Adaptive Control of a Class of Nonlinear Systems with Nonlinearly Parameterized Fuzzy Approximators

Hugang Han, Chun-Yi Su, and Yury Stepanenko

***Abstract*—Recently, through the use of parameterized fuzzy approximators, various adaptive fuzzy control schemes have been developed to deal with nonlinear systems whose dynamics are poorly understood. An important class of parameterized fuzzy approximators is constructed using radial basis function (RBF) as a membership function. However, some tuneable parameters in RBF appear nonlinearly and the determination of the adaptive law for such parameters is a nontrivial task. In this paper, we propose a new adaptive control method in an effort to tune all the RBFs parameters thereby reducing the approximation error and improving control performance. Global boundedness of the overall adaptive system and tracking to within a desired precision are established with the new adaptive controller. Simulations performed on a simple nonlinear system illustrate the approach.**

***Index Terms*—Adaptive control, fuzzy approximators, global stability, nonlinear parameterization, nonlinear systems.**

I. Introduction

THE APPLICATION of fuzzy set theory to control problems has been the focus of numerous studies [1]. The motivation is often that fuzzy set theory provides an alternative to the traditional modeling and design of control systems where system knowledge and dynamic models in the traditional sense are uncertain and time varying. Despite achieving many practical successes, fuzzy control has not been viewed as a rigorous approach due to the lack of formal synthesis techniques that can guarantee global stability among other basic requirements for control systems. Recently, some research has been directed at the use of the Lyapunov synthesis approach to construct stable adaptive fuzzy controllers [2]–[6]. A key element of this success has been the merger of robust adaptive systems theory with fuzzy approximation theory [7], where the unknown plants are approximated by parameterized fuzzy approximators. In [2], [3], [5], and [6], the parameterized fuzzy approximator is expressed as a series of radial basis functions (RBF) expansion due to its excellent approximation properties [9], [10].

In the RBF expansion, three parameter vectors are used: connection weights, variances, and centers. It is obvious that as these parameters change, the bell-shaped radial functions will vary accordingly, and will exhibit various forms of shapes. This property could be employed to capture the fast-changing system dynamics, reduce approximation error, and improve control performance. However, in recently developed adaptive fuzzy control schemes [2], [3], [5], [6], only connection weights are updated in the RBF expansion, while the variances are fixed and the centers are simply placed on a regular mesh covering a relevant region of system space. This can be attributed to the connection weights appearing linearly, whereas the variances and centers appear nonlinearly in the RBF expansion. Currently, very few results are available for the adjustment of nonlinearly parametrized systems [11]. Though the gradient approaches were used in [12] and [13], the way of fusing them into the adaptive fuzzy control schemes to generate global stability is still an open problem.

In this paper, a new control method is introduced in an effort to tune all parameters in the RBF expansion, thereby improving tracking performance. The approximation error between the plant function and the parameterized fuzzy approximators can be described as a linearly parameterizable form modulo a residual term. Control methods to deal with the residual term and adaptive laws to adjust the nonlinear parameters are then synthesized using a Lyapunov function. It is demonstrated that the proposed fuzzy adaptive controller guarantees the tracking to within a desired precision. Simulations performed on a simple nonlinear system illustrate and clarify the approach.

II. Problem Formulation

This paper focuses on the design of adaptive control algorithms for a class of dynamic systems whose equation of motion can be expressed in the canonical form:

$$x^{(n)}(t)+f\Big(x(t),\,\dot{x}(t),\,\ldots,\,x^{(n-1)}(t)\Big)=b\Big(x(t),\,\dot{x}(t),\,\ldots,\,x^{(n-2)}(t)\Big)\,u(t) \tag{1}$$

where

- $u(t)$ control input;
- f unknown linear or nonlinear function;
- b control gain.

The control objective is to force the state $X=[x,\,\dot{x},\,\ldots,\,x^{(n-1)}]^T$ to follow a specified desired trajectory $X_d=[x_d,\,\dot{x}_d,\,\ldots,\,x_d^{(n-1)}]^T$. Defining the tracking error vector as, $\tilde{X}=X-X_d$, the problem is to design a control law $u(t)$ which ensures that $\tilde{X}\rightarrow 0$, as $t\rightarrow\infty$.

The majority of solutions using adaptive control results have focused on the situation where an explicit linear parameterization of the unknown function $f(X)$ is possible. The parameterization $f(X)$ can be expressed as $f(X)=\sum_{j=1}^{N}\theta_j Y_j(X)$, where θ_j is a set of unknown parameters which appear linearly, and $Y_j(X)$ is a set of known regressors or basis functions. The function $f(X)$ can be approximated as $\hat{f}(X)=$

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$\sum_{j=1}^{N} \hat{\theta}_j Y_j(X)$. As the approximation error only occurs on the parameters $\hat{\theta}_j$, adaptive methods can then be used to adjust the parameters $\hat{\theta}_j$ to achieve the control objective. The challenge addressed in this paper is the development of adaptive controllers when an explicit linear parameterization of the function $f(X)$ is *either unknown or impossible.*

Fuzzy systems through use of fuzzy IF–THEN rules have been proven to have capabilities of nonlinear function approximation [7]. The feasibility of applying fuzzy approximation results to unknown dynamic system control has been demonstrated in many studies [2]–[6]. In this paper, a *universal fuzzy approximator,* as described in [3] and [7], shall be used to approximate the function $f(X)$. Consider a subset of the fuzzy systems with *singleton fuzzifier, product inference,* and *Gaussian membership function.* In this case, such a fuzzy system can be expressed as a series of RBF expansion [3], [7],

$$\begin{aligned} C(X) &= \sum_{j=1}^{N} \omega_j g_j(\sigma_j \|X - \xi_j\|) \\ &= W^T \cdot G(X, \xi, \sigma) \end{aligned} \tag{2}$$

where

$C: U \subset R^n \to R$;
$X = (x_1, x_2, \ldots, x_n) \in U$;
$g_j(\sigma_j \|X - \xi_j\|) = \prod_{i=1}^{n} \exp[-(\sigma_j^i (x_i - \xi_j^i))^2]$;
$\sigma_j = (\sigma_j^1, \sigma_j^2, \ldots, \sigma_j^n) \in U$;
$\xi_j = (\xi_j^1, \xi_j^2, \ldots, \xi_j^n) \in U$;
ξ_j^i and σ_j^i are real-valued parameters;
ω_j are connection weights;
$G(X, \xi, \sigma) = [g_1(\sigma_1 \|X - \xi_1\|), g_2(\sigma_2 \|X - \xi_2\|), \ldots, g_N(\sigma_N \|X - \xi_N\|)]^T$;
$W = [\omega_1, \omega_2, \ldots, \omega_N]^T$, $\xi = [\xi_1, \xi_2, \ldots, \xi_N]^T$, $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_N]^T$, $j = 1, 2, \ldots, N$.

Remark: Contrary to the traditional notation, in this paper, $1/\sigma_j^i$ is used to represent the variance for the convenience of later development.

It has been proven [3] that the RBF expansion (2) satisfies the conditions of the Stone–Weierstrass theorem and is capable of uniformly approximating any real continuous nonlinear function on the compact set $U \subset R^n$. This implies that RBF expansion (2) is a *universal approximator* on a compact set. Since the *universal approximator* (2) is characterized by the parameters W, ξ, and σ, where ξ and σ appear nonlinearly, we thus call it a *nonlinearly parameterized fuzzy approximator.* It should be emphasized that this result is local, since it is valid only on the compact set $U \subset R^n$. We should note that the above membership function could be replaced by other functions such as sigmoidal function [8]. However, it is shown in [9] and [10] that Gaussian basis functions do have the best approximation property. This is the principal reason being the selection of Gaussian functions to characterize the membership functions in this paper.

Assume that the desired trajectories are contained in the compact subset of the state space A_d. Then, on this subset A_d, the unknown function $f(X)$ is reconstructed (expansion (2) provides a fuzzy structure). As shown in [14], an important aspect of the above approximation is that there exist optimal constants W^*, ξ^*, and σ^* such that the function approximation error is bounded by a constant on the set A_d. Let this *optimal fuzzy approximator* be expressed as $f^*(X) = W^{*T} \cdot G(X, \xi^*, \sigma^*)$ and this constant be denoted as ε^*. The result of [14] states that $|f(X) - f^*(X)| \leq \varepsilon^*$ if $X \in A_d$. Since A_d only presents a compact subset of the state space, outside this compact subset A_d the approximation ability for the optimal fuzzy approximator is of great concern in the controller design. Let the approximation error on the entire state space be expressed as $f(X) - f^*(X) = \varepsilon_f(X)$, where $\varepsilon_f(X)$ represents the fuzzy reconstruction error. It is then reasonable to make the following assumption.

Assumption 1:

$$|\varepsilon_f(X)| \leq \varepsilon^* + \alpha(X),\ \alpha(X) = 0 \qquad \text{if } X \in A_d \tag{3}$$

where $\alpha(X)$ represents the approximation error outside the compact subset A_d.

Remarks: The approximation error $\alpha(X)$ for $X \in A_d^c$ is of great importance, since the relationship $|f(X) - f^*(X)| \leq \varepsilon^*$ may not hold outside the compact subset A_d. This fact will significantly influence the controller design strategy, which the following section will present in detail.

To construct $f^*(X)$, the values of the parameters W^*, ξ^*, and σ^* are required. Unfortunately, they are unavailable. Normally, the unknown parameter values W^*, ξ^*, and σ^* are replaced by their estimates $\hat{W}$, $\hat{\xi}$, and $\hat{\sigma}$. Then the estimation function $\hat{f}(X) = \hat{W}^T \cdot G(X, \hat{\xi}, \hat{\sigma})$ is used instead of f^* to approximate the unknown function $f(X)$. The parameters in the estimate $\hat{f}(X)$ should then be stably tuned to provide effective tracking control architecture.

However, in adaptive fuzzy control schemes [2], [3], [5], [6], only connection weights $\hat{W}$ are tuned, whereas $\hat{\sigma}$ is fixed and $\hat{\xi}$ is simply chosen as a regular mesh covering a relevant region of system space. This is because the connection weights $\hat{W}$ appear linearly whereas $\hat{\sigma}$ and $\hat{\xi}$ appear nonlinearly in $\hat{f}(X)$. The main drawback of such designs is that the center and shape of the fuzzy membership function is fixed before controller design. This may lead to large approximation error and degrade the control performance. Since the parameters $\hat{\sigma}$ and $\hat{\xi}$ appear nonlinearly in $\hat{f}$, the determination of the adaptive law for nonlinearly parametrized systems is an involved task. Currently, very few results are available in the literature to address this problem [11]. In [12] and [13] the gradient approach was used. However, the way of fusing this approach with the adaptive fuzzy control schemes to guarantee global stability of the closed-loop systems is still an open problem. If the parameters $\hat{W}$, $\hat{\xi}$, $\hat{\sigma}$ can be stably tuned simultaneously to approach W^*, ξ^*, and σ^*, the approximation error could be reduced and the control performance could thus be improved. This motivates the need for a method that allows a stable estimation of the parameters $\hat{W}$, $\hat{\xi}$, and $\hat{\sigma}$ and simultaneously yields tracking to within a desired precision. This is precisely what is accomplished in this paper. Along the same line, an approach for tuning the parameters $\hat{W}$, $\hat{\xi}$, $\hat{\sigma}$ was also proposed in [11]. The main differences will be stated in the following section.

Since W^*, ξ^*, and σ^* are unknown, the approximation function $f^*(X) = W^{*T} \cdot G(X, \xi^*, \sigma^*)$ cannot be used directly to construct the control law. Using the estimation function $\hat{f}(X) = \hat{W}^T \cdot G(X, \hat{\xi}, \hat{\sigma})$ of f^*, the approximation error between f and

$\hat{f}$ needs to be established. Theorem 1 explains how this is accomplished.

Theorem 1: Define the estimation errors of the parameter vectors as

$$\tilde{W} \equiv W^* - \hat{W}, \quad \tilde{\xi} \equiv \xi^* - \hat{\xi}, \quad \tilde{\sigma} \equiv \sigma^* - \hat{\sigma}. \tag{4}$$

The function approximation error $\varepsilon(X) \equiv f(X) - \hat{f}(X)$ can be expressed as

$$\varepsilon(X) = \tilde{W}^T \cdot \left(\hat{G} - G'_\xi \hat{\xi} - G'_\sigma \hat{\sigma}\right) + \hat{W}^T \cdot \left(G'_\xi \tilde{\xi} + G'_\sigma \tilde{\sigma}\right) + d_f \tag{5}$$

where

$$G'_\xi\left(X, \hat{\xi}, \hat{\sigma}\right) = \left[g'_{\xi 1}\left(\hat{\sigma}_1 \left\|X - \hat{\xi}_1\right\|\right), g'_{\xi 2}\left(\hat{\sigma}_2 \left\|X - \hat{\xi}_2\right\|\right), \ldots, g'_{\xi N}\left(\hat{\sigma}_N \left\|X - \hat{\xi}_N\right\|\right)\right]^T \in R^{N \times (Nn)}$$

and

$$G'_\sigma\left(X, \hat{\xi}, \hat{\sigma}\right) = \left[g'_{\sigma 1}\left(\hat{\sigma}_1 \left\|X - \hat{\xi}_1\right\|\right), g'_{\sigma 2}\left(\hat{\sigma}_2 \left\|X - \hat{\xi}_2\right\|\right), \ldots, g'_{\sigma N}\left(\hat{\sigma}_N \left\|X - \hat{\xi}_N\right\|\right)\right]^T \in R^{N \times (Nn)}$$

are derivatives of $G(X, \xi^*, \sigma^*)$ with respect to ξ^* and σ^* at $(\hat{\xi}, \hat{\sigma})$, respectively. Therein,

$$g'_{\sigma j}\left(\hat{\sigma}_j \left\|X - \hat{\xi}_j\right\|\right) = -2\hat{\sigma}_j \left\|X - \hat{\xi}_j\right\|^2 g_j\left(\hat{\sigma}_j \left\|X - \hat{\xi}_j\right\|\right)$$

$$g'_{\xi j}\left(\hat{\sigma}_j \left\|X - \hat{\xi}_j\right\|\right) = 2\hat{\sigma}_j \left\|X - \hat{\xi}_j\right\| \hat{\sigma}_j^T g_j\left(\hat{\sigma}_j \left\|X - \hat{\xi}_j\right\|\right)$$

$j = 1, 2, \ldots, N$, and d_f is a residual term. Moreover, when $\forall X \in A_d$, d_f satisfies

$$|d_f| < \theta_f^{*T} \cdot Y_f \tag{6}$$

where $\theta_f^* \in R^4$ is an unknown constant vector composed of optimal weight matrices and some bounded constants and $Y_f = [1, \|\hat{W}\|, \|\hat{\xi}\|, \|\hat{\sigma}\|]^T$ is a known function vector.

Proof: Denote $\varepsilon_f(X)$ as an approximation error between f and f^*. In this case the function approximation error $\varepsilon(X) \equiv f(X) - \hat{f}(X)$ can be written as

$$\begin{aligned}\varepsilon(X) &= f(X) - \hat{W}^T \hat{G} \\ &= f^*(X) - \hat{W}^T \hat{G} + \varepsilon_f(X) \\ &= W^{*T} G^* - W^{*T} \hat{G} + W^{*T} \hat{G} - \hat{W}^T \hat{G} + \varepsilon_f(X) \\ &= W^{*T} \tilde{G} + \tilde{W}^T \hat{G} + \varepsilon_f(X) \\ &= W^{*T} \tilde{G} - \hat{W}^T \tilde{G} + \hat{W}^T \tilde{G} + \tilde{W}^T \hat{G} + \varepsilon_f(X) \\ &= \tilde{W}^T \tilde{G} + \hat{W}^T \tilde{G} + \tilde{W}^T \hat{G} + \varepsilon_f(X)\end{aligned} \tag{7}$$

where $\tilde{G} \equiv G(X, \xi^*, \sigma^*) - G(X, \hat{\xi}, \hat{\sigma}) = G^* - \hat{G}$. In order to deal with $\tilde{G}$, the Taylor's series expansion of G^* is taken about $\xi^* = \hat{\xi}$ and $\sigma^* = \hat{\sigma}$. This produces

$$G(X, \xi^*, \sigma^*) = G\left(X, \hat{\xi}, \hat{\sigma}\right) + G'_\xi \cdot \left(\xi^* - \hat{\xi}\right) + G'_\sigma \cdot (\sigma^* - \hat{\sigma}) + o\left(X, \tilde{\xi}, \tilde{\sigma}\right) \tag{8}$$

where $o(\cdot)$ denotes the sum of high-order arguments in a Taylor's serious expansion, and $G'_\xi \in R^{N \times (Nn)}$ and $G'_\sigma \in R^{N \times (Nn)}$ are derivatives of $G(X, \xi^*, \sigma^*)$ with respect to ξ^* and σ^* at $(\hat{\xi}, \hat{\sigma})$. They are expressed as

$$G'_\xi \equiv G'_\xi\left(X, \hat{\xi}, \hat{\sigma}\right) = \left.\frac{\partial G(X, \xi^*, \sigma^*)}{\partial \xi^*}\right|_{\substack{\xi^* = \hat{\xi} \\ \sigma^* = \hat{\sigma}}} \tag{9}$$

$$G'_\sigma \equiv G'_\sigma\left(X, \hat{\xi}, \hat{\sigma}\right) = \left.\frac{\partial G(X, \xi^*, \sigma^*)}{\partial \sigma^*}\right|_{\substack{\xi^* = \hat{\xi} \\ \sigma^* = \hat{\sigma}}}. \tag{10}$$

Equation (8) can then be expressed as

$$\tilde{G} = G'_\xi \tilde{\xi} + G'_\sigma \tilde{\sigma} + o\left(X, \tilde{\xi}, \tilde{\sigma}\right). \tag{11}$$

Using (11), $\varepsilon(X)$ in (7) can be rewritten as

$$\begin{aligned}\varepsilon(X) &= \tilde{W}^T \tilde{G} + \hat{W}^T \tilde{G} + \tilde{W}^T \hat{G} + \varepsilon_f(X) \\ &= \tilde{W}^T \left(G'_\xi \tilde{\xi} + G'_\sigma \tilde{\sigma} + o\left(X, \tilde{\xi}, \tilde{\sigma}\right)\right) \\ &\quad + \hat{W}^T \left(G'_\xi \tilde{\xi} + G'_\sigma \tilde{\sigma} + o\left(X, \tilde{\xi}, \tilde{\sigma}\right)\right) \\ &\quad + \tilde{W}^T \hat{G} + \varepsilon_f(X) \\ &= \tilde{W}^T G'_\xi \left(\xi^* - \hat{\xi}\right) + \tilde{W}^T G'_\sigma (\sigma^* - \hat{\sigma}) + \hat{W}^T G'_\xi \tilde{\xi} \\ &\quad + \hat{W}^T G'_\sigma \tilde{\sigma} + W^{*T} o\left(X, \tilde{\xi}, \tilde{\sigma}\right) + \tilde{W}^T \hat{G} + \varepsilon_f(X) \\ &= \tilde{W}^T \left(\hat{G} - G'_\xi \hat{\xi} - G'_\sigma \hat{\sigma}\right) + \hat{W}^T \left(G'_\xi \tilde{\xi} + G'_\sigma \tilde{\sigma}\right) + d_f\end{aligned} \tag{12}$$

where

$$d_f = \tilde{W}^T \left(G'_\xi \xi^* + G'_\sigma \sigma^*\right) + W^{*T} o\left(X, \tilde{\xi}, \tilde{\sigma}\right) + \varepsilon_f(X). \tag{13}$$

Now let us examine the term d_f. First, using (11), the high-order term $o(X, \tilde{\xi}, \tilde{\sigma})$ is bounded by

$$\begin{aligned}\left\|o\left(X, \tilde{\xi}, \tilde{\sigma}\right)\right\| &= \left\|\tilde{G} - G'_\xi \tilde{\xi} - G'_\sigma \tilde{\sigma}\right\| \\ &\le \left\|\tilde{G}\right\| + \|G'_\xi\| \left\|\tilde{\xi}\right\| + \|G'_\sigma\| \, \|\tilde{\sigma}\| \\ &\le c_1 + c_2 \left\|\tilde{\xi}\right\| + c_3 \|\tilde{\sigma}\|\end{aligned} \tag{14}$$

where c_1, c_2, and c_3 are some bounded constants due to the fact that RBF and its derivative are always bounded by constants (the proof is omitted here to save space). Second, it is obvious that there should exist constants $\overline{W}$, $\overline{\xi}$, and $\overline{\sigma}$ satisfying $\|W^*\| \le \overline{W}$, $\|\xi^*\| \le \overline{\xi}$, and $\|\sigma^*\| \le \overline{\sigma}$. Finally, based on the facts:

$$\begin{aligned}\left\|\tilde{W}\right\| &\le \|W^*\| + \left\|\hat{W}\right\| \le \overline{W} + \left\|\hat{W}\right\| \\ \left\|\tilde{\xi}\right\| &\le \|\xi^*\| + \left\|\hat{\xi}\right\| \le \overline{\xi} + \left\|\hat{\xi}\right\| \\ \|\tilde{\sigma}\| &\le \|\sigma^*\| + \|\hat{\sigma}\| \le \overline{\sigma} + \|\hat{\sigma}\|.\end{aligned} \tag{15}$$

The term $d_f(X)$ can be bounded, on the set A_d, as

$$\begin{aligned}|d_f| &= \left\|\tilde{W}^T G'_\xi \xi^* + \tilde{W}^T G'_\sigma \sigma^* + W^{*T} o\left(X, \tilde{\xi}, \tilde{\sigma}\right) + \varepsilon_f(X)\right\| \\ &\le \left\|\tilde{W}\right\| \|G'_\xi\| \, \|\xi^*\| + \left\|\tilde{W}\right\| \|G'_\sigma\| \, \|\sigma^*\| + \|W^*\| \\ &\quad \cdot \left(c_1 + c_2\left\|\tilde{\xi}\right\| + c_3\|\tilde{\sigma}\|\right) + \varepsilon^* \\ &\le \left(\overline{W} + \left\|\hat{W}\right\|\right) c_2\overline{\xi} + \left(\overline{W} + \left\|\hat{W}\right\|\right) c_3\overline{\sigma} + \overline{W}c_1 \\ &\quad + \overline{W}c_2\left(\overline{\xi} + \left\|\hat{\xi}\right\|\right) + \overline{W}c_3(\overline{\sigma} + \|\hat{\sigma}\|) + \varepsilon^* \\ &= 2c_2\overline{W}\overline{\xi} + 2c_3\overline{W}\overline{\sigma} + c_1\overline{W} + \varepsilon^* + \left(c_2\overline{\xi} + c_3\overline{\sigma}\right)\left\|\hat{W}\right\| \\ &\quad + c_2\overline{W}\left\|\hat{\xi}\right\| + c_3\overline{W}\|\hat{\sigma}\| \\ &= \left[\theta^*_{f1}, \theta^*_{f2}, \theta^*_{f3}, \theta^*_{f4}\right] \cdot \left[1, \left\|\hat{W}\right\|, \left\|\hat{\xi}\right\|, \|\hat{\sigma}\|\right]^T \\ &\equiv \theta_f^{*T} \cdot Y_f \end{aligned} \tag{16}$$

where the fact $|\varepsilon_f(X)| \le \varepsilon^*$, and ε^* is a constant for $X \in A_d$, given in Assumption 1, has been used, and

$$\begin{aligned}\theta^*_{f_1} &= 2c_2\overline{W}\overline{\xi} + 2c_3\overline{W}\overline{\sigma} + c_1\overline{W} + \varepsilon^* \\ \theta^*_{f_2} &= c_2\overline{\xi} + c_3\overline{\sigma} \\ \theta^*_{f_3} &= c_2\overline{W} \\ \theta^*_{f_4} &= c_3\overline{W}.\end{aligned}$$

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Remarks:

1) The novelty of this approach as compared with [3], is that through the first-order Taylor's expansion of $f^*(X)$ near $\hat{\xi}$ and $\hat{\sigma}$, the function approximation error $\varepsilon(X) \equiv f(X) - \hat{f}(X)$ has been expressed in a linearly parameterizable form with respect to $\tilde{W}$, $\tilde{\xi}$, and $\tilde{\sigma}$, which makes the updates of $\hat{W}$, $\hat{\xi}$, and $\hat{\sigma}$ possible. If compared with the approach given in [11], where the Taylor series expansion was also used to deal with nonlinearly parameterized fuzzy approximators, there is an important difference. In [11], the higher-order terms were dealt with by the Mean Value Theorem, whereas in this paper, the error equation (5) is expressed as a linearly parameterizable form modulo a residual term. Moreover, the residual term is bounded by a linear expression with a known function vector. Thus, adaptive control techniques can be applied to deal with this residual term.
2) Most of fuzzy adaptive control approaches in the literature assume that the residual term, $d_f(X)$, is a constant. This is obviously a very restrictive assumption, since the term $d_f(X)$, based on the above result, is not bounded by a constant, even if $X \in A_d$. Therefore, the motivation of Theorem 1 is twofold. First, it gives an expression for the approximation error between f and $\hat{f}$, and the assumption on the constant bound is not imposed in the developed control method. This implies that the applicability of the scheme is greatly broadened when compared with [3]. Second, as will be clarified shortly, it is this expression that makes the adjustment of nonlinear parameterized systems possible. This constitutes the major objective of this paper.
3) It should be noted that the explicit expressions for ξ^*, σ^*, ω^*, θ^*_f are not required since these values will be stably learned through the use of adaptive algorithm developed in the following section.

III. Adaptive Control Using Nonlinearly Parameterized Fuzzy Approximators

To develop the controller, the compact set A_d needs to be set up. In this paper, A_d is defined as an n-dimensional hypercube on which the unknown function $f(X)$ is reconstructed

$$A_d = \{X \mid \|X - X_0\|_{p,\,\pi} \le 1\}. \tag{17}$$

The variable X_0 is a fixed vector in the state space of the plant, and $\|X\|_{p,\,\pi}$ is a weighted p-norm of the form:

$$\|X\|_{p,\,\pi} = \left\{\sum_{i=1}^{n}\left(\frac{|x_i|}{\pi_i}\right)^p\right\}^{1/p}$$

for a set of strictly positive weights $\{\pi_i\}_{i=1}^n$. Based on this definition, as in [3] and [17], we assume that a prior upper bound is known on the magnitude of f for points outside of the set A_d, i.e.

$$|f(X)| \le D(X) \qquad \text{when } X \in A_d^c. \tag{18}$$

We are now ready to develop the control law to achieve the control objective. As in [3] and [15], an error metric is first defined as

$$s(t) = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}(t) \qquad \text{with } \lambda > 0 \tag{19}$$

which can be rewritten as $s(t) = \Lambda^T \tilde{X}(t)$, with $\Lambda^T = [\lambda^{n-1}, (n-1)\lambda^{n-2}, \ldots, 1]$. The equation $s(t) = 0$ defines a time-varying hyperplane in R^n on which the tracking error vector decays exponentially to zero, so that perfect tracking can be asymptotically obtained by maintaining this condition [15]. In this case, the control objective becomes the design of a controller to force $s(t) = 0$. The time derivative of the error metric can be written as

$$\dot{s}(t) = -x_d^{(n)}(t) + \Lambda_V^T \tilde{X}(t) + bu - f(X) \tag{20}$$

where $\Lambda_V^T = [0, \lambda^{n-1}, (n-1)\lambda^{n-2}, \ldots, (n-1)\lambda]$.

Using $\hat{f}(X) = \hat{W}^T \cdot G(X, \hat{\xi}, \hat{\sigma})$, which is an estimate of $f^*(X)$, (20) can then be expressed as

$$\dot{s}(t) = -x_d^{(n)}(t) + \Lambda_V^T \tilde{X}(t) + bu - \hat{f}(X) - \varepsilon(X) \tag{21}$$

where $\varepsilon(X) = f(X) - \hat{f}(X)$ represents the fuzzy reconstruction error. The property of $\varepsilon(X)$ has been given by Theorem 1. Now, our focus is on the error equation (21) and the determination of the adaptive laws for $\hat{W}$, $\hat{\xi}$, and $\hat{\sigma}$ so that all signals remain bounded and $s(t) = 0$.

A. Controller Structure Using Nonlinearly Parameterized Fuzzy Approximator

We now present the main results in two stages for ease of exposition. In the first stage, we take $b = 1$ in (1) and establish the global boundedness of all system signals. This allows us to focus on the structure of the adaptive controller, which stably tunes all parameters in the nonlinearly parameterized fuzzy approximator (2). In the second stage, we extend the results to case of nonunity control gain. This gives a complete solution to the control objective.

Our adaptive control law is now described below:

$$u(t) = -k_d s_\phi(t) + u_{fd}(t) + (1-m(t))u_{fu}(t) + m(t)u_{su}(t) \tag{22}$$

$$u_{fd}(t) = x_d^{(n)}(t) - \Lambda_V^T \tilde{X}(t) \tag{23}$$

$$u_{fu}(t) = \hat{W}^T \cdot G\left(X, \hat{\xi}, \hat{\sigma}\right) - \hat{\theta}_f^T Y_f \operatorname{sat}\left(\frac{s(t)}{\phi}\right) \tag{24}$$

$$u_{su}(t) = -k_{su}(t) \cdot \operatorname{sat}\left(\frac{s(t)}{\phi}\right) \tag{25}$$

$$s_\phi(t) = s(t) - \phi \operatorname{sat}\left(\frac{s(t)}{\phi}\right) \tag{26}$$

$$\dot{\hat{W}} = (1-m(t)) \cdot s_\phi(t) \cdot \Gamma_1 \left(G'_\xi \hat{\xi} + G'_\sigma \hat{\sigma} - \hat{G}\right) \tag{27}$$

$$\dot{\hat{\xi}} = -(1-m(t)) \cdot s_\phi(t) \cdot \Gamma_2 \left(\hat{W}^T G'_\xi\right)^T \tag{28}$$

$$\dot{\hat{\sigma}} = -(1-m(t)) \cdot s_\phi(t) \cdot \Gamma_3 \left(\hat{W}^T G'_\sigma\right)^T \tag{29}$$

$$\dot{\hat{\theta}}_f = (1-m(t)) \cdot |s_\phi(t)| \cdot \Gamma_4 Y_f \tag{30}$$

where $\hat{W}$, $\hat{\xi}$, $\hat{\sigma}$, $\hat{\theta}_f$ are the estimates of ω^*, ξ^*, σ^*, θ_f^*, $G'_\xi(X, \hat{\xi}, \hat{\sigma})$ and $G'_\sigma(X, \hat{\xi}, \hat{\sigma})$ are derivatives of $G(X, \xi^*, \sigma^*)$ with respect to ξ^* and σ^* at $(\hat{\xi}, \hat{\sigma})$ and are given in Theorem 1, $k_{su}(t)$ is the gain satisfying $k_{su}(t) \geq D(X)$, $\operatorname{sat}(\cdot)$ is a saturation function, $\phi(>0)$ is a small constant, $\Gamma_1 \in R^{N\times N}$, $\Gamma_2 \in R^{(Nn)\times(Nn)}$, $\Gamma_3 \in R^{(Nn)\times(Nn)}$, and $\Gamma_4 \in R^{4\times 4}$ are the symmetric positive definite matrices which determine the rates of adaptation. The modulation function $m(t)$ is chosen as follows:

$$m(t) = \begin{cases} 0 & \text{if } X \in A_d \\ \dfrac{\|X - X_0\|_{p,\pi} - 1}{\Psi} & \text{if } X \in A_\Psi - A_d \\ 1 & \text{if } X \in A_\Psi^c \end{cases} \tag{31}$$

where A_Ψ is chosen as $A_\Psi = \{X \mid \|X - X_0\|_{p,\pi} \leq 1 + \Psi\}$ and Ψ is a small positive constant, representing the width of the transition region.

Remarks:

1) The control law (22) consists of three components. The first component is $(-k_d s_\phi(t) + u_{fd}(t))$, representing a negative feedback of the measured tracking error states. The second component $u_{fu}(t)$ represents the adaptive control of the control law, and $u_{su}(t)$ is a sliding-mode component. To switch between the adaptive and sliding-mode modes, a switch operation $m(t)$ is introduced in the control law (22). If the state X is constrained in the set A_d, which implies that the unknown function f can be approximated by the fuzzy IF–THEN rules, the adaptive control u_{fu} behaves approximately like the conventional adaptive controller. When the state X is not constrained in the set A_d, the robust control $u_{su}(t)$ takes over from the adaptive component through the module function $m(t)$ and forces the state back into A_d. The purpose of introducing a new set A_Ψ, containing A_d, in $m(t)$ is to generate a *smooth switching* between the robust and adaptive modes. In this case, the pure adaptive operation is restricted to the interior of the set A_d, whereas the pure robust operation is restricted to the exterior of the set A_Ψ. In between the region $A_\Psi - A_d$, the two modes are effectively blended using a continuous modulation function [i.e., $m(t) = 0$ when $X \in A_d$, $m(t) = 1$ when $X \in A_\Psi^c$, and $0 < m(t) < 1$, otherwise].
2) Compared with the controller given in [3], in addition to adjusting the weighting parameter, the parameters, which appear nonlinearly in the FBF expansion, are also tuned. In this case, the approximation capability for the fuzzy systems to capture the fast changing system dynamics is further enhanced and better control performance can be expected.
3) Compared with the control schemes given in [11] where the Taylor series expansion is also used to deal with nonlinearly parameterized fuzzy approximators, there is an important difference. In [11], the convergence of the tracking error depends on the condition that signal $v(t)$ is squared integrable. However, since the signal $v(t)$ is a combination of the function approximation errors and the parameter approximation errors, this condition may not be easy to verify. In our scheme such a condition is not required and the control scheme guarantees the convergence of the tracking error.

B. Stability Analysis

The stability of the closed-loop system described by (1), (19), and (22)–(31) is established in the following theorem.

Theorem 2: If the robust adaptive control law (22)–(31) is applied to the nonlinear plant (1) with $b = 1$, then all states in the adaptive system will remain bounded, and the tracking errors will be asymptotically bounded by

$$\left|\tilde{x}^{(i)}(t)\right| \leq 2^i \lambda^{i-n+1}\phi, \qquad i = 0, 1, \ldots, n-1.$$

Proof: From (20) and (22), $\dot{s}(t)$ is rewritten as

$$\begin{aligned}\dot{s}(t) &= -k_d s_\phi(t) + (1-m(t)) \cdot u_{fu}(t) + m(t) \cdot u_{su}(t) - f(X) \\ &= -k_d s_\phi(t) + (1-m(t)) \cdot (u_{fu} - f(X)) + m(t)(u_{su} - f(X)).\end{aligned} \tag{32}$$

Using (24) and (5), $u_{fu}(t)$ can be rewritten as

$$\begin{aligned}u_{fu}(t) &= f(X) - \varepsilon(X) - \hat{\theta}_f^T Y_f \operatorname{sat}\left(\frac{s(t)}{\phi}\right) \\ &= f(X) + \tilde{W}^T\left(G'_\xi \hat{\xi} + G'_\sigma \hat{\sigma} - \hat{G}\right) \\ &\quad - \hat{W}^T\left(G'_\xi \tilde{\xi} + G'_\sigma \tilde{\sigma}\right) - d_f \\ &\quad - \hat{\theta}_f^T Y_f \operatorname{sat}\left(\frac{s(t)}{\phi}\right).\end{aligned} \tag{33}$$

Equation (32) can then be expressed as

$$\begin{aligned}\dot{s}(t) =& -k_d s_\phi(t) + (1-m(t)) \\ &\cdot \left\{ \tilde{W}^T \left(G'_\xi \hat{\xi} + G'_\sigma \hat{\sigma} - \hat{G} \right) - \hat{W}^T \left(G'_\xi \tilde{\xi} + G'_\sigma \tilde{\sigma} \right) \right\} \\ &- (1-m(t)) \cdot \left(\hat{\theta}_f^T Y_f \, \mathrm{sat}\left(\frac{s(t)}{\phi} \right) + d_f \right) \\ &+ m(t) \cdot (u_{su} - f(X)). \end{aligned} \tag{34}$$

Consider the Luapunov function candidate:

$$\begin{aligned} V(t) = \tfrac{1}{2} \Big(& s_\phi^2(t) + \tilde{W}^T \Gamma_1^{-1} \tilde{W} + \tilde{\xi}^T \Gamma_2^{-1} \tilde{\xi} \\ & + \tilde{\sigma}^T \Gamma_3^{-1} \tilde{\sigma} + \tilde{\theta}_f^T \Gamma_4^{-1} \tilde{\theta}_f \Big). \end{aligned} \tag{35}$$

Taking the derivative of the both sides of (35), one has

$$\begin{aligned} \dot{V}(t) =& s_\phi(t) \dot{s}_\phi(t) - \tilde{W}^T \Gamma_1^{-1} \dot{\tilde{W}} - \dot{\tilde{\xi}}^T \Gamma_2^{-1} \tilde{\xi} \\ &- \dot{\tilde{\sigma}}^T \Gamma_3^{-1} \tilde{\sigma} - \tilde{\theta}_f^T \Gamma_4^{-1} \dot{\tilde{\theta}}_f \\ =& -k_d s_\phi^2(t) + (1-m(t)) \cdot s_\phi(t) \\ &\cdot \tilde{W}^T \left(G'_\xi \hat{\xi} + G'_\sigma \hat{\sigma} - \hat{G} \right) - (1-m(t)) \\ &\cdot s_\phi(t) \cdot \hat{W}^T G'_\xi \tilde{\xi} - (1-m(t)) \cdot s_\phi(t) \cdot \hat{W}^T G'_\sigma \tilde{\sigma} \\ &- (1-m(t)) \cdot s_\phi(t) \cdot \left(\hat{\theta}_f^T Y_f \, \mathrm{sat}\left(\frac{s(t)}{\phi} \right) + d_f \right) \\ &+ m(t) \cdot s_\phi(t) \cdot (u_{su} - f(X)) - \tilde{W}^T \Gamma_1^{-1} \\ &\cdot \left\{ (1-m(t)) \cdot s_\phi(t) \cdot \Gamma_1 \left(G'_\xi \hat{\xi} + G'_\sigma \hat{\sigma} - \hat{G} \right) \right\} \\ &+ (1-m(t)) \cdot s_\phi(t) \cdot \left(\hat{W}^T G'_\xi \right) \cdot \Gamma_2 \cdot \Gamma_2^{-1} \tilde{\xi} \\ &+ (1-m(t)) \cdot s_\phi(t) \cdot \left(\hat{W}^T G'_\sigma \right) \cdot \Gamma_3 \cdot \Gamma_3^{-1} \tilde{\sigma} \\ &- (1-m(t)) \cdot |s_\phi(t)| \cdot \tilde{\theta}_f^T \Gamma_4^{-1} \cdot \Gamma_4 Y_f \\ =& -k_d s_\phi^2(t) + m(t) s_\phi(t) \cdot (u_{su} - f(X)) \\ &- (1-m(t)) \cdot |s_\phi(t)| \cdot \hat{\theta}_f^T Y_f - (1-m(t)) \cdot s_\phi(t) d_f \\ &- (1-m(t)) \cdot |s_\phi(t)| \cdot \left(\theta_f^* - \hat{\theta}_f \right)^T Y_f \\ \le& -k_d s_\phi^2(t) + m(t)(-k_{su} |s_\phi(t)| + |s_\phi(t)| |f(X)|) \\ &- (1-m(t)) \cdot s_\phi(t) d_f - (1-m(t)) \cdot |s_\phi(t)| \cdot \theta_f^{*T} Y \\ \le& -k_d s_\phi^2(t) - (1-m(t)) \cdot s_\phi(t) \cdot d_f - (1-m(t)) \\ &\cdot |s_\phi(t)| \cdot \theta_f^{*T} Y_f \\ \le& -k_d s_\phi^2(t) \end{aligned} \tag{36}$$

where the facts for $\forall X \in A_d$, $|d_f| < \theta_f^{*T} Y_f$, $\dot{s}_\phi(t) = \dot{s}(t)$, and $s_\phi(t)\,\mathrm{sat}(s(t)\phi) = |s_\phi(t)|$ have been used. Therefore, all signals in the system (1) are bounded. Since $s_\phi(t)$ is uniformly bounded, it is easily shown that, if $\tilde{X}(0)$ is bounded, then $\tilde{X}(t)$ is also bounded for all t, and since $X_d(t)$ is bounded by design, $X(t)$ is as well. To complete the proof and establish asymptotic convergence of the tracking error, it is necessary to show that $s_\phi(t) \to 0$ as $t \to \infty$. This can be accomplished by applying Barbalat's Lemma to the continuous, nonnegative function:

$$\begin{aligned} V_1(t) = V(t) - \int_0^t \left(\dot{V}(\tau) + k_d s_\phi^2(\tau) \right) d\tau \\ \text{with } \dot{V}_1(t) = -k_d s_\phi^2(t). \end{aligned} \tag{37}$$

It can easily be shown that every term on the right-hand side (RHS) of (34) is bounded, hence $\dot{s}_\phi(t)$ is bounded. This implies $\dot{V}_1(t)$ is a uniformly continuous function of time. Since $V_1(t)$ is bounded below by 0, and $\dot{V}_1(t) \le 0$ for all t, use of Barbalat's lemma proves that $\dot{V}_1(t) \to 0$ as $t \to \infty$. This means that the inequality $|s(t)| \le \phi$ is obtained asymptotically and the asymptotic tracking errors can be shown [15] to be asymptotically bounded by

$$\left| \tilde{x}^{(i)}(t) \right| \le 2^i \lambda^{i-n+1} \phi, \qquad i = 0, 1, \ldots, n-1. \quad \nabla\nabla\nabla$$

Remarks:

1) From the above inequality, it is shown how ϕ affects the size of the tracking error. If $\phi \to 0$, then $\tilde{X}(t) \to 0$. In such a case,

$$\mathrm{sat}\left(\frac{s(t)}{\phi} \right)$$

in (24) and (25) becomes $\mathrm{sgn}(s(t))$, which is a typical sliding-mode control law [18]. As a matter of fact, the control law (22)–(31) is just a smoothing realization of the switch function $\mathrm{sgn}(s(t))$. In doing this, chatter is overcome which makes this method more easily implemented in practical situations.

2) Theorem 2 demonstrates that a globally stable adaptive system can be established by tuning the parameters that appear nonlinearly in the system. This was possible principally due to the linear expression of $\tilde{\xi}$ and $\tilde{\sigma}$ in Theorem 1.

C. Extension to Nonunity Gain Case

In the control law (22)–(31), it is required that the control gain $b(x(t), \dot{x}(t), \ldots, x^{(n-2)}(t)) = 1$. The results are now extended to plants with nonunity control gain. As in [16], we make the following general assumptions regarding the control gain b.

Assumption 2:

1) The control gain b is finite, nonzero.
2) The functions $h(X) = f(X)/b(X)$ and $g(X) = 1/b(X)$ are bounded outside the set A_d by known positive functions $M_0(X)$ and $M_1(X)$

$$|h(X)| \le M_0(X), \qquad X \in A_d^c \tag{38}$$

$$|g(X)| \le M_1(X), \qquad X \in A_d^c. \tag{39}$$

3) There exits a known positive function $M_2(X)$, such that

$$\left| \frac{d}{dt} g(X) \right| \le M_2(X) \|X\|. \tag{40}$$

Let us denote $\hat{h}(X) = \hat{W}_h^T \cdot G(X, \hat{\xi}_h, \hat{\sigma}_h)$ and $\hat{g}(X) = \hat{W}_g^T \cdot G(X, \hat{\xi}_g, \hat{\sigma}_g)$ to be the estimates of the optimal fuzzy approximators $h^*(X) = W_h^{*T} \cdot G(X, \xi_h^*, \sigma_h^*)$ and $g^*(X) = W_g^{*T} \cdot G(X, \xi_g^*, \sigma_g^*)$, respectively. Theorem 1 can still be applied to obtain the following approximation error properties:

$$\begin{aligned} \tilde{h} = h - \hat{h} =& \tilde{W}_h^T \cdot \left(\hat{G}_h - G'_{h_\xi} \hat{\xi}_h - G'_{h_\sigma} \hat{\sigma}_h \right) \\ &+ \hat{W}_h^T \cdot \left(G'_{h_\xi} \tilde{\xi}_h + G'_{h_\sigma} \tilde{\sigma}_h \right) + d_h \end{aligned} \tag{41}$$

$$\begin{aligned} \tilde{g} = g - \hat{g} =& \tilde{W}_g^T \cdot \left(\hat{G}_g - G'_{g_\xi} \hat{\xi}_g - G'_{g_\sigma} \hat{\sigma}_g \right) \\ &+ \hat{W}_g^T \cdot \left(G'_{g_\xi} \tilde{\xi}_g + G'_{g_\sigma} \tilde{\sigma}_g \right) + d_g. \end{aligned} \tag{42}$$

Furthermore, $|d_h| < \theta_h^{*T} \cdot Y_h$ and $|d_g| < \theta_g^{*T} \cdot Y_g$ for $\forall X \in A_d$.

With these definitions, the proposed robust adaptive control law for the case of the nonunity gain is

$$u(t) = -k_d s_\phi(t) - \tfrac{1}{2} M_2(X)\|X\| s_\phi(t) + (1-m(t))u_{fu}(t) + m(t)u_{su}(t) \tag{43}$$

$$u_{fu}(t) = \hat{W}_h^T \cdot G\left(X, \hat{\xi}_h, \hat{\sigma}_h\right) + \hat{W}_g^T \cdot G\left(X, \hat{\xi}_g, \hat{\sigma}_g\right) a_r - \left(\hat{\theta}_h^T Y_h + \hat{\theta}_g^T Y_g |a_r|\right) \operatorname{sat}\left(\frac{s(t)}{\phi}\right) \tag{44}$$

$$u_{su}(t) = -k_{su}(t) \cdot \operatorname{sat}\left(\frac{s(t)}{\phi}\right) \tag{45}$$

$$\dot{\hat{W}}_h = (1-m(t)) \cdot s_\phi(t) \cdot \Gamma_{h1}\left(G'_{h\xi}\hat{\xi}_h + G'_{h\sigma}\hat{\sigma}_h - \hat{G}_h\right) \tag{46}$$

$$\dot{\hat{\xi}}_h = -(1-m(t)) \cdot s_\phi(t) \cdot \Gamma_{h2}\left(\hat{W}_h^T G'_{h\xi}\right)^T \tag{47}$$

$$\dot{\hat{\sigma}}_h = -(1-m(t)) \cdot s_\phi(t) \cdot \Gamma_{h3}\left(\hat{W}_h^T G'_{h\sigma}\right)^T \tag{48}$$

$$\dot{\hat{\theta}}_h = (1-m(t)) \cdot |s_\phi(t)| \cdot \Gamma_{h4} Y_h \tag{49}$$

$$\dot{\hat{W}}_g = (1-m(t)) \cdot s_\phi(t) \cdot \Gamma_{g1}\left(G'_{g\xi}\hat{\xi}_g + G'_{g\sigma}\hat{\sigma}_g - \hat{G}_g\right) a_r \tag{50}$$

$$\dot{\hat{\xi}}_g = -(1-m(t)) \cdot s_\phi(t) \cdot \Gamma_{g2}\left(\hat{W}_g^T G'_{g\xi}\right)^T a_r \tag{51}$$

$$\dot{\hat{\sigma}}_g = -(1-m(t)) \cdot s_\phi(t) \cdot \Gamma_{g3}\left(\hat{W}_g^T G'_{g\sigma}\right)^T a_r \tag{52}$$

$$\dot{\hat{\theta}}_g = (1-m(t)) \cdot |s_\phi(t)| \cdot \Gamma_{g4} Y_g |a_r| \tag{53}$$

where $\hat{W}_h, \hat{\xi}_h, \hat{\sigma}_h, \hat{\theta}_h, \hat{W}_g, \hat{\xi}_g, \hat{\sigma}_g, \hat{\theta}_g$ are the estimates of $W_h^*, \xi_h^*, \sigma_h^*, \theta_g^*, W_g^*, \xi_g^*, \sigma_g^*, \theta_g^*$;
$k_{su}(t) = M_0(X) + M_1(X)|a_r|$;
$a_r = x_d^{(n)}(t) - \Lambda_V^T \tilde{X}(t)$ with $\Lambda_V^T = [0, \lambda^{n-1}, (n-1)\lambda^{n-2}, \ldots, (n-1)\lambda]$;
$\Gamma_{h1} \in R^{N\times N}$, $\Gamma_{h2} \in R^{(Nn)\times(Nn)}$, $\Gamma_{h3} \in R^{(Nn)\times(Nn)}$, $\Gamma_{h4} \in R^{4\times 4}$, $\Gamma_{g1} \in R^{N\times N}$, $\Gamma_{g2} \in R^{(Nn)\times(Nn)}$, $\Gamma_{g3} \in R^{(Nn)\times(Nn)}$, and $\Gamma_{g4} \in R^{4\times 4}$ are the symmetric positive definite matrices which determine the rates of adaptation.

The stability of the closed-loop system described by (1), (38)–(40), and (43)–(53) is established in the following theorem.

Theorem 3: For the nonlinear plant (1), under the assumption 2, the robust adaptive control law given in (43)–(53) assures that all states in the adaptive system will remain bounded. Moreover, the tracking errors will be asymptotically bounded by: $|\tilde{x}^{(i)}(t)| \le 2^i \lambda^{i-n+1}\phi$, $i = 0, 1, \ldots, n-1$.

Proof: From (20) and (43), $g(X)\dot{s}(t)$ can be written as

$$\begin{aligned} g(X)\dot{s}(t) &= -h(X) + u(t) - g(X)a_r \\ &= -k_d s_\phi(t) - \tfrac{1}{2}M_2(X)\|X\|s_\phi(t) \\ &\quad + (1-m(t)) \cdot (u_{fu}(t) - h(X) - g(X)a_r) \\ &\quad + m(t) \cdot (u_{su}(t) - h(X) - g(X)a_r). \end{aligned} \tag{54}$$

Using (41) and (42), the term $u_{fu}(t)$ in (44) can be rewritten as

$$\begin{aligned} u_{fu}(t) &= \hat{h}(X) - \hat{\theta}_h^T Y_h \operatorname{sat}\left(\frac{s(t)}{\phi}\right) + \hat{g}(X)a_r \\ &\quad - \hat{\theta}_g^T Y_g |a_r| \operatorname{sat}\left(\frac{s(t)}{\phi}\right) \\ &= h(X) - \tilde{h}(X) - \hat{\theta}_h^T Y_h \operatorname{sat}\left(\frac{s(t)}{\phi}\right) \\ &\quad + (g(X) - \tilde{g}(X))a_r - \hat{\theta}_g^T Y_g |a_r| \operatorname{sat}\left(\frac{s(t)}{\phi}\right) \\ &= h(X) + \tilde{W}_h^T\left(G'_{h\xi}\hat{\xi}_h + G'_{h\sigma}\hat{\sigma}_h - \hat{G}_h\right) \\ &\quad - \hat{W}_h^T\left(G'_{h\xi}\tilde{\xi}_h + G'_{h\sigma}\tilde{\sigma}_h\right) - d_h - \hat{\theta}_h^T Y_h \operatorname{sat}\left(\frac{s(t)}{\phi}\right) \\ &\quad + \Big[g(X) + \tilde{W}_g^T\left(G'_{g\xi}\hat{\xi}_g + G'_{g\sigma}\hat{\sigma}_g - \hat{G}_g\right) \\ &\qquad - \hat{W}_g^T\left(G'_{g\xi}\tilde{\xi}_g + G'_{g\sigma}\tilde{\sigma}_g\right) - d_g\Big] a_r \\ &\quad - \hat{\theta}_g^T Y_g |a_r| \operatorname{sat}\left(\frac{s(t)}{\phi}\right). \end{aligned} \tag{55}$$

Equation (54) can then be expressed as

$$\begin{aligned} g(X)\dot{s}(t) &= -k_d s_\phi(t) - \tfrac{1}{2}M_2(X)\|X\|s_\phi(t) + (1-m(t)) \\ &\quad \cdot \Big[\tilde{W}_h^T\left(G'_{h\xi}\hat{\xi}_h + G'_{h\sigma}\hat{\sigma}_h - \hat{G}_h\right) \\ &\qquad - \hat{W}_h^T\left(G'_{h\xi}\tilde{\xi}_h + G'_{h\sigma}\tilde{\sigma}_h\right)\Big] - (1-m(t)) \\ &\quad \cdot \left[d_h + \hat{\theta}_h^T Y_h \operatorname{sat}\left(\frac{s(t)}{\phi}\right)\right] + (1-m(t)) \\ &\quad \cdot \Big[\tilde{W}_g^T\left(G'_{g\xi}\hat{\xi}_g + G'_{g\sigma}\hat{\sigma}_g - \hat{G}_g\right) \\ &\qquad - \hat{W}_g^T\left(G'_{g\xi}\tilde{\xi}_g + G'_{g\sigma}\tilde{\sigma}_g\right)\Big] a_r - (1-m(t)) \\ &\quad \cdot \left[d_g a_r + \hat{\theta}_g^T Y_g |a_r| \operatorname{sat}\left(\frac{s(t)}{\phi}\right)\right] \\ &\quad + m(t) \cdot (u_{su}(t) - h(X) - g(X)a_r). \end{aligned} \tag{56}$$

Consider the following Luapunov function candidate:

$$\begin{aligned} V(t) &= \tfrac{1}{2}\Big(g(X)s_\phi^2(t) + \tilde{W}_h^T \Gamma_{h1}^{-1}\tilde{W}_h + \tilde{\xi}_h^T \Gamma_{h2}^{-1}\tilde{\xi}_h + \tilde{\sigma}_h^T \Gamma_{h3}^{-1}\tilde{\sigma}_h \\ &\quad + \tilde{\theta}_h^T \Gamma_4^{-1}\tilde{\theta}_h + \tilde{W}_g^T \Gamma_{g1}^{-1}\tilde{W}_g + \tilde{\xi}_g^T \Gamma_{g2}^{-1}\tilde{\xi}_g \\ &\quad + \tilde{\sigma}_g^T \Gamma_{g3}^{-1}\tilde{\sigma}_h + \tilde{\theta}_g^T \Gamma_{g4}^{-1}\tilde{\theta}_g\Big). \end{aligned} \tag{57}$$

The derivative of the both sides of (57) yields

$$\begin{aligned} \dot{V}(t) &= \tfrac{1}{2}\dot{g}(X)s_\phi^2(t) + s_\phi g(X)\dot{s}_\phi(t) - \tilde{W}_h^T \Gamma_{h1}^{-1}\dot{\hat{W}}_h \\ &\quad - \dot{\hat{\xi}}_h^T \Gamma_{h2}^{-1}\tilde{\xi}_h - \dot{\hat{\sigma}}_h^T \Gamma_{h3}^{-1}\tilde{\sigma}_h - \tilde{\theta}_h^T \Gamma_{h4}^{-1}\dot{\hat{\theta}}_h - \tilde{W}_g^T \Gamma_{g1}^{-1}\dot{\hat{W}}_g \\ &\quad - \dot{\hat{\xi}}_g^T \Gamma_{g2}^{-1}\tilde{\xi}_g - \dot{\hat{\sigma}}_g^T \Gamma_{g3}^{-1}\tilde{\sigma}_h - \tilde{\theta}_g^T \Gamma_{g4}^{-1}\dot{\hat{\theta}}_g \\ &= -k_d s_\phi^2(t) + \tfrac{1}{2}(\dot{g}(X) - M_2(X)\|X\|)s_\phi^2(t) \\ &\quad - (1-m(t))|s_\phi(t)|\hat{\theta}_h^T Y_h - (1-m(t))s_\phi(t)d_h \\ &\quad - (1-m(t))|s_\phi(t)|\hat{\theta}_g^T Y_g|a_r| - (1-m(t))s_\phi(t)d_g \\ &\quad \cdot a_r + m(t)(u_{su}(t) - h(X) - g(X)a_r) \\ &\quad - (1-m(t))|s_\phi(t)|\tilde{\theta}_h^T Y_h \\ &\quad - (1-m(t))|s_\phi(t)|\tilde{\theta}_g^T Y_g|a_r| \\ &= -k_d s_\phi^2(t) + \tfrac{1}{2}(\dot{g}(X) - M_2(X)\|X\|)s_\phi^2(t) \\ &\quad - (1-m(t))s_\phi(t)d_h - (1-m(t))|s_\phi(t)|\theta_h^{*T} Y_h \\ &\quad - (1-m(t))s_\phi(t)d_g a_r - (1-m(t))|s_\phi(t)|\theta_g^{*T} Y_g|a_r| \\ &\quad + m(t)(u_{su}(t) - h(X) - g(X)a_r) \\ &< -k_d s_\phi^2(t) \end{aligned} \tag{58}$$

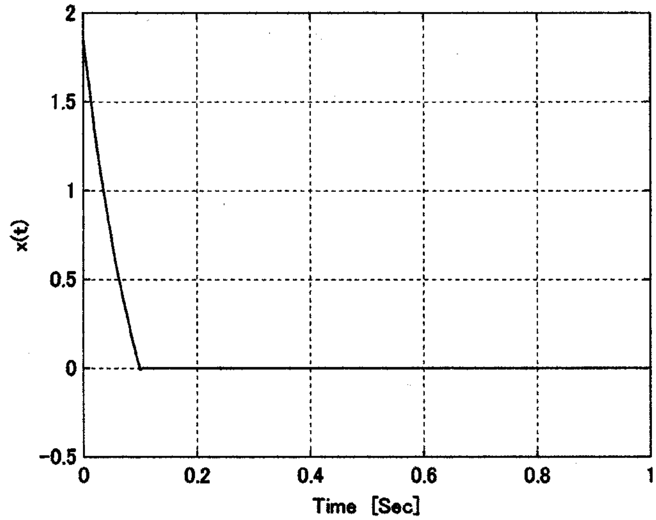

Fig. 1. Closed-loop state $x(t)$ using the controller law (22)–(31).

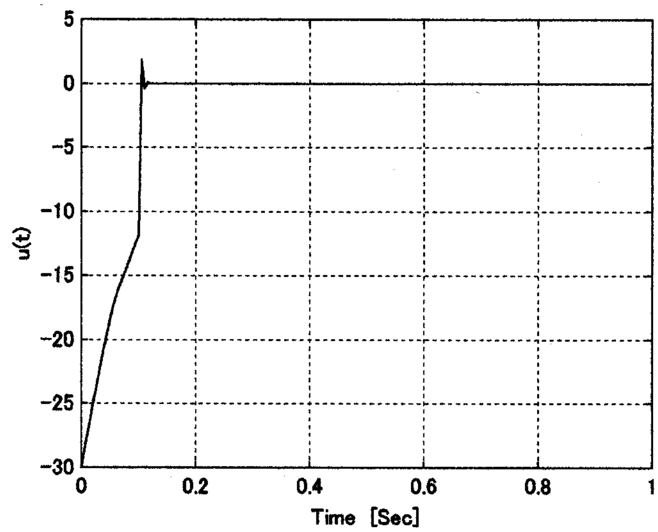

Fig. 2. Control signal $u(t)$ using the controller law (22)–(31).

where the facts $|\dot{g}(X)| \leq M_2(X)\|X\|$, $|d_h| < \theta_h^{*T} Y_h$ and $|d_g| < \theta_g^{*T} Y_g$ for $\forall X \in A_d$, $s_\phi(t)\,\mathrm{sat}(s(t)/\phi) = |s_\phi(t)|$ and (13) have been used. This implies that the proof is completed via the same argument given in the proof of Theorem 2. $\nabla\nabla\nabla$

IV. Simulation Example

To illustrate and clarify the proposed design procedure, we apply the adaptive fuzzy controller developed in Section III to control a nonlinear system as used in [2] and [3]

$$\dot{x}(t) = \frac{1 - e^{-x(t)}}{1 + e^{-x(t)}} + u(t). \tag{59}$$

The control objective is to force the system state $x(t)$ to the origin; i.e., $x_d = 0$. The set A_d was chosen to be $[-3, 3]$ interval with respect to the weighted infinity norm $\|X\|_{\infty,\pi} = |x|/3$. A thin transition region between the adaptive and sliding operation was chosen to a value of $\Psi = 0.05$ so that $A_\Psi = \{x \| \|X\|_{\infty,\pi} \leq 1.05\}$. The simulation is conducted with the following linguistic descriptions:

$$R_j\text{: IF } x \text{ is near } k, \quad \text{THEN } f \text{ is near } B_k$$

where k, $k = -3, -2, -1, 0, 1, 2, 3$, is a fuzzy set with membership functions $\mu_k(x) = \exp(-(x-k)^2)$. B_k are obtained by evaluating f at points $x = -3, -2, -1, 0, 1, 2, 3$. The values of B_k are not required here since the exact W^*, ξ^*, and σ^* are not required in the control law. However, the knowledge of B_k will be helpful in the choice of initial $\hat{W}(0)$, $\hat{\xi}(0)$, $\hat{\sigma}(0)$, and $\hat{\theta}_f(0)$ to speed up the adaptation process. In this example, these initial values $\hat{W}(0)$, $\hat{\xi}(0)$, $\hat{\sigma}(0)$, $\hat{\theta}_f(0)$, are selected as

$$\begin{aligned}
\hat{W}(0) &= [-0.8,\ -0.6,\ -0.4,\ 0,\ 0.4,\ 0.6,\ 0.8]^T \\
\hat{\xi}(0) &= [-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3]^T \\
\hat{\sigma}(0) &= [2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2]^T \\
\hat{\theta}_f(0) &= [4,\ 1,\ 1,\ 1]^T.
\end{aligned}$$

Control law (22) was used where $k_d = 10$. The component $u_{fu}(t)$ is synthesized by (24) where $\Gamma_1 = 0.2$, $\Gamma_2 = 0.2$, $\Gamma_3 = 0.3$, and $\Gamma_4 = 0.01$. Since the nonlinearity f is uniformly upper bounded on A_d^C, satisfying

$$D(X) = 1 \geq \frac{1 - e^{-x(t)}}{1 + e^{-x(t)}}$$

the gain $k_{su}(t) = 1$ is used in the sliding controller in (25), where boundary $\phi = 0.05$ is introduced to avoid the control chatter. The initial state is chosen as $x(0) = 2$.

The result of simulation is shown in Fig. 1, where the evolution of $x(t)$ is presented. A drastic improvement on the system performance is observed, if compared with the result in [3]. Therefore, tuning all the parameters in FBF expansion clearly results in a superior tracking performance. The amount of control effort required to achieve the above level of performance is illustrated in Fig. 2, which confirms the smoothness of the control signal.

In addition, the final tuned $\hat{W}$, $\hat{\xi}$, and $\hat{\sigma}$ become as

$$\begin{aligned}
\hat{W} &= [-0.8,\ -0.6,\ -0.39,\ -0.04,\ 0.51,\ 0.8,\ 0.60]^T \\
\hat{\xi} &= [-3,\ -2\ -0.99\ -0.01\ 0.39\ 0.81\ 2.11]^T \\
\hat{\sigma} &= [2,\ 2,\ 1.98,\ 2.02,\ 2.56,\ 4.32,\ 2.5]^T.
\end{aligned}$$

Comparing with the initial values, some of the parameters in the FBF's expansion have been changed. The reason is that since the state x has a positive initial value $x(0) = 2$ and speedily converged to the origin $x_d = 0$, the parameters regarding the positive region are tuned and the others regarding the negative region are not tuned.

V. Conclusion

We have presented in this paper a new fuzzy adaptive control law that is capable of stably tuning the parameters, which appear nonlinearly in the fuzzy approximators in an effort to reduce approximation error and improve control performance. The developed controller guarantees the global stability of the resulting closed-loop system in the sense that all signals involved are uniformly bounded and tracked to within a desired precision. Simulation results verified the theoretical analysis.

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