Adaptive Variable Structure Set-Point Control of Underactuated Robots

Chun-Yi Su and Yury Stepanenko

***Abstract*—Control of underactuated mechanical systems (robots) represents an important class of control problem. In this correspondence, a model-based adaptive variable structure control scheme is introduced, where the uncertainty bounds only depend on the inertia parameters of the system. Global asymptotic stability is established in the Lyapunov sense. Numerical simulations are conducted to validate the theoretical analysis.**

***Index Terms*—Adaptive control, underactuated robots, variable structure control.**

I. Introduction

In recent years, the control of underactuated mechanical systems has attracted growing attention and is a topic of great interest [1]–[8], [10]–[12], [16]. Examples of such systems are illustrated, for example, in [10]. Interest in studying the underactuated mechanical systems is motivated by their role as a class of strongly nonlinear systems where complex internal dynamics, nonholonomic behavior, and lack of feedback linearizability are often exhibited [11], [15], for which traditional nonlinear control methods are insufficient and new approaches must be developed.

Although dynamics of the underactuated mechanical systems is well understood, the difficulty of the control problem for underactuated mechanisms is obviously due to the reduced dimension of the input space. The literature on the control of underactuated systems is mainly recent [1]–[8], [10]–[12], [17], and the discussion mainly focuses on two-degree-of-freedom examples [3], [4], [6], [11], [15]. Earlier work that deals with control of underactuated robotic systems is described in [1]. Seto and Baillieul [10] developed a general backstepping control method for the system with a chain structure. Underactuated mechanical systems have also been investigated from a nonholonomic constraint point of view [2], [8], [12], [15], [17], where, for instance, Oriolo and Nakamura [8] and Wichlund *et al.* [17] established the conditions for partial integrability of second-order nonholonomic constraints and discussed control problems.

While some interesting techniques and results have been presented in the above-mentioned publications, the control of such systems still remains an open problem. For example, most of the control schemes mentioned above either failed to provide a thorough analysis of the overall system stability or assumed that gravitation forces do not act on the passive joints. Furthermore, the precise knowledge of the dynamic model is generally required except in [5] and [12]. However, the schemes in [5] and [12] suffer their own drawbacks. For example, the scheme in [5] needs inverse matrix calculations, and the scheme in [12] can only stabilize the system to a manifold.

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In this correspondence, a robust nonlinear control law is derived for underactuated robots based on the variable structure theory [16]. More specifically, the results given in [13] and [14] are extended to the underactuated case. The proposed scheme keeps the advantages of [12] and [13]: the uncertainty bounds, needed to design the control law and to prove the global asymptotic stability, depend only on the inertia parameters. As a result, precise bounds on the uncertainty can easily be computed. Stability analysis shows that the closed-loop system is global asymptotic stable.

II. Underactuated Robot Dynamics

The dynamic model of a mechanical system (robot) can be written as

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + B(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q}) = T\boldsymbol{u} \tag{1}$$

where $\boldsymbol{q} \in R^n$ is the generalized coordinates (joint positions); $D(\boldsymbol{q}) \in R^{n\times n}$ is the inertia matrix; vector $B(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in R^n$ presents the centripetal, Coriolis forces; $G(\boldsymbol{q})$ represents the gravitational forces; T is an input transformation matrix; and $\boldsymbol{u} \in R^n$ is the vector of applied joint torques. The dynamic equation has a structure property: the matrix $[\dot{M}(\boldsymbol{q}) - 2B(\boldsymbol{q}, \dot{\boldsymbol{q}})]$ is skew symmetric with a suitable definition of $B(\boldsymbol{q}, \dot{\boldsymbol{q}})$.

If only m joints are equipped with actuators, vector $\boldsymbol{q}$ can be partitioned without loss of generality as $(\boldsymbol{q}_a, \boldsymbol{q}_u)$, where $\boldsymbol{q}_a \in R^m$ represents the actuated joints, while $\boldsymbol{q}_u \in R^{(n-m)}$ represents the unactuated ones. The dynamic model (1) is then written as

$$\begin{bmatrix} M_{aa}(\boldsymbol{q}) & M_{au}(\boldsymbol{q}) \\ M_{ua}(\boldsymbol{q}) & M_{uu}(\boldsymbol{q}) \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_a \\ \ddot{\boldsymbol{q}}_u \end{bmatrix} + \begin{bmatrix} b_{aa}(\boldsymbol{q}, \dot{\boldsymbol{q}}) & b_{au}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ b_{ua}(\boldsymbol{q}, \dot{\boldsymbol{q}}) & b_{uu}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}_a \\ \dot{\boldsymbol{q}}_u \end{bmatrix} + \begin{bmatrix} \boldsymbol{G}_a(\boldsymbol{q}) \\ \boldsymbol{G}_u(\boldsymbol{q}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_a \\ 0 \end{bmatrix} \tag{2}$$

and, in particular, the dynamic equation relative to the unactuated joints is

$$[M_{ua}(\boldsymbol{q}) \quad M_{uu}(\boldsymbol{q})] \begin{bmatrix} \ddot{\boldsymbol{q}}_a \\ \ddot{\boldsymbol{q}}_u \end{bmatrix} + [b_{ua}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \quad b_{uu}(\boldsymbol{q}, \dot{\boldsymbol{q}})] \begin{bmatrix} \dot{\boldsymbol{q}}_a \\ \dot{\boldsymbol{q}}_u \end{bmatrix} + \boldsymbol{G}_u(\boldsymbol{q}) = 0. \tag{3}$$

We note that the structure property still holds for (2).

From (2) we can see that the robot system has n generalized coordinates, but only m control inputs. No input term explicitly appears in (3), which may thus be interpreted as an $(n - m)$-dimensional constraints involving generalized coordinates as well as their first and second time derivatives.

III. Controller Design

The control objective can be specified as: given desired $\boldsymbol{q}_d$, $\dot{\boldsymbol{q}}_d$, and $\ddot{\boldsymbol{q}}_d$, which are assumed to be bounded and should satisfy the constraint equation (3), determine a control law for $\boldsymbol{u}_a$ such that $\boldsymbol{q}$ asymptotically converge to $\boldsymbol{q}_d$. However, for given $\boldsymbol{q}_d$, the constraint equation (3) may not be verifiable since the parameters are unknown. In this correspondence, we will only focus our attention on the set-point control (global regulation) problem (i.e., $\dot{\boldsymbol{q}}_d = 0$).

To achieve the above regulation objective, we denote $\boldsymbol{q}_d^T = [\boldsymbol{q}_{ad}^T \quad \boldsymbol{q}_{ud}^T]^T$, and define

$$\boldsymbol{e} = \begin{bmatrix} \boldsymbol{e}_a \\ \boldsymbol{e}_u \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_a - \boldsymbol{q}_{ad} \\ \boldsymbol{q}_u - \boldsymbol{q}_{ud} \end{bmatrix} \tag{4}$$

$$\dot{\boldsymbol{q}}_r = \begin{bmatrix} \dot{\boldsymbol{q}}_{ar} \\ \dot{\boldsymbol{q}}_{ur} \end{bmatrix} = \begin{bmatrix} -\lambda_a \boldsymbol{e}_a \\ -\lambda_u \boldsymbol{e}_u \end{bmatrix} \tag{5}$$

$$\boldsymbol{s} = \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_r = \begin{bmatrix} \dot{\boldsymbol{q}}_a - \dot{\boldsymbol{q}}_{ar} \\ \dot{\boldsymbol{q}}_u - \dot{\boldsymbol{q}}_{ur} \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_a \\ \boldsymbol{s}_u \end{bmatrix} \tag{6}$$

where $\lambda_a > 0$ and $\lambda_u > 0$ are design parameters. Then, using (2), the dynamics in terms of the newly defined signals $\boldsymbol{s}_a$ and $\boldsymbol{s}_u$ can be derived as

$$\begin{bmatrix} M_{aa}(\boldsymbol{q}) & M_{au}(\boldsymbol{q}) \\ M_{ua}(\boldsymbol{q}) & M_{uu}(\boldsymbol{q}) \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{s}}_a \\ \dot{\boldsymbol{s}}_u \end{bmatrix} + \begin{bmatrix} b_{aa}(\boldsymbol{q}, \dot{\boldsymbol{q}}) & b_{au}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ b_{ua}(\boldsymbol{q}, \dot{\boldsymbol{q}}) & b_{uu}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_a \\ \boldsymbol{s}_u \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_a + \boldsymbol{u}_r \\ \boldsymbol{u}_f \end{bmatrix} \tag{7}$$

where

$$\begin{aligned} \boldsymbol{u}_r &= -M_{aa}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{ar} - M_{ua}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{ur} - b_{aa}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_{ar} \\ &\quad - b_{au}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_{ur} - G_a(\boldsymbol{q}) \\ \boldsymbol{u}_f &= -M_{ua}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{ar} - M_{uu}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{ur} - b_{ua}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_{ar} \\ &\quad - b_{uu}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_{ur} - G_u(\boldsymbol{q}). \end{aligned}$$

Based on the well-known linear-in-parameter property, the nonlinear terms, such as $M_{aa}(\boldsymbol{q})$, $b_{au}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, and $G_a(\boldsymbol{q})$, can be expressed as a product of a *known* matrix and an unknown parameter vector. Therefore, one has

$$\begin{aligned} \boldsymbol{u}_r &= -M_{aa}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{ar} - M_{ua}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{ur} - b_{aa}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_{ar} \\ &\quad - b_{au}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_{ur} - G_a(\boldsymbol{q}) \\ &= \Phi_r(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}_{ar}, \ddot{\boldsymbol{q}}_{ur}, \dot{\boldsymbol{q}}_{ar}, \dot{\boldsymbol{q}}_{ur})\alpha_r \end{aligned} \tag{8}$$

$$\begin{aligned} \boldsymbol{u}_f &= -M_{ua}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{ar} - M_{uu}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{ur} - b_{ua}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_{ar} \\ &\quad - b_{uu}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_{ur} - G_u(\boldsymbol{q}) \\ &= \Phi_f(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}_{ar}, \ddot{\boldsymbol{q}}_{ur}, \dot{\boldsymbol{q}}_{ar}, \dot{\boldsymbol{q}}_{ur})\alpha_f \end{aligned} \tag{9}$$

where $\alpha_r \in R^{p1}$ and $\alpha_f \in R^{p2}$ are constant vectors of inertia parameters; $\Phi_r \in R^{m \times p1}$ and $\Phi_f \in R^{(n-m)\times p2}$ are matrices of known functions of the generalized coordinates and their velocities, known as the regressors.

Then, (7) can be written as

$$M(\boldsymbol{q})\dot{\boldsymbol{s}} + B(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{s} = \begin{bmatrix} \boldsymbol{u}_a + \Phi_r\alpha_r \\ \Phi_f\alpha_f \end{bmatrix}. \tag{10}$$

We supposed only that the parameter vectors α_r and α_f are uncertain, by which we mean that there exist $\rho_r \in R_+$ and $\rho_f \in R_+$, both known, such that

$$\|\alpha_r\| \le \rho_r; \qquad \|\alpha_f\| \le \rho_f \tag{11}$$

where the norm of vector x, throughout, is defined as $\|x\| = \sum_{i=1}^n |x_i|$, and that of matrix A is defined as the corresponding induced norm.

With the above in mind, a regressor based adaptive variable structure set-point control law is defined as

$$\boldsymbol{u}_a = -K_a\boldsymbol{s}_a - \rho_r\Phi_r\left[\operatorname{sgn}\left(\boldsymbol{s}_a^T\Phi_r\right)\right]^T - \boldsymbol{u}_c \tag{12}$$

$$\boldsymbol{u}_c = \frac{(1+k)\boldsymbol{s}_a}{\|\boldsymbol{s}_a\|^2 + \delta}\left[\rho_f\left\|\boldsymbol{s}_u^T\Phi_f\right\| + \boldsymbol{s}_u^T K_u \boldsymbol{s}_u\right] \tag{13}$$

$$\dot{k} = \begin{cases} \dfrac{\eta}{k}\left(\dfrac{k\|\boldsymbol{s}_a\|^2 - \delta_1}{\|\boldsymbol{s}_a\|^2 + \delta}\right) \left[\rho_f\left\|\boldsymbol{s}_u^T\Phi_f\right\| + \boldsymbol{s}_u^T K_u\boldsymbol{s}_u\right], & \text{if } k \ne 0 \\ \delta, & \text{if } k = 0 \end{cases} \tag{14}$$

where

$K_a \in R^{m\times m}$ and $K_u \in R^{(n-m)\times(n-m)}$ are positive definite (diagonal) matrices; $\delta > 0$ and $\delta_1 > 0$ are small constants, satisfying $\delta < \delta_1$; and $\eta > 0$ is a constant, determining the rate of the adaptations.

The system formed by (2) and (12)–(14) is discontinuous. In such a case, the solution concept for the closed-loop system is in the sense of Filippov [18].

It is important to point out that the direct application of discontinuous control in mechanical systems is almost always impractical since the effects of switching forces on the actuators and gear trains can be destructive. Thus, in real systems, the control discontinuity is smoothed [19] so that the system trajectory moves to a neighborhood of the approximate discontinuity. The study of the idealized discontinuous control scheme, however, gives a clear picture of the salient properties of the system dynamics.

The following theorem can then be stated.

Theorem 1: If the variable structure control law given by (12)–(14) is applied to the underactuated robots (2), then in the closed-loop system, $\lim_{t\to\infty}\boldsymbol{q}(t) = \boldsymbol{q}_d$.

Proof: By denoting $K_s = \operatorname{diag}(K_a\ K_u)$, and adding a term $K_s\boldsymbol{s}$ to both sides of (10), one has

$$M(\boldsymbol{q})\dot{\boldsymbol{s}} + B(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{s} + K_s\boldsymbol{s} = \begin{bmatrix} \boldsymbol{u}_a + \Phi_r\alpha_r + K_a\boldsymbol{s}_a \\ \Phi_f\alpha_f + K_u\boldsymbol{s}_u \end{bmatrix}. \tag{15}$$

Let us consider the positive function

$$V = \tfrac{1}{2}\boldsymbol{s}^T M(\boldsymbol{q})\boldsymbol{s} + \tfrac{1}{2}k^2/\eta. \tag{16}$$

A simple calculation shows that along solutions of (15)

$$\begin{aligned} \dot{V} &= \boldsymbol{s}^T\left(\tfrac{1}{2}\dot{M} - B\right)\boldsymbol{s} \\ &\quad + \boldsymbol{s}^T\left(-K_s\boldsymbol{s} + \begin{bmatrix} \boldsymbol{u}_a + \Phi_r\alpha_r + K_a\boldsymbol{s}_a \\ \Phi_f\alpha_f + K_u\boldsymbol{s}_u \end{bmatrix}\right) + k\dot{k}/\eta \\ &= -\boldsymbol{s}^T K_s\boldsymbol{s} + k\dot{k}/\eta + \begin{bmatrix} \boldsymbol{s}_a^T & \boldsymbol{s}_u^T \end{bmatrix} \\ &\quad \times \begin{bmatrix} \Phi_r\alpha_r - \rho_r\Phi_r\left[\operatorname{sgn}\left(\boldsymbol{s}_a^T\Phi_r\right)\right]^T - \boldsymbol{u}_c \\ \Phi_f\alpha_f + K_u\boldsymbol{s}_u \end{bmatrix} + k\dot{k}/\eta \\ &\le -\boldsymbol{s}^T K_s\boldsymbol{s} + \|\alpha_r\|\left\|\boldsymbol{s}_a^T\Phi_r\right\| - \rho_r\left\|\boldsymbol{s}_a^T\Phi_r\right\| - \boldsymbol{s}_a^T\boldsymbol{u}_c \\ &\quad + \boldsymbol{s}_u^T\Phi_f\alpha_f + \boldsymbol{s}_u^T K_u\boldsymbol{s}_u + k\dot{k}/\eta \\ &\le -\boldsymbol{s}^T K_s\boldsymbol{s} - \boldsymbol{s}_a^T\boldsymbol{u}_c + \|\alpha_f\|\left\|\boldsymbol{s}_u^T\Phi_f\right\| + \boldsymbol{s}_u^T K_u\boldsymbol{s}_u + k\dot{k}/\eta \end{aligned} \tag{17}$$

where the identity $\boldsymbol{s}^T(\frac{1}{2}\dot{M} - B)\boldsymbol{s} = 0$ has been used to derive (17).

Using (13) and (14) it can easily be verified that

$$\begin{aligned} &-\boldsymbol{s}_a^T\boldsymbol{u}_c + \boldsymbol{s}_u^T K_u\boldsymbol{s}_u + k\dot{k}/\eta \\ &\quad = \frac{1}{\|\boldsymbol{s}_a\|^2 + \delta}\Big[-(1+k)\rho_f\|\boldsymbol{s}_a\|^2\left\|\boldsymbol{s}_u^T\Phi_f\right\| \\ &\quad + (k\|\boldsymbol{s}_a\|^2 - \delta_1)\rho_f\left\|\boldsymbol{s}_u^T\Phi_f\right\| + (\delta - \delta_1)\boldsymbol{s}_u^T K_u\boldsymbol{s}_u\Big] \\ &\quad \le \frac{1}{\|\boldsymbol{s}_a\|^2 + \delta}\left[-\rho_f\|\boldsymbol{s}_a\|^2\left\|\boldsymbol{s}_u^T\Phi_f\right\| - \delta_1\rho_f\left\|\boldsymbol{s}_u^T\Phi_f\right\|\right]. \end{aligned} \tag{18}$$

Substituting (18) into (17), one obtains

$$\begin{aligned} \dot{V} &\le -\boldsymbol{s}^T K_s\boldsymbol{s} + \frac{1}{\|\boldsymbol{s}_a\|^2 + \delta}\Big[-\rho_f\|\boldsymbol{s}_a\|^2\left\|\boldsymbol{s}_u^T\Phi_f\right\| - \delta_1\rho_f\left\|\boldsymbol{s}_u^T\Phi_f\right\| \\ &\quad + \|\alpha_f\|\|\boldsymbol{s}_a\|^2\left\|\boldsymbol{s}_u^T\Phi_f\right\| + \delta\|\alpha_f\|\left\|\boldsymbol{s}_u^T\Phi_f\right\|\Big] \\ &\le -\boldsymbol{s}^T K_s\boldsymbol{s}. \end{aligned} \tag{19}$$

This shows that $\boldsymbol{s} \in L_2^n \cap L_\infty^n$ and $k \in L_\infty$. Combining this fact and the fact that $\dot{\boldsymbol{s}} \in L_\infty^n$ from (15), we can directly infer that $\lim_{t\to\infty}\boldsymbol{s}(t) = 0$ via Barbalat's Lemma, which implies that $\lim_{t\to\infty}\boldsymbol{q}(t) = \boldsymbol{q}_d(t)$ from the standard stable filter theory. This concludes our proof.

Remarks: 1) The control law is, in a simple fashion, only related to the bounds of the inertia parameters α_r and α_f. So, the parameter variations in the plant can be taken into account easily. Similar to [13] and [14], we may assign different gains to the components of $\boldsymbol{u}_a$ to avoid using only α_r and α_f which may lead to overly conservative design. Then, the gain condition (11) becomes

$$|\alpha_{ri}| \le \rho_{ri}; \qquad |\alpha_{fi}| \le \rho_{fi} \tag{20}$$

and the ith component of the control input $\boldsymbol{u}_a$ becomes

$$\boldsymbol{u}_{ai} = -[K_a \boldsymbol{s}_a]_i - \rho_{ri} \sum_{j=1}^{p1} [\Phi_r]_{ij} \left[\mathrm{sgn}\left(\boldsymbol{s}_a^T \Phi_r\right)\right]_j^T - \boldsymbol{u}_{ci} \tag{21}$$

$$\boldsymbol{u}_{ci} = \frac{(1+k)\boldsymbol{s}_{ai}}{\|\boldsymbol{s}_a\|^2 + \delta} \left[\sum_{i=1}^{p2} \left(\rho_{fi} \left|\sum_{j=1}^{n-m} \boldsymbol{s}_{uj} [\Phi_f]_{ji}\right|\right) + \boldsymbol{s}_u^T K_u \boldsymbol{s}_u\right] \tag{22}$$

$$\dot{k} = \begin{cases} \dfrac{\eta}{k}\left(\dfrac{k\|\boldsymbol{s}_a\|^2 - \delta_1}{\|\boldsymbol{s}_a\|^2 + \delta}\right) \left[\displaystyle\sum_{i=1}^{p2}\left(\rho_{fi}\left|\sum_{j=1}^{n-m} \boldsymbol{s}_{uj}[\Phi_f]_{ji}\right|\right) + \boldsymbol{s}_u^T K_u \boldsymbol{s}_u\right], & \text{if } k \ne 0 \\ \delta, & \text{if } k = 0. \end{cases} \tag{23}$$

In this case, it can still be shown that $\dot{V} \le -\boldsymbol{s}^T K_s \boldsymbol{s}$; therefore, $\lim_{t\to\infty} \boldsymbol{q}(t) = \boldsymbol{q}_d(t)$.

2) The control law given above is discontinuous and needs to be smoothed for implementation. As usual, we can replace signum nonlinearity by a saturation nonlinearity, which is specified as

$$\mathrm{sat}(s/\phi) = \begin{cases} \mathrm{sgn}(s), & \text{if } |s| > \phi \\ s/\phi, & \text{if } |s| \le \phi \end{cases}$$

where ϕ is boundary layer thickness. With this boundary layer, the adaptive variable structure control law given by (12)–(14) becomes

$$\boldsymbol{u}_a = -K_a \boldsymbol{s}_a - \rho_r \Phi_r \left[\mathrm{sat}\left(\boldsymbol{s}_a^T \Phi_r / \phi\right)\right]^T - \boldsymbol{u}_c \tag{24}$$

$$\boldsymbol{u}_c = \frac{(1+k)\boldsymbol{s}_a}{\|\boldsymbol{s}_a\|^2 + \delta}\left[\rho_f \left\|\boldsymbol{s}_u^T \Phi_f\right\| + \boldsymbol{s}_u^T K_u \boldsymbol{s}_u\right] \tag{25}$$

$$\dot{k} = \begin{cases} \dfrac{\eta}{k}\left(\dfrac{k\|\boldsymbol{s}_a\|^2 - \delta_1}{\|\boldsymbol{s}_a\|^2 + \delta}\right)\left[\rho_f \left\|\boldsymbol{s}_u^T \Phi_f\right\| + \boldsymbol{s}_u^T K_u \boldsymbol{s}_u\right], & \text{if } k \ne 0 \\ \delta, & \text{if } k = 0. \end{cases} \tag{26}$$

Such a smooth method generally leads to the conclusion that the regulation error is globally uniformly ultimately bounded. However, for this proposed algorithm, the proof of the boundedness of the regulation error seems not to be straightforward and still needs further investigation. We have conducted extensive simulation studies using two link underactuated manipulator as an example (see the next section), and the simulation results indeed confirm the boundedness of the smoothed algorithm.

IV. Example: The Two-Link Manipulator

In this section, the developed method is applied to a two-link manipulator with rotational joints in a horizontal plane. Suppose that the first joint is actuated while the second is not. The equation of motion for this system can be written in the following form:

$$\begin{bmatrix} m_{aa}(\boldsymbol{q}) & m_{au}(\boldsymbol{q}) \\ m_{ua}(\boldsymbol{q}) & m_{uu}(\boldsymbol{q}) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} h\dot{q}_2 & h(\dot{q}_1 + \dot{q}_2) \\ -h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} G_a \\ G_u \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \tag{27}$$

with

$$\begin{aligned} m_{aa} &= m_2 l_{c2}^2 + I_2 \\ m_{au} &= m_{ua} = m_2(l_{c2}^2 + l_1 l_{c2}\cos q_2) + I_2 \\ m_{uu} &= m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos q_2) + I_1 + I_2 \\ h &= -m_2 l_1 l_{c2} \sin q_2 \\ G_a &= (m_1 l_{c1} + m_2 l_1) g \cos q_1 + m_2 l_{c2} g \cos(q_1 + q_2) \\ G_u &= m_2 l_{c2} \cos(q_1 + q_2). \end{aligned} \tag{28}$$

Denote $q_a = q_1$ and $q_u = q_2$. If q_{ad} is the desired position for q_1, the choice of the desired position q_{ud} for q_2 is not independent. It should satisfy the constrained equation

$$m_{au}\ddot{q}_1 + m_{uu}\ddot{q}_2 - h\dot{q}_1 + G_u = 0. \tag{29}$$

In this example, as an illustration, we simply steer the robot from an initial position $\boldsymbol{q}^o = (q_a^o, q_u^o)$ to a given position $\boldsymbol{q}_d = (q_{ad}, q_{ud})$, with initial and final zero velocity. Then, (29) leads to $\cos(q_1 + q_2) = 0$, which implies that given a particular q_{1d}, the corresponding q_{2d} can be calculated.

The unknown parameters α_r and α_f are chosen as

$$\alpha_r = \begin{pmatrix} m_1 l_{c1}^2 + m_2 l_1^2 + I_1 \\ m_2 l_{c2}^2 + I_2 \\ m_2 l_1 l_{c2} \\ m_1 l_{c1} \\ m_2 l_1 \\ m_2 l_{c2} \end{pmatrix} \qquad \alpha_f = \begin{pmatrix} m_2 l_{c2}^2 + I_2 \\ m_2 l_1 l_{c2} \\ m_2 l_{c2} \end{pmatrix} \tag{30}$$

which lead to regressor matrices

$$\Phi_r^T = \begin{pmatrix} -\ddot{q}_{ar} \\ -\ddot{q}_{ar} - \ddot{q}_{ur} \\ -2\ddot{q}_{ar}\cos q_2 - \ddot{q}_{ur}\cos q_2 + \dot{q}_2\dot{q}_{ar}\sin q_2 + (\dot{q}_1 + \dot{q}_2)\dot{q}_{ur}\sin q_2 \\ -g\cos q_1 \\ -g\cos q_1 \\ -g\cos(q_1 + q_2) \end{pmatrix} \tag{31}$$

$$\Phi_f^T = \begin{pmatrix} -\ddot{q}_{ar} - \ddot{q}_{ur} \\ -\ddot{q}_{ar}\cos q_2 - \dot{q}_1\dot{q}_{ar}\sin q_2 \\ -g\cos(q_1 + q_2) \end{pmatrix}. \tag{32}$$

The true values for α_r and α_f are $\alpha_r^T = [8.33 \quad 1.67 \quad 2.5 \quad 5 \quad 5 \quad 2.5]$ and $\alpha_f^T = [1.67 \quad 2.5 \quad 2.5]$. Thus, we choose $\rho_r = 30$ and $\rho_f = 8$.

The desired (q_{ad}, q_{ud}) are chosen as $(q_{ad}, q_{ud}) = (-90^\circ, 0^\circ)$, and the initial positions and velocities of the robot are chosen as

$$\begin{aligned} \dot{q}_a(0) &= \dot{q}_u(0) = 0 \\ q_a(0) &= 70^\circ \\ q_u(0) &= 20^\circ. \end{aligned}$$

Ignoring any friction and attenuation effects, the smoothed control algorithm was tested using Simulink. Therein, the boundary layer ϕ is chosen as $\phi = 0.1$.

The task was to bring the manipulator to -90° position for joint 1 and 0° for joint 2. The results for two controllers with slightly different parameters are presented in Figs. 1 and 2. The

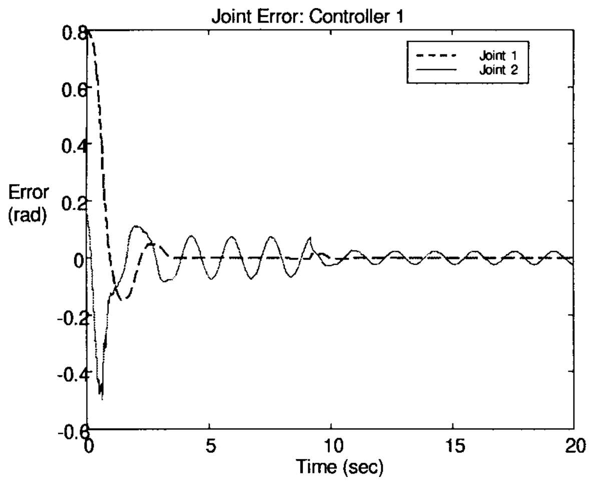

Fig. 1. Joint errors with the control parameters $q_{ad} = -90°$, $q_{ud} = 0°$, $\lambda_{a1} = 0.2$, $\lambda_{u1} = 250$, $\lambda_{a2} = 10$, $\lambda_{u2} = 0.5 K_a = 20I$, $K_u = 40I$, $\eta = 0.000\,01$, $\delta_1 = 0.75$, $\delta = 0.1$, $\rho_r = 30$, $\rho_f = 8$.

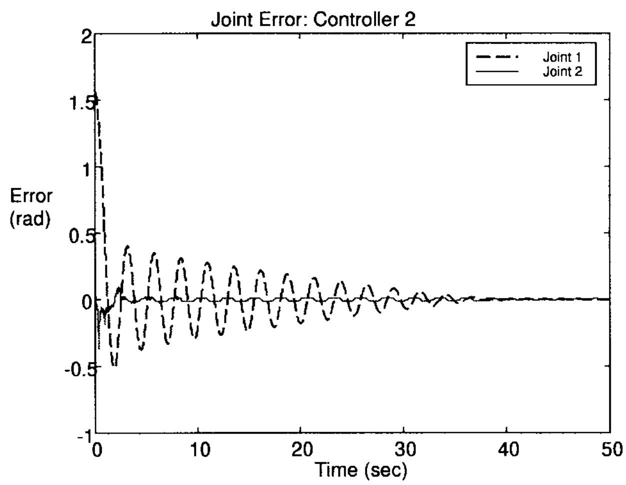

Fig. 2. Joint errors with the control parameters $q_{ad} = -90°$, $q_{ud} = 0°$, $\lambda_{a1} = 0.2$, $\lambda_{u1} = 75$, $\lambda_{a2} = 0.5$, $\lambda_{u2} = 0.5 K_a = 20I$, $K_u = 40I$, $\eta = 0.000\,01$, $\delta_1 = 0.75$, $\delta = 0.1$, $\rho_r = 30$, $\rho_f = 8$.

figures show the versatility of the control algorithm. From Figs. 1 and 2, we see that selection of controller parameters can affect the system performance. Unfortunately, there is no systematic approach for the selection of these values. They must be chosen using iterative simulations, and a tradeoff between system response and control gains should be made.

V. Conclusion

A regressor-based adaptive variable structure control algorithm has been proposed for unactuated mechanical systems, in the case of arbitrary uncertain inertia parameters. The controller ensures the global regulation and the switching gains depend only on the inertia parameters of the mechanical system. Simulation results were presented to demonstrate the regulation performance of the closed-loop system.

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