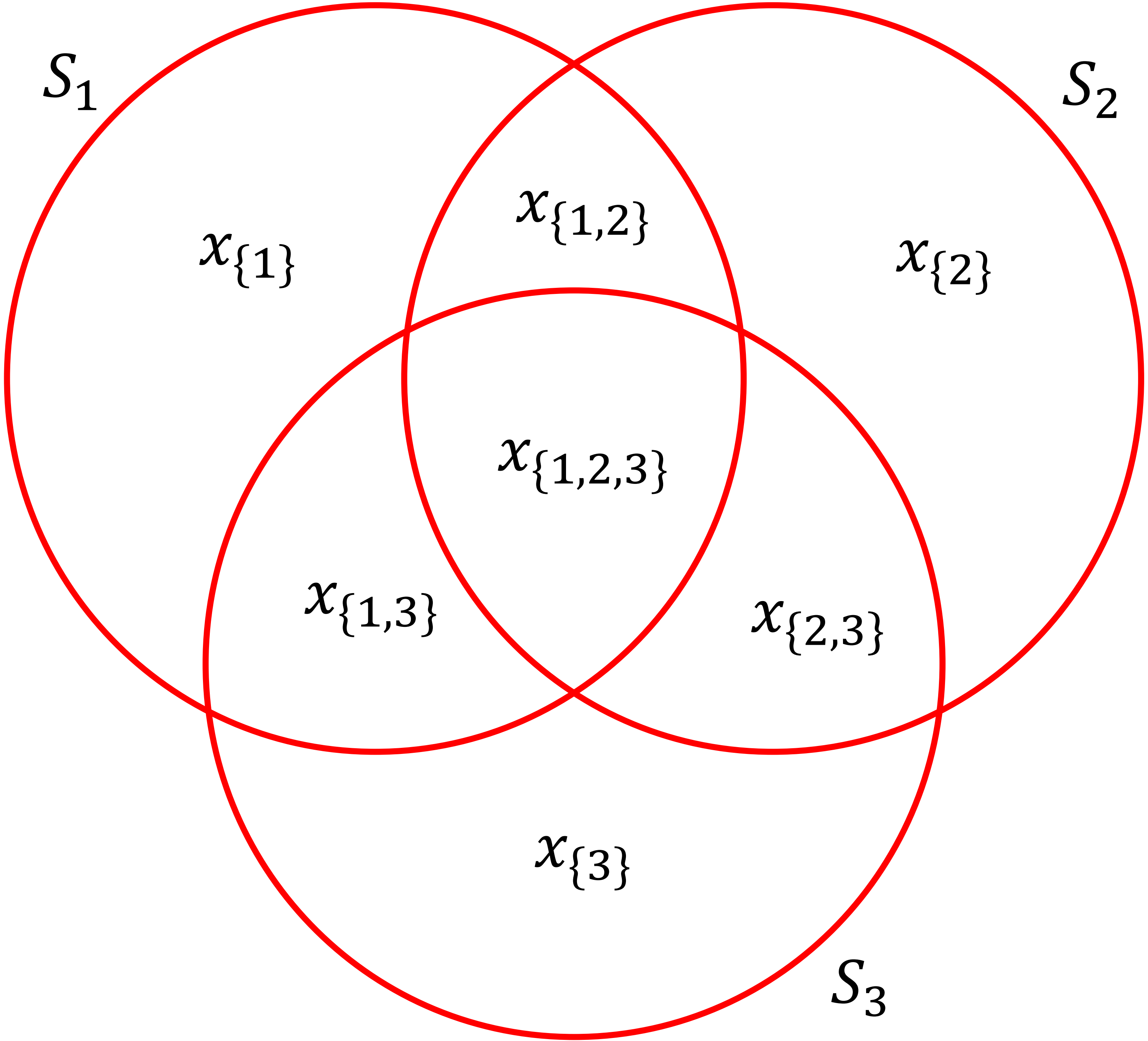
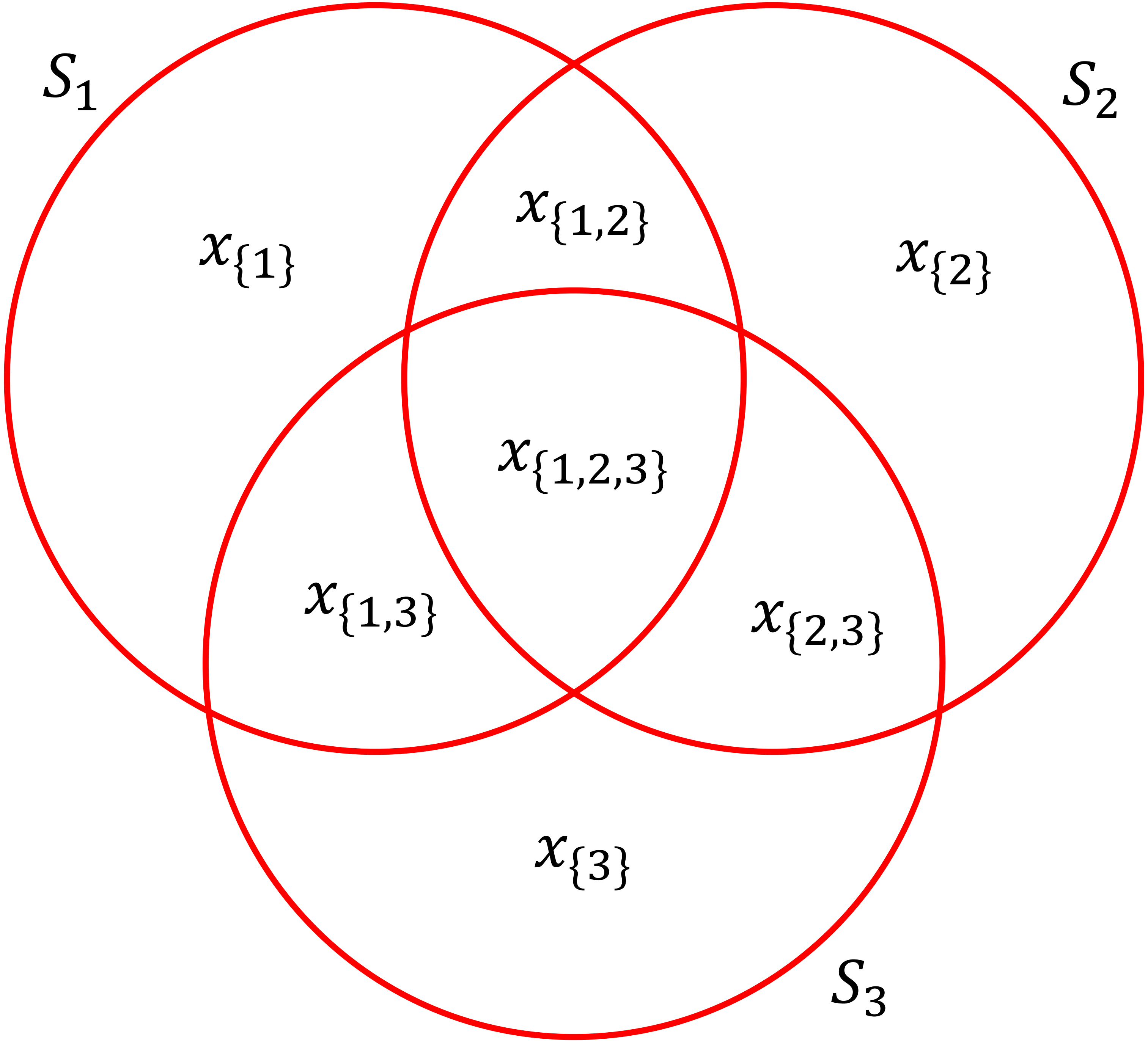
$x_{\{3\}}$



A family of sets $S_1, S_2, \ldots S_m$ is described by the sizes of its $2^m - 1$ *atoms.*



Δ-system
if and only if
$x_{\{1,2\}} = x_{\{1,3\}} = x_{\{2,3\}} = 0$



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Weak Δ-system
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Δ-system iff $x_A = 0$ whenever $1 < |A| < m$

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Weak Δ-system iff $\sum(x_A : A \ni i, j) = \lambda$ whenever $1 \leq i < j \leq m$

Atom sizes x_A describe:

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Its ILP formulation: When $m = k^2 - k + 2$, every integer solution of

$$x_A \ge 0 \text{ for all subsets } A \text{ of } \{1,2,\dots m\},$$

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Its restatement: For every subset B of $\{1,2,\ldots m\}$ such that $1 < |B| < m$, the optimum value of the problem

Maximize x_B subject to

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is 0.

Battle plan: Drop the integrality constraint and solve the resulting LP relaxation. If its optimum value is less than 1, then the optimum value of the ILP problem is zero.

Battle plan: Given a subset B of $\{1,2,\ldots m\}$,

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(and hope that the maximum is less than 1).

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LP duality → want a linear combination of the constraints that reads $\sum c_A x_A \leq d$ with $c_A \geq 0$ for all A and $c_B = 1$ and $d < 1$.

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We have to handle $m + m(m-1)/2$ multipliers.

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Symmetry to the rescue:

multiplier p at each $\sum(x_A : A \ni i) = k$ with $i \in B$,
multiplier q at each $\sum(x_A : A \ni i) = k$ with $i \notin B$,
multiplier r at each $\sum(x_A : A \ni i,j) = \lambda$ with $i \in B,\; j \in B$,
multiplier s at each $\sum(x_A : A \ni i,j) = \lambda$ with $i \notin B,\; j \notin B$,
multiplier t at each $\sum(x_A : A \ni i,j) = \lambda$ with $i \in B,\; j \notin B$.

Revised battle plan: Given a subset B of $\{1,2,\dots m\}$, find

multiplier p at each $\sum(x_A : A \ni i) = k$ with $i \in B$,
multiplier q at each $\sum(x_A : A \ni i) = k$ with $i \notin B$,
multiplier r at each $\sum(x_A : A \ni i, j) = \lambda$ with $i \in B,\ j \in B$, and $i \neq j$,
multiplier s at each $\sum(x_A : A \ni i, j) = \lambda$ with $i \notin B,\ j \notin B$, and $i \neq j$,
multiplier t at each $\sum(x_A : A \ni i, j) = \lambda$ with $i \in B,\ j \notin B$, and $i \neq j$,

such that the resulting linear combination of the constraints reads
$\sum c_A x_A \leq d$ with d minimized subject to $c_A \geq 0$ for all A and $c_B = 1$.

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such that the resulting linear combination of the constraints reads

$\sum c_A x_A \leq d$ with d minimized subject to $c_A \geq 0$ for all A and $c_B = 1$.

Now $c_A = r|A \cap B|^2 + s|A - B|^2$

$$+t|A \cap B||A - B| + (p-r)|A \cap B| + (q-s)|A - B|$$

and $d = \lambda r|B|^2 + \lambda s(m - |B|)^2$

$$+\lambda t|B|(m-|B|) + (pk - r\lambda)|B| + (qk - s\lambda)(m - |B|)$$

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Conclusion: In every weak Δ-system with $m = k^2 - k + 2$, every point belongs to at most $k - \lambda$ sets or to at least $m - (k - \lambda)$ sets.

What we wanted: In every weak Δ-system with $m = k^2 - k + 2$, every point belongs to at most one set or to all m sets.

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Easy exercise: In every weak Δ-system with $m = k^2 - k + 2$, each of the m sets includes at least λ rich points.

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Another easy exercise: In every weak Δ-system with $m = k^2 - k + 2$, there are at most λ rich points.

At least six years before the appearance of Cook's epoch-making paper,
Edmonds discussed the classes P and NP (the latter in terms of an "absolute supervisor").
Where we say today that recognizing pairs (G, k) such that $\alpha(G) \geq k$ is a problem in NP,
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Paper [2] Some linear programming aspects of combinatorics, *Congressus Numerantium* **13** (1975), 2-30 contains the following conjecture, where $c(G)$ stands for the minimum length of a cutting-plane proof of $\alpha(G) \leq k$ from $T(G)$:

<u>CONJECTURE</u>. For every polynomial p there is a graph G with n vertices such that c(G) > p(n) .

This conjecture is somewhat related to the conjecture that there is no good characterization for (5.2); the differences between the two go as follows.

1. It is conceivable that the above conjecture is true and yet there is a good characterization for (5.2). (Necessarily, such a characterization would have to use more powerful inference rules than those based on our cutting planes.)

2. It is conceivable that the above conjecture is false and yet the shortest ILP proofs of $\alpha(G) \leq k$ do not provide a good characterization for (5.2). (Necessarily, these shortest ILP proofs would have to involve excessively large coefficients.)

In 1971, Stephen Cook ("The complexity of theorem proving procedures", *Proceedings of the Third Annual ACM Symposium on Theory of Computing*. pp. 151–158) introduced the notion of NP-complete problems. Two of his examples are

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INPUT: Graph G and positive integer d
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PARTITION
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Exhibiting such pairs explicitly would be paradoxical (you would certify that $\alpha(G) < d$ and at the same time prove that such a certification is hard), but proving their existence is a different matter. In particular, it is tempting to conjecture that, under some probability distribution, almost all pairs (G, d) have the desired properties.

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THEOREM (Pavel Pudlák, "Lower Bounds for Resolution and Cutting Plane Proofs and Monotone Computations", *The Journal of Symbolic Logic* **62** (1997), 981- 998):

For arbitrarily large integers n there are unsatisfiable sets of $O\left(n^{7/6}\right)$ clauses in n variables such that every cutting-plane proof of $0 \geq 1$ from

$$0 \leq x \leq 1 \text{ for all } x,$$

$$\sum(x\colon x \in C) \geq 1 \text{ for all clauses } C$$

has length $\exp(\Omega(n^{1/6}))$.

THANK YOU!