[7] M. Ghausi and J. Kelly, *Introduction to Distributed-Parameter Networks*. New York: Krieger, 1977.
[8] P. Sotiriadis and Y. Tsividis, "Integrators using a single distributed RC element," in *Proc. IEEE 2002 Int. Symp. Circuits and Systems*, vol. 2, Phoenix, AZ, May 2002, pp. 21–24.

Design and Analysis of Nonlinear Control for Uncertain Linear Systems

Xinkai Chen and Chun-Yi Su

***Abstract*—By using the input–output information, the problem of robust output tracking control is addressed for linear dynamical systems with arbitrary relative degrees. The considered systems are confined to minimum phase systems with unknown parameters, and unmatched disturbances composed of a bounded part and a class of unmodeled dynamics. The *a priori* knowledge concerning the disturbance bounds is unknown. The development of the nonlinear robust controller involves three steps. First, a special signal is generated, which can be thought of as an estimate of a filter of the input signal. Second, the derivatives up to a certain order of this special signal are derived. Third, the output tracking control input is synthesized by using the derivatives of the special signal. In the above process, the upper bounds of the disturbances are adaptively updated on-line. The proposed control law ensures the uniform boundedness of all the signals in the closed-loop system and achieves the output tracking to within a desired precision. The effectiveness of the proposed method is demonstrated through simulation.**

***Index Terms*—Input–output information, minimum phase systems, output tracking, relative degree, robust control, unmatched uncertainty.**

I. Introduction

In robust output tracking control, a central problem is to design a feedback control for a plant such that the output of the plant can asymptotically track a class of reference signals and reject a class of disturbances while maintaining closed-loop stability. For the class of linear systems, the solvability of the output tracking problem was thoroughly studied in [3], [4], and [7]–[11]. However, the system disturbances are generally assumed to be either constant or bounded. For minimum phase systems with unknown parameters and bounded disturbances, several typical adaptive methods achieving output tracking were suggested in [5], [6].

For systems with uncertainties, variable structure control has been investigated in robust control literature because of its effective performances [12], [13], [15]. However, in this kind of approach, the system uncertainties or disturbances are still assumed both bounded and matched. Also, the results are restricted to minimum phase dynamical systems with relative degree one. The proposed formulations cannot cope with systems of higher relative degrees, and cannot deal with unmatched disturbances or uncertainties. In the variable structure control, the unmatched disturbances become part of the equivalent control and must be estimated for the construction of the equivalent control.

Manuscript received February 28, 2002; revised August 4, 2002. This paper was recommended by Associate Editor X. Yu.

X. Chen is with the Department of Intelligent Mechanics, Kinki University, Naga-gun, Wakayama 649-6493, Japan (e-mail: chen@mec.waka.kindai.ac.jp).

C.-Y. Su is with the Department of Mechanical Engineering, Concordia University, Montreal, H3G 1M8 Canada (e-mail: cysu@me.concordia.ca).

Digital Object Identifier 10.1109/TCSI.2002.808231

For systems with unknown parameters and unmatched disturbances, an interesting robust approach is developed in [14] based on state-space techniques, where the input–output information and the *a priori* knowledge concerning the disturbance bounds are used. The overall system can be ensured to be globally uniformly ultimately bounded (GUUB) which can be made arbitrarily close to exponential stability if the control energy permits. However, the perfect *a priori* knowledge concerning the disturbance bounds may not be easily obtained in practice.

This brief demonstrates the design of a nonlinear output tracking controller for systems with both unknown parameters and unmatched disturbances. The unmatched disturbances are composed of a bounded part and a class of unmodeled dynamics. The perfect *a priori* knowledge concerning the disturbance bounds is not required. The disturbance bounds are adaptively updated online. The considered systems may have higher relative degrees. The proposed formulation is inspired by the "nonlinear differentiator" proposed in [1], and [2], which is motivated by the variable structure control and adaptive control methods. The design procedure in this brief can be summarized as three steps. First, a special signal is generated, which can be thought of as an estimate of a filter of the input signal. Second, the derivatives up to a certain order of this special signal are derived, where a backstepping idea [4] is used. Third, the output tracking control input is synthesized by using the derivatives of the special signal. The proposed nonlinear control law ensures the uniform boundedness of all the signals in the closed-loop system and achieves output tracking within a desired precision. The effectiveness of the proposed method is demonstrated through simulation.

This brief is organized as follows. Section II gives the problem formulation. In Section III, firstly, a special signal (which can be thought of as an estimate of a filter of the input signal) is generated. Secondly, the derivatives up to a certain order of the special signal are derived. Finally, the output tracking control input is determined, and the stability of the closed-loop system is analyzed. Section IV gives a design example to illustrate the proposed formulation. Section V provides conclusions.

II. Problem Statement

Consider an uncertain system of the form

$$a(s)y(t) = b(s)u(t) + v(t) \tag{1}$$

where s denotes the differential operator; $u(t)$ and $y(t)$ are scalar input and output, respectively; $v(t)$ is an unknown signal composed of model uncertainties, nonlinearities and disturbances, etc.; $a(s)$ and $b(s)$ are described by

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \tag{2}$$
$$b(s) = b_r s^{n-r} + b_{r+1}s^{n-r-1} + \cdots + b_{n-1}s + b_n. \tag{3}$$

It can be easily seen that $v(t)$ is an unmatched unknown signal. For simplicity, the signal $v(t)$ is called the "disturbance" of the system. It is assumed that the initial time is t_0.

The following assumptions are made.

(A1) $b(s)$ is a Hurwitz polynomial. $a(s)$ and $b(s)$ are coprime.

(A2) The indexes n and r are known. $b_r \neq 0$ and the sign of it is known. Without loss of generality, it is assumed $b_r > 0$.

(A3) The parameters in $a(s)$ and $b(s)$ are unknown constants but they are bounded in known compact sets. More specifically, there are known constants $\underline{a}_i$, $\overline{a}_i$, $\underline{b}_j$ and $\overline{b}_j$ such that for $1 \leq i \leq n$ and $r \leq j \leq n$

$$\underline{a}_i \leq a_i \leq \overline{a}_i \quad \underline{b}_j \leq b_j \leq \overline{b}_j \tag{4}$$

where $\underline{b}_r > 0$.

(A4) The desired output signal $y_d(t)$ is differentiable to a necessary order. Further, it is assumed that $y_d(t)$ and its derivatives are uniformly bounded.

The object is to control the output to follow the desired signal $y_d(t)$ by using the input–output information for the uncertain system (1).

III. Design Procedure of Robust Controller

A. Some Preliminaries

First, the "filter" used is defined. For a real constant $\Gamma > 0$ and a signal $\mu(t)$, $(1/(s+\Gamma))\mu(t)$ is defined as the solution of the following differential equation:

$$\dot{\xi}(t) + \Gamma\xi(t) = \mu(t) \quad \xi(t_0) = 0. \tag{5}$$

Thus, the filter $(h_2(s)/h_1(s))\mu(t)$ can be analogously defined, where $h_1(s)$ is a Hurwitz polynomial and $h_2(s)/h_1(s)$ is proper.

Now, introduce a monic $(n-1)$th order Hurwitz polynomial

$$d(s) = d_1(s)(s+\lambda)^{r-1} \tag{6}$$

where $\lambda > 0$ is an introduced design parameter, and $d_1(s)$ is a monic $(n-r)$th order Hurwitz polynomial.

Then, system (1) can be rewritten as

$$\dot{y}(t) + \lambda y(t) = \frac{(s+\lambda)d(s) - a(s)}{d(s)}\, y(t) + \frac{b(s)}{d_1(s)}\, \overline{u}(t) + \overline{v}(t) + \varepsilon(t) \tag{7}$$

where $\overline{u}(t)$ and $\overline{v}(t)$ are, respectively, defined as

$$\overline{u}(t) = \frac{1}{(s+\lambda)^{r-1}}\, u(t) \quad \overline{v}(t) = \frac{1}{d(s)}\, v(t). \tag{8}$$

$\varepsilon(t)$ is an exponentially decaying term which arises from the possibility of nonzero initial conditions since zero initial conditions were assumed in the definition of the filter [see (5)].

In the following, $\overline{u}(t)$ is called "intermediate input."

Concerning the uncertainty $\overline{v}(t)$, the following assumption is made.

(A5) There is a known nonnegative function $\rho(\xi)$ [$\rho(\xi) \geq 0$ for all ξ] with the property that, if ξ is bounded, then $\rho(\xi)$ is bounded, such that

$$|\overline{v}(t)| \leq \overline{K}_1 + K_2\rho(y) \tag{9}$$

where $\overline{K}_1$ and K_2 are unknown positive constants.

Remark 1: In this brief, the structure of the upper bound of the uncertainty $\overline{v}(t)$ [instead of the disturbance $v(t)$] is given. By definition of $\overline{v}(t)$, assumption (A5) means that the disturbance $v(t)$ may include some bounded dynamics, the filters of the output, and some dynamics of the derivatives of the output up to $(n-1)$th order, etc. The function $\rho(y)$ may be 0, $|y(t)|$, $|(l_2(s)/l_1(s))y(t)|$ [where $l_1(s)$ is a Hurwitz polynomial, $l_2(s)/l_1(s)$ is proper], etc.

Now, rewrite (7) as

$$\dot{y}(t) + \lambda y(t) = \frac{(s+\lambda)d_2(s) + \lambda s^{n-1}}{d(s)}\, y(t) + \theta^T\phi(t) + b_r\overline{u}(t) - \frac{b_r d_3(s)}{d_1(s)}\, \overline{u}(t) + \overline{\overline{v}}(t) \tag{10}$$

where $d_2(s)$ and $d_3(s)$ are, respectively, $(n-2)$th and $(n-r-1)$th order polynomials which are defined by

$$d(s) = s^{n-1} + d_2(s) \tag{11}$$

$$d_1(s) = s^{n-r} + d_3(s). \tag{12}$$

θ, $\phi(t)$ and $\overline{\overline{v}}(t)$ are, respectively, defined as

$$\theta = [-a_1, \ldots, -a_n, b_{r+1}, \ldots, b_n]^T \tag{13}$$

$$\phi(t) = \left[\frac{s^{n-1}}{d(s)}\, y(t), \ldots, \frac{1}{d(s)}\, y(t), \frac{s^{n-r-1}}{d_1(s)}\, \overline{u}(t), \ldots, \frac{1}{d_1(s)}\, \overline{u}(t)\right]^T \tag{14}$$

$$\overline{\overline{v}}(t) = \overline{v}(t) + \varepsilon(t). \tag{15}$$

It can be easily seen that $\phi(t)$ is an available signal if $\overline{u}(t)$ is determined.

In (10), $(((s+\lambda)d_2(s) + \lambda s^{n-1})/d(s))y(t)$ is an available signal. Since $d_3(s)/d_1(s)$ is strictly proper, the filtered signal $(d_3(s)/d_1(s))\overline{u}(t)$ is also available if $\overline{u}(t)$ is determined.

By assumption (A5), it can be seen that there exists $K_1 > 0$ such that

$$\left|\overline{\overline{v}}(t)\right| \leq K_1 + K_2\rho(y) \tag{16}$$

where K_1 is an unknown positive constant.

The following upper bounds, which will be used in the remainder of the brief, are estimated as follows.

The upper bound of $\|\theta\|_2$ is estimated as

$$\|\theta\|_2 = \sqrt{\sum_{i=1}^{n} a_i^2 + \sum_{j=r+1}^{n} b_j^2} \leq \sqrt{\sum_{i=1}^{n} \max\left(\underline{a}_i^2, \overline{a}_i^2\right) + \sum_{j=r+1}^{n} \max\left(\underline{b}_j^2, \overline{b}_j^2\right)} \triangleq \Theta \tag{17}$$

where assumption (A3) is used.

The upper bound of

$$\left(\frac{(s+\lambda)d_2(s) + \lambda s^{n-1}}{d(s)}\, y(t) + \theta^T\phi(t) - \dot{y}_d(t) - \lambda y_d(t)\right)$$

is estimated as

$$\left|\frac{(s+\lambda)d_2(s) + \lambda s^{n-1}}{d(s)}\, y(t) + \theta^T\phi(t) - \dot{y}_d(t) - \lambda y_d(t)\right| \leq \left|\frac{(s+\lambda)d_2(s) + \lambda s^{n-1}}{d(s)}\, y(t)\right| + \Theta\|\phi(t)\|_2 + |\dot{y}_d(t) + \lambda y_d(t)| \triangleq \omega_1(t). \tag{18}$$

In the following parts, firstly, a special signal $\sigma_1(t)$ is synthesized such that the output tracking control can be approximately achieved if the intermediate input $\overline{u}(t)$ is chosen as $\overline{u}(t) = \sigma_1(t)$. Secondly, a signal $\sigma_r(t)$ is derived such that $(1/(s+\lambda)^{r-1})\sigma_r(t)$ is very close to $\sigma_1(t)$. Thirdly, the output tracking control $u(t)$ is synthesized.

B. Determination of the Special Signal $\sigma_1(t)$

Let

$$\sigma_1(t) = \frac{d_3(s)}{d_1(s)}\, \overline{u}(t) - \frac{1}{\underline{b}_r}\, \frac{\hat{\chi}_1^2(t)(y(t) - y_d(t))}{\hat{\chi}_1(t)|y(t) - y_d(t)| + \delta_1} \tag{19}$$

where $\hat{\chi}_1(t)$ is defined as

$$\hat{\chi}_1(t) = \omega_1(t) + \hat{K}_{11}(t) + \hat{K}_{12}(t)\rho(y). \tag{20}$$

$\omega_1(t)$ is defined in (18), $\hat{K}_{11}(t)$ and $\hat{K}_{12}(t)$ are respectively defined as

$$\dot{\hat{K}}_{11}(t) = \begin{cases} \alpha_{11}|y(t) - y_d(t)|, & \text{if } |y(t) - y_d(t)| > \sum_{i=1}^{r} \lambda^{-i+1}\sqrt{\frac{2\delta_i}{\lambda}} \\ 0, & \text{otherwise} \end{cases} \quad (21)$$

$$\dot{\hat{K}}_{12}(t) = \begin{cases} \alpha_{12}|y(t) - y_d(t)|\rho(y), & \text{if } |y(t) - y_d(t)| > \sum_{i=1}^{r} \lambda^{-i+1}\sqrt{\frac{2\delta_i}{\lambda}} \\ 0, & \text{otherwise.} \end{cases} \quad (22)$$

$\hat{K}_{11}(t_0)$ and $\hat{K}_{12}(t_0)$ can be chosen as any small positive constants, α_{11} and α_{12} are positive constants; $\delta_i > 0$ $(i = 1, \ldots, r)$ are design parameters.

Remark 2: The relative degree of (7) [equivalently, (10)] is one with respect to the relation between the output and the intermediate input $\overline{u}(t)$. If the intermediate control input $\overline{u}(t)$ is chosen as $\overline{u}(t) = \sigma_1(t)$, it can be proved that the output tracking control can be approximately achieved. This fact can be verified by referring the proof of Theorem 1.

C. Derivation of the Signal $\sigma_r(t)$ Such That $(1/(s+\lambda)^{r-1})\sigma_r(t)$ is Very Close to $\sigma_1(t)$

In the first step, a signal $\sigma_2(t)$ is found such that $(1/(s+\lambda))\sigma_2(t)$ is very close to $\sigma_1(t)$. In the second step, a signal $\sigma_3(t)$ is found such that $(1/(s+\lambda))\sigma_3(t)$ is very close to $\sigma_2(t)$. Consequently, in the final $(r-1)$th step, a signal $\sigma_r(t)$ is found such that $(1/(s+\lambda))\sigma_r(t)$ is very close to $\sigma_{r-1}(t)$. Thus, it can be seen that $(1/(s+\lambda)^{r-1})\sigma_r(t)$ is very close to $\sigma_1(t)$.

Step 1: Derivation of a Signal $\sigma_2(t)$ Such That $(1/(s+\lambda))\sigma_2(t)$ is Very Close to $\sigma_1(t)$: Based on the trivial differential equation

$$\dot{\sigma}_1(t) + \lambda\sigma_1(t) = (s+\lambda)\sigma_1(t) \quad (23)$$

construct the corresponding differential equation

$$\dot{\hat{\sigma}}_1(t) + \lambda\hat{\sigma}_1(t) = \sigma_2(t) \quad \hat{\sigma}_1(t_0) = \sigma_1(t_0) \quad (24)$$

where $\hat{\sigma}_1(t)$ is a signal which can be obtained by solving equation (24), and $\sigma_2(t)$ is the input to be determined.

First, derive the upper bound of $|(s+\lambda)\sigma_1(t)|$. Because $\sigma_1(t)$ is an available signal, it is only needed to estimate the upper bound of $|\dot{\sigma}_1(t)|$. It can be calculated by first developing bounds for the first-order partial derivatives of $\sigma_1(t)$ with respect to its variables, and then by determining the bounds for the first-order time derivatives of its variables. Since $d_3(s)/d_1(s)$ is strictly proper, eventually, it is only needed to derive the upper bound of $|\dot{y}(t)|$.

Based on (10), the upper bound of $|\dot{y}(t)|$ can be estimated as

$$|\dot{y}(t)| \le \left|\frac{sd_2(s)}{d(s)}\, y(t)\right| + \Theta\|\phi(t)\|_2 + \overline{b}_r \left|\frac{s^{n-r}}{d_1(s)}\, \overline{u}(t)\right| + K_1 + K_2\rho(y) \quad (25)$$

where relations (16) and (17) are employed.

Thus, the upper bound of $|(s+\lambda)\sigma_1(t)|$ can be estimated as

$$|(s+\lambda)\sigma_1(t)| \le \chi_{21}(y, \overline{u}) + \chi_{22}(y, \overline{u})(K_1 + K_2\rho(y)) \triangleq \chi_2(t) \quad (26)$$

where $\chi_{21}(y, \overline{u})$ and $\chi_{22}(y, \overline{u})$ are known positive functions of $y(t)$, $\overline{u}(t)$ and their filters, and have the property that, if $y(t)$ and $\overline{u}(t)$ are bounded, then $\chi_{21}(y, \overline{u})$ and $\chi_{22}(y, \overline{u})$ are bounded.

Lemma 1: For equation (24), $\sigma_2(t)$ is chosen as

$$\sigma_2(t) = \frac{\{\sigma_1(t) - \hat{\sigma}_1(t)\}\hat{\chi}_2^2(t)}{|\sigma_1(t) - \hat{\sigma}_1(t)|\hat{\chi}_2(t) + \frac{\delta_2}{\overline{b}_r^2}} \quad (27)$$

where $\hat{\chi}_2(t)$ is defined as

$$\hat{\chi}_2(t) = \chi_{21}(y, \overline{u}) + \chi_{22}(y, \overline{u})\left(\hat{K}_{21}(t) + \hat{K}_{22}(t)\rho(y)\right). \quad (28)$$

$\hat{K}_{21}(t)$ and $\hat{K}_{22}(t)$ are, respectively, defined by

$$\dot{\hat{K}}_{21}(t) = \begin{cases} \alpha_{21}|\sigma_1(t) - \hat{\sigma}_1(t)|\chi_{22}(y, \overline{u}), & \text{if } |\sigma_1(t) - \hat{\sigma}_1(t)| > \frac{1}{\overline{b}_r}\sqrt{\frac{2\delta_2}{\lambda}} \\ 0, & \text{otherwise} \end{cases} \quad (29)$$

$$\dot{\hat{K}}_{22}(t) = \begin{cases} \alpha_{22}|\sigma_1(t) - \hat{\sigma}_1(t)|\chi_{22}(y, \overline{u})\cdot\rho(y), & \text{if } |\sigma_1(t) - \hat{\sigma}_1(t)| > \frac{1}{\overline{b}_r}\sqrt{\frac{2\delta_2}{\lambda}} \\ 0, & \text{otherwise} \end{cases} \quad (30)$$

α_{21} and α_{22} are positive constants, $\hat{K}_{21}(t_0)$ and $\hat{K}_{22}(t_0)$ can be chosen to be any positive constant. Then, it can concluded that $|\sigma_1(t) - \hat{\sigma}_1(t)|$, $\hat{K}_{21}(t)$ and $\hat{K}_{22}(t)$ are uniformly bounded, and there exists $t_1 > t_0$ such that

$$|\sigma_1(t) - \hat{\sigma}_1(t)| \le \frac{1}{\overline{b}_r}\sqrt{\frac{2\delta_2}{\lambda}} \quad (31)$$

as $t \ge t_1$.

Proof: See Appendix A.

Step i $(1 < i \le r-1)$: Derivation of a signal $\sigma_{i+1}(t)$ such that $(1/(s+\lambda))\sigma_{i+1}(t)$ is very close to $\sigma_i(t)$.

Based on the trivial differential equation

$$\frac{d}{dt}\{\sigma_i(t)\} + \lambda\sigma_i(t) = (s+\lambda)\sigma_i(t) \quad (32)$$

construct the corresponding differential equation

$$\dot{\hat{\sigma}}_i(t) + \lambda\hat{\sigma}_i(t) = \sigma_{i+1}(t) \quad \hat{\sigma}_i(t_0) = \sigma_i(t_0) \quad (33)$$

where $\hat{\sigma}_i(t)$ is a signal which can be obtained by solving equation (33), and $\sigma_{i+1}(t)$ is the input to be determined.

First, derive the upper bound of $|(s+\lambda)\sigma_i(t)|$. As $\sigma_i(t)$ has been determined in the $(i-1)$th step, it is only needed to derive the upper bound of $|(d/dt)\sigma_i(t)|$ in this step. It can be calculated by first developing bounds for the first-order partial derivatives of $(d/dt)\sigma_i(t)$ with respect to its variables, and then by determining the bounds for the first-order time derivatives of its variables. The results in the $(i-1)$th step can be employed to estimate the bounds for the first-order time derivatives of the variables of $\sigma_i(t)$. Thus, the upper bound of $|(s+\lambda)\sigma_i(t)|$ can be estimated as

$$\begin{aligned}|(s+\lambda)\sigma_i(t)| &\le \chi_{i+1,1}\left(y, \overline{u}^{(i-1)}\right) + \chi_{i+1,2}\left(y, \overline{u}^{(i-1)}\right)(K_1 + K_2\rho(y)) \\ &\triangleq \chi_{i+1}(t)\end{aligned} \quad (34)$$

where $\overline{u}^{(i-1)}(t)$ denotes the $(i-1)$th order derivative of $\overline{u}(t)$; $\chi_{i+1,1}(y, \overline{u}^{(i-1)})$ and $\chi_{i+1,2}(y, \overline{u}^{(i-1)})$ are known positive functions of $y(t)$, $\overline{u}^{(i-1)}(t)$ and their filters, and have the property that, if $y(t)$ and $\overline{u}^{(i-1)}(t)$ are bounded, then $\chi_{i+1,1}(y, \overline{u}^{(i-1)})$ and $\chi_{i+1,2}(y, \overline{u}^{(i-1)})$ are bounded.

Similar to Lemma 1, $\sigma_{i+1}(t)$ can be chosen as

$$\sigma_{i+1}(t) = \frac{\{\sigma_i(t) - \hat{\sigma}_i(t)\}\hat{\chi}_{i+1}^2(t)}{|\sigma_i(t) - \hat{\sigma}_i(t)|\hat{\chi}_{i+1}(t) + \frac{\delta_{i+1}}{\overline{b}_r^2}} \quad (35)$$

where $\hat{\chi}_{i+1}(t)$ is defined as

$$\hat{\chi}_{i+1}(t) = \chi_{i+1,1}\left(y, \overline{u}^{(i-1)}\right) + \chi_{i+1,2}\left(y, \overline{u}^{(i-1)}\right)\cdot\left(\hat{K}_{i+1,1}(t) + \hat{K}_{i+1,2}(t)\rho(y)\right) \quad (36)$$

where $\hat{K}_{i+1,1}(t)$ and $\hat{K}_{i+1,2}(t)$ are, respectively, defined by

$$\dot{\hat{K}}_{i+1,1}(t) = \begin{cases} \alpha_{i+1,1}|\sigma_i(t) - \hat{\sigma}_i(t)|\chi_{i+1,2}\left(y, \overline{u}^{(i-1)}\right), \\ \qquad \text{if } |\sigma_i(t) - \hat{\sigma}_i(t)| > \frac{1}{\overline{b}_r}\sqrt{\frac{2\delta_{i+1}}{\lambda}} \\ 0, \qquad \text{otherwise} \end{cases} \tag{37}$$

$$\dot{\hat{K}}_{i+1,2}(t) = \begin{cases} \alpha_{i+1,2}|\sigma_i(t) - \hat{\sigma}_i(t)|\chi_{i+1,2}\left(y, \overline{u}^{(i-1)}\right)\cdot\rho(y), \\ \qquad \text{if } |\sigma_i(t) - \hat{\sigma}_i(t)| > \frac{1}{\overline{b}_r}\sqrt{\frac{2\delta_{i+1}}{\lambda}} \\ 0, \qquad \text{otherwise.} \end{cases} \tag{38}$$

$\alpha_{i+1,1}$ and $\alpha_{i+1,2}$ are positive constants, $\hat{K}_{i+1,1}(t_0)$ and $\hat{K}_{i+1,2}(t_0)$ can be chosen to be any positive constant. Then, it can be concluded that $|\sigma_i(t) - \hat{\sigma}_i(t)|$, $\hat{K}_{i+1,1}(t)$ and $\hat{K}_{i+1,2}(t)$ are uniformly bounded, and there exists $t_i > t_0$ such that

$$|\sigma_i(t) - \hat{\sigma}_i(t)| \le \frac{1}{\overline{b}_r}\sqrt{\frac{2\delta_{i+1}}{\lambda}} \tag{39}$$

as $t \ge t_i$.

By forwarding the above process to the $(r-1)$th step, the next lemma can be obtained.

Lemma 2: For $i = 1, \ldots, r-1$, construct the differential equations

$$\begin{aligned} \dot{\hat{\sigma}}_i(t) + \lambda\hat{\sigma}_i(t) &= \sigma_{i+1}(t) \\ \sigma_{i+1}(t) &= \frac{\{\sigma_i(t) - \hat{\sigma}_i(t)\}\hat{\chi}_{i+1}^2(t)}{|\sigma_i(t) - \hat{\sigma}_i(t)|\hat{\chi}_{i+1}(t) + \frac{\delta_{i+1}}{\overline{b}_r^2}} \\ \hat{\sigma}_i(t_0) &= \sigma_i(t_0) \end{aligned} \tag{40}$$

where $\hat{\sigma}_i(t)$ are the signals which can be obtained by solving the differential equations in (40); $\delta_{i+1} > 0$ are design parameters; $\hat{\chi}_{i+1}(t)$ are defined in (36). Then, it can be concluded that $|\sigma_i(t) - \hat{\sigma}_i(t)|$, $\hat{K}_{i+1,1}(t)$ and $\hat{K}_{i+1,2}(t)$ are uniformly bounded, and there exist $t_i > t_0$ such that

$$|\sigma_i(t) - \hat{\sigma}_i(t)| \le \frac{1}{\overline{b}_r}\sqrt{\frac{2\delta_{i+1}}{\lambda}} \tag{41}$$

as $t \ge t_i$.

Proof: The lemma can be proved by mimicking the proof of Lemma 1.

Remark 3: The upper bound of $|(s+\lambda)\sigma_i(t)|$ is roughly estimated in the above analysis. Thus, it can be argued that $\hat{\chi}_{i+1}(t)$ may be much larger than $|(s+\lambda)\sigma_i(t)|$. In this case, the next corollary can be obtained.

Corollary 1: If the difference $\hat{\chi}_{i+1}(t) - |(s+\lambda)\sigma_i(t)|$ is very large, then the magnitude of $|\sigma_i(t) - \hat{\sigma}_i(t)|$ can be controlled to be very small even though δ_{i+1} is not very small and λ is not so large.

Proof: See Appendix B.

By the results of Lemma 2, the next lemma can be obtained.

Lemma 3: For $i = 1, \ldots, r-1$ and the signal $\sigma_r(t)$ generated in Lemma 2, $|\sigma_i(t) - (1/(s+\lambda)^{r-i})\sigma_r(t)|$ are uniformly bounded, and there exist $\tau_i > t_0$ such that

$$\left|\sigma_i(t) - \frac{1}{(s+\lambda)^{r-i}}\sigma_r(t)\right| \le \frac{\lambda^{i+1}}{\overline{b}_r}\sum_{j=i+1}^{r}\lambda^{-j}\sqrt{\frac{2\delta_j}{\lambda}} \tag{42}$$

for $t > \tau_i$, i.e., the difference $\sigma_i(t) - (1/(s+\lambda)^{r-i})\sigma_r(t)$ can be controlled by the designed parameters λ and δ_j (for $j = i+1, \ldots, r$).

Proof: By using the relations $\hat{\sigma}_i(t) = (1/(s+\lambda))\sigma_{i+1}(t)$ (for $i = 1, \ldots, r-1$) and the results in Lemma 2, the lemma can be easily proved, where τ_i is defined as $\tau_i = \max\{t_i, \ldots, t_{r-1}\}$.

Corollary 2: For $i = 1, \ldots, r-1$, if the differences $\hat{\chi}_j(t) - |(s+\lambda)\sigma_{j-1}(t)|$ are very large for all $j = i+1, \ldots, r$, then the magnitude of $|\sigma_i(t) - (1/(s+\lambda)^{r-i})\sigma_r(t)|$ can be controlled to be very small even though δ_j (for $j = i+1, \ldots, r$) are not very small and λ is not so large.

Proof: The corollary can be easily proved by using Corollary 1 and Lemma 3.

D. The Robust Control Input and the Global Stability of the Closed-Loop System

In the proposed formulation, the control input is chosen as

$$u(t) = \sigma_r(t). \tag{43}$$

Therefore, by Lemma 3 (for the case $i = 1$), it can be seen that $\overline{u}(t)$ is very close to $\sigma_1(t)$. Further, by the choice of $\sigma_1(t)$ in Section III-B, it can be guessed that the output tracking may be approximately achieved by using this control. The next theorem describes the stability of the closed-loop system.

Theorem 1: Consider system (1) satisfying assumptions (A1)–(A5). If the control input is chosen as $u(t) = \sigma_r(t)$, where the signal $\sigma_r(t)$ is generated in Lemma 2, then $y(t)$, $\hat{K}_{i1}(t)$ (for $i = 1, \ldots, r$), $\hat{K}_{i2}(t)$ (for $i = 1, \ldots, r$), $\sigma_i(t)$ (for $i = 1, \ldots, r$), $\hat{\sigma}_i(t)$ (for $i = 1, \ldots, r-1$), $\hat{\chi}_i(t)$ (for $i = 1, \ldots, r$) are all uniformly bounded. Further, the robust control $u(t)$ is continuous and globally uniformly bounded, and there exists $T > t_0$ such that

$$|y(t) - y_d(t)| \le \sum_{i=1}^{r}\lambda^{-i+1}\sqrt{\frac{2\delta_i}{\lambda}} \tag{44}$$

for all $t \ge T$, i.e., the magnitude of the output tracking error can be controlled by the design parameters λ and δ_i ($i = 1, \ldots, r$).

Proof: See Appendix C.

Corollary 3: If the differences

$$\hat{\chi}_1(t) - \left(\left|\frac{(s+\lambda)d_2(s) + \lambda s^{n-1}}{d(s)}y(t) + \theta^T\phi(t) - \dot{y}_d - \lambda y_d\right| + \left|\overline{\overline{v}}(t)\right|\right)$$

and $\hat{\chi}_i(t) - |(s+\lambda)\sigma_{i-1}(t)|$ (for $i = 2, \ldots, r$) are very large, then the output tracking error can be controlled to be very small even though δ_i ($i = 1, \cdots, r$) are not very small and λ is not so large.

Proof: The corollary can be easily proved by observing the result of Corollary 2 and the proof of Theorem 1.

Remark 4: The design parameters $\delta_i > 0$ ($i = 1, \ldots, r$), which should be chosen very small, determine the output tracking precision. However, by Corollary 3, the output tracking error may also be controlled to be very small even though the parameters $\delta_i > 0$ are not so small.

Remark 5: The parameters $\alpha_{i1} > 0$ and $\alpha_{i2} > 0$ should be chosen large enough to adjust the estimated upper bounds $\hat{K}_{i1}(t)$ and $\hat{K}_{i2}(t)$ rapidly for $i = 1, \ldots, r$.

Remark 6: The design parameter $\lambda > 0$ determines the converging speed of $y(t) - y_d(t)$ and $\sigma_i(t) - \hat{\sigma}_i(t)$ ($i = 1, \ldots, r-1$). It also influences the output tracking precision. Thus, by Theorem 1, the parameter $\lambda > 0$ should not be chosen very small.

Remark 7: As is well known that, if b_r is known, the term

$$\left(\frac{(s+\lambda)d_2(s) + \lambda s^{n-1}}{d(s)}y(t) - \dot{y}_d(t) - \lambda y_d(t)\right)$$

can be canceled and the control gain may be reduced.

Remark 8: If we have the *a priori* information that $\overline{v}(t)$ is bounded, then the controller may become simple, where the adaptation laws of updating $\hat{K}_{i2}(t)$ are not needed because $\rho(y) = 0$.

IV. Example and Simulation Results

Consider the system described by

$$s(s+1)y(t) = u(t) + v(t), \qquad y(0) = 0 \tag{45}$$

where the disturbance $v(t)$ is governed by

$$v(t) = (s-1)\left\{(\sin t)\, \frac{\dot{y}(t) + 0.5u(t) + 0.6y(t)}{|\dot{y}(t) + 0.5u(t) + 0.6y(t)| + 1}\, y(t)\right\}. \tag{46}$$

The purpose of the control is to drive the output to follow the signal $y_d(t) = \sin t$.

Suppose the bounds of the unknown system parameters are known as

$$0.5 \le a_1 \le 1.5 \quad -0.5 \le a_2 \le 0.5 \quad 0.5 \le b_2 \le 1.5. \tag{47}$$

Choose $\lambda = 1$ and $d_1(s) = 1$. Thus, $d(s) = s + 1$, $d_2(s) = 1$ and $d_3(s) = 0$. Suppose it is known that $\overline{v}(t) = (1/(s+1))v(t)$ is bounded.

Thus, $\phi(t)$ defined in (14) can be expressed as

$$\phi(t) = \left[\frac{s}{s+1}\, y(t),\ \frac{1}{s+1}\, y(t)\right]^T. \tag{48}$$

The special signal $\sigma_1(t)$ can be chosen as

$$\sigma_1(t) = -\frac{1}{0.5}\, \frac{\hat{\chi}_1^2(t)(y(t) - \sin t)}{\hat{\chi}_1(t)|y(t) - \sin t| + \delta_1} \tag{49}$$

where $\hat{\chi}_1(t)$ is defined as

$$\hat{\chi}_1(t) = \omega_1(t) + \hat{K}_{11}(t) \tag{50}$$

where $\omega_1(t)$ is defined as

$$\omega_1(t) = |\cos t + \sin t| + 1.5\left|\frac{s}{s+1}\, y(t)\right| + 1.5\left|\frac{1}{s+1}\, y(t)\right|. \tag{51}$$

$\hat{K}_{11}(t)$ is defined in

$$\dot{\hat{K}}_{11}(t) = \begin{cases} \alpha_{11}|y(t) - y_d(t)|, & \text{if } |y(t) - y_d(t)| > \sqrt{2\delta_1} + \sqrt{2\delta_2} \\ 0 & \text{otherwise} \end{cases}$$
$$\hat{K}_{11}(0) = 0.1 \tag{52}$$

and α_{11} is a positive constant; $\delta_1 > 0$ and $\delta_2 > 0$ are design parameters.

Now, construct the following differential equation:

$$\dot{\hat{\sigma}}_1(t) + \hat{\sigma}_1(t) = \sigma_2(t) \qquad \sigma_2(t) = \frac{\{\sigma_1(t) - \hat{\sigma}_1(t)\}\hat{\chi}_2^2(t)}{|\sigma_1(t) - \hat{\sigma}_1(t)|\hat{\chi}_2(t) + \frac{4}{9}\delta_2} \tag{53}$$

where $\hat{\chi}_2(t)$ is defined as

$$\hat{\chi}_2(t) = \chi_{21}(y, \overline{u}) + \chi_{22}(y, \overline{u})\hat{K}_{21}(t) \tag{54}$$

where $\chi_{21}(y, \overline{u})$ and $\chi_{22}(y, \overline{u})$ are, respectively, defined by

$$\begin{aligned} \chi_{21}(y, \overline{u}) &= |\sigma_1(t)| + \frac{2\delta_1\hat{\chi}_1^2(t)}{(\hat{\chi}_1(t)|y(t) - \sin t| + \delta_1)^2} \\ &\quad \cdot \left(|\cos t| + 0.5\left|\frac{s}{s+1}\, y(t)\right| + 0.5\left|\frac{1}{s+1}\, y(t)\right| + 1.5\left|\frac{1}{s+1}\, u(t)\right|\right) \\ &\quad + \frac{2\hat{\chi}_1(t)|y(t) - \sin t|(\hat{\chi}_1(t)|y(t) - \sin t| + 2\delta_1)}{(\hat{\chi}_1(t)|y(t) - \sin t| + \delta_1)^2} \\ &\quad \cdot \left(3.75\left|\frac{s}{s+1}\, y(t)\right| + 0.75\left|\frac{1}{s+1}\, y(t)\right| + 2.25\left|\frac{1}{s+1}\, u(t)\right| + |\cos t - \sin t| + \dot{\hat{K}}_{11}(t)\right) \end{aligned} \tag{55}$$

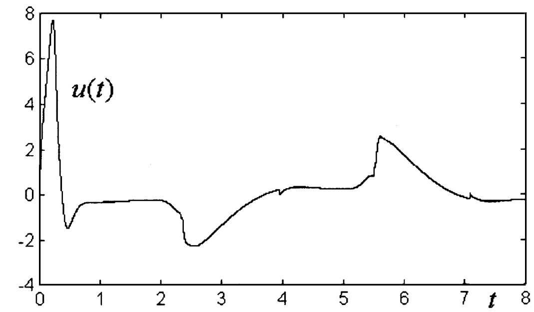

Fig. 1. Robust output tracking control input.

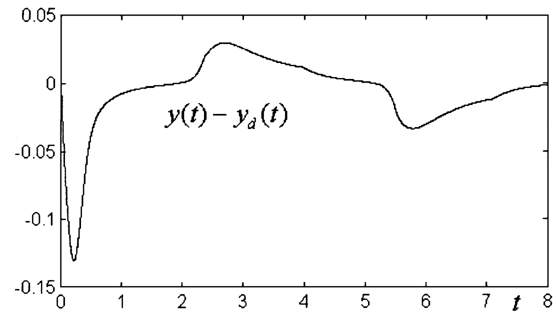

Fig. 2. Difference between the controlled output and the desired output.

$$\begin{aligned} \chi_{22}(y, \overline{u}) &= \frac{2\delta_1\hat{\chi}_1^2(t)}{(\hat{\chi}_1(t)|y(t) - \sin t| + \delta_1)^2} \\ &\quad + \frac{3\hat{\chi}_1(t)|y(t) - \sin t|(\hat{\chi}_1(t)|y(t) - \sin t| + 2\delta_1)}{(\hat{\chi}_1(t)|y(t) - \sin t| + \delta_1)^2} \end{aligned} \tag{56}$$

where $\hat{K}_{21}(t)$ is defined in

$$\dot{\hat{K}}_{21}(t) = \begin{cases} \alpha_{21}|\sigma_1(t) - \hat{\sigma}_1(t)|\chi_{22}(y, \overline{u}), & \text{if } |\sigma_1(t) - \hat{\sigma}_1(t)| > \frac{1}{1.5}\sqrt{2\delta_2} \\ 0, & \text{otherwise} \end{cases}$$
$$\hat{K}_{21}(0) = 0.1 \tag{57}$$

and α_{21} is a positive constant.

By Theorem 1, the control input can be chosen as

$$u(t) = \sigma_2(t). \tag{58}$$

In the simulation, the sampling period is chosen as 6×10^{-4} s. The design parameters are chosen as $\delta_1 = 0.25$, $\delta_2 = 15$, and $\alpha_1 = \alpha_2 = 5$. The starting time is $t_0 = 0$.

Fig. 1 shows the output tracking control input which remains uniformly bounded. Fig. 2 shows the difference between the controlled output and the desired output. It can be seen that the proposed control functions well with very small error. The parameter δ_2 need not be very small. This is because $\hat{\chi}_2(t)$ is much larger than $|(s+\lambda)\sigma_1(t)|$ (see Corollary 3). If the parameters δ_1 and δ_2 are chosen to be much smaller, the output tracking performance may become much improved.

Remark 9: Even though some dynamics are included in $\overline{v}(t)$ (actually, it is not bounded in the open-loop system), it was simply assumed to be bounded. The reason for dealing with the disturbance in this way is based on the adaptation algorithm of estimating the upper bound of the disturbance. Thus, a wide range of disturbance can be assumed to be bounded and simplify by using the proposed algorithm.

V. Conclusion

In this brief, a new nonlinear output tracking control is proposed for minimum phase dynamical systems with unknown parameters and unmatched disturbances. The unmatched disturbance is composed of a bounded part and a class of unmodeled dynamics. The perfect *a priori* knowledge concerning the disturbance bounds is unknown. The considered system could have higher relative degree. The proposed formulation is inspired by the backstepping method. The design procedure of the new nonlinear controller can be summarized as three steps. Firstly, a special signal (which can be regarded as the estimate of a filter of the input signal) is generated. Secondly, the derivatives up to a certain order of the special signal are derived. Finally, the output tracking control input is determined based on the derived derivatives of the special signal. The new robust control law ensures the uniform boundedness of all the signals in the closed-loop system. The output tracking error can be controlled as small as necessary by choosing the design parameters. Simulation results show the effectiveness of the proposed algorithm.

Appendix A
Proof of Lemma 1

Consider the Lyapunov candidate

$$L_2(t) = \frac{1}{2}(\sigma_1(t) - \hat{\sigma}_1(t))^2 + \frac{1}{2\alpha_{21}}\left(\hat{K}_{21}(t) - K_1\right)^2 + \frac{1}{2\alpha_{22}}\left(\hat{K}_{22}(t) - K_2\right)^2. \quad (59)$$

If $|\sigma_1(t) - \hat{\sigma}_1(t)| > (1/\bar{b}_r)\sqrt{2\delta_2/\lambda}$, then differentiating $L_2(t)$ yields

$$\begin{aligned}\dot{L}_2(t) =& (\sigma_1(t)-\hat{\sigma}_1(t))(-\lambda(\sigma_1(t)-\hat{\sigma}_1(t)) + (s+\lambda)\sigma_1(t) - \sigma_2(t)) \\ &+ \left(\hat{K}_{21}(t) - K_1\right)|\sigma_1(t) - \hat{\sigma}_1(t)|\chi_{22}(y, \overline{u}) \\ &+ \left(\hat{K}_{22}(t) - K_2\right)|\sigma_1(t) - \hat{\sigma}_1(t)|\chi_{22}(y, \overline{u})\rho(y) \\ =& -\lambda(\sigma_1(t) - \hat{\sigma}_1(t))^2 + (\sigma_1(t) - \hat{\sigma}_1(t))((s+\lambda)\sigma_1(t)) \\ &- |\sigma_1(t) - \hat{\sigma}_1(t)|\chi_2(t) + |\sigma_1(t) - \hat{\sigma}_1(t)|\hat{\chi}_2(t) \\ &- \frac{\hat{\chi}_2^2(t)(\sigma_1(t) - \hat{\sigma}_1(t))^2}{\hat{\chi}_2(t)|\sigma_1(t) - \hat{\sigma}_1(t)| + \frac{\delta_2}{\bar{b}_r^2}} \\ \le& -\lambda(\sigma_1(t) - \hat{\sigma}_1(t))^2 + \frac{\hat{\chi}_2(t)|\sigma_1(t) - \hat{\sigma}_1(t)|\frac{\delta_2}{\bar{b}_r^2}}{\hat{\chi}_2(t)|\sigma_1(t) - \hat{\sigma}_1(t)| + \frac{\delta_2}{\bar{b}_r^2}} \\ \le& -\lambda(\sigma_1(t) - \hat{\sigma}_1(t))^2 + \frac{\delta_2}{\bar{b}_r^2} \\ <& -\frac{\delta_2}{\bar{b}_r^2}. \end{aligned} \quad (60)$$

Thus, $L_2(t)$ will decrease monotonically at a speed faster than $\delta_2/\bar{b}_r^2$. Therefore, it can be seen that the condition $|\sigma_1(t) - \hat{\sigma}_1(t)| \le (1/\bar{b}_r)\sqrt{2\delta_2/\lambda}$ can be satisfied in finite time. Thus, there exists $t_1 > t_0$ such that

$$|\sigma_1(t) - \hat{\sigma}_1(t)| \le \frac{1}{\bar{b}_r}\sqrt{\frac{2\delta_1}{\lambda}} \quad (61)$$

for $t > t_1$, and $L_2(t)$ [i.e., $|\sigma_1(t) - \hat{\sigma}_1(t)|$, $\hat{K}_{21}(t)$ and $\hat{K}_{22}(t)$] is uniformly bounded for $t_0 \le t \le t_1$. By (29) and (30), it can be seen that $\hat{K}_{21}(t) = \hat{K}_{21}(t_1)$ and $\hat{K}_{22}(t) = \hat{K}_{22}(t_1)$ for $t > t_1$. Thus, it can be concluded that $|\sigma_1(t) - \hat{\sigma}_1(t)|$, $\hat{K}_{21}(t)$ and $\hat{K}_{22}(t))$ are uniformly bounded for all $t \ge t_0$.

Appendix B
Proof of Corollary 1

Let

$$\gamma_{i+1}(t) = \hat{\chi}_{i+1}(t) - |(s+\lambda)\sigma_i(t)|. \quad (62)$$

Then, for $t > t_i$, differentiating $(1/2)(\sigma_i(t) - \hat{\sigma}_i(t))^2$ yields

$$\begin{aligned}\frac{d}{dt}&\left(\frac{1}{2}(\sigma_i(t) - \hat{\sigma}_i(t))^2\right) \\ &= (\sigma_i(t) - \hat{\sigma}_i(t))(-\lambda(\sigma_i(t) - \hat{\sigma}_i(t)) + (s+\lambda)\sigma_i(t) - \sigma_{i+1}(t)) \\ &= -\lambda(\sigma_i(t) - \hat{\sigma}_i(t))^2 + (\sigma_i(t) - \hat{\sigma}_i(t))((s+\lambda)\sigma_i(t)) \\ &\quad - |\sigma_i(t) - \hat{\sigma}_i(t)|\hat{\chi}_{i+1}(t) + |\sigma_i(t) - \hat{\sigma}_i(t)|\hat{\chi}_{i+1}(t) \\ &\quad - \frac{\hat{\chi}_{i+1}^2(t)(\sigma_i(t) - \hat{\sigma}_i(t))^2}{\hat{\chi}_{i+1}(t)|\sigma_i(t) - \hat{\sigma}_i(t)| + \frac{\delta_{i+1}}{\bar{b}_r^2}} \\ &\le -\lambda(\sigma_i(t) - \hat{\sigma}_i(t))^2 - \gamma_{i+1}(t)|\sigma_i(t) - \hat{\sigma}_i(t)| + \frac{\delta_{i+1}}{\bar{b}_r^2}. \end{aligned} \quad (63)$$

From (63), it can be seen that if

$$|\sigma_i(t) - \hat{\sigma}_i(t)| > \frac{1}{2}\left(\sqrt{\frac{\gamma_{i+1}^2(t)}{\lambda^2} + \frac{4\delta_{i+1}}{\lambda\bar{b}_r^2}} - \frac{\gamma_{i+1}(t)}{\lambda}\right)$$

then

$$\frac{d}{dt}\left(\frac{1}{2}(\sigma_i(t) - \hat{\sigma}_i(t))^2\right) < 0 \quad (64)$$

i.e., $|\sigma_i(t) - \hat{\sigma}_i(t)|$ will decrease monotonically until the condition

$$\begin{aligned}|\sigma_i(t) - \hat{\sigma}_i(t)| &\le \frac{1}{2}\left(\sqrt{\frac{\gamma_{i+1}^2(t)}{\lambda^2} + \frac{4\delta_{i+1}}{\lambda\bar{b}_r^2}} - \frac{\gamma_{i+1}(t)}{\lambda}\right) \\ &= \frac{2\delta_{i+1}}{\lambda\bar{b}_r^2} \cdot \frac{1}{\sqrt{\frac{\gamma_{i+1}^2(t)}{\lambda^2} + \frac{4\delta_{i+1}}{\lambda\bar{b}_r^2}} + \frac{\gamma_{i+1}(t)}{\lambda}} \end{aligned} \quad (65)$$

is satisfied. If $\gamma_{i+1}(t)$ is very large, from (65), it can be concluded that $|\sigma_i(t) - \hat{\sigma}_i(t)|$ will be very small in finite time.

Appendix C
Proof of Theorem 1

For $i = 2, \ldots, r$, the uniform boundedness of $\hat{K}_{i1}(t)$ and $\hat{K}_{i2}(t)$ is proved in Lemma 2. If the input is chosen as (43), then

$$\overline{u}(t) = \frac{1}{(s+\lambda)^{r-1}}\sigma_r(t). \quad (66)$$

Consider the Lyapunov candidate

$$V(t) = \frac{1}{2}(y(t) - y_d(t))^2 + \frac{1}{2\alpha_{11}}\left(\hat{K}_{11}(t) - K_1\right)^2 + \frac{1}{2\alpha_{12}}\left(\hat{K}_{12}(t) - K_2\right)^2. \quad (67)$$

For $t \ge \tau_1$, if

$$|y(t) - y_d(t)| > \sum_{i=1}^{r}\lambda^{-i+1}\sqrt{\frac{2\delta_i}{\lambda}}$$

then, differentiating $V(t)$ along equation (10) yields

$$\begin{aligned}\dot{V}(t) =& (y(t)-y_d(t))\Bigg(-\lambda y(t) + \frac{(s+\lambda)d_2(s)+\lambda s^{n-1}}{d(s)}\, y(t) \\ &+ \theta^T\phi(t) + b_r\overline{u}(t) - \frac{b_r d_3(s)}{d_1(s)}\,\overline{u}(t) + \overline{\overline{v}}(t) - \dot{y}_d(t)\Bigg) \\ &+ \Big(\hat{K}_{11}(t) - K_1\Big)|y(t)-y_d(t)| \\ &+ \Big(\hat{K}_{12}(t) - K_2\Big)|y(t)-y_d(t)|\rho(y) \\ =& -\lambda(y(t)-y_d(t))^2 + b_r(y(t)-y_d(t))(\overline{u}(t)-\sigma_1(t)) \\ &+ (y(t)-y_d(t))\Bigg(\frac{(s+\lambda)d_2(s)+\lambda s^{n-1}}{d(s)}\, y(t) + \theta^T\phi(t) \\ &+ b_r\sigma_1(t) - \frac{b_r d_3(s)}{d_1(s)}\,\overline{u}(t) + \overline{\overline{v}}(t) - \dot{y}_d(t) - \lambda y_d(t)\Bigg) \\ &+ \Big(\hat{K}_{11}(t) - K_1\Big)|y(t)-y_d(t)| \\ &+ \Big(\hat{K}_{12}(t) - K_2\Big)|y(t)-y_d(t)|\rho(y) \\ =& -\lambda(y(t)-y_d(t))^2 + b_r(y(t)-y_d(t))(\overline{u}(t)-\sigma_1(t)) \\ &+ (y(t)-y_d(t))\cdot\Bigg(\frac{(s+\lambda)d_2(s)+\lambda s^{n-1}}{d(s)}\, y(t) \\ &+ \theta^T\phi(t) - \dot{y}_d(t) - \lambda y_d(t)\Bigg) - |y(t)-y_d(t)|\omega_1(t) \\ &+ (y(t)-y_d(t))\overline{\overline{v}}(t) - |y(t)-y_d(t)|(K_1+K_2\rho(y)) \\ &+ |y(t)-y_d(t)|\omega_1(t) + |y(t)-y_d(t)| \\ &\cdot\Big(\hat{K}_{11}(t) + \hat{K}_{12}(t)\rho(y)\Big) \\ &- \frac{b_r}{\underline{b}_r}\,\frac{\hat{\chi}_1^2(t)(y(t)-y_d(t))^2}{\hat{\chi}_1(t)|y(t)-y_d(t)|+\delta_1} \\ \le& -\lambda(y(t)-y_d(t))^2 + b_r(y(t)-y_d(t))(\overline{u}(t)-\sigma_1(t)) \\ &+ |y(t)-y_d(t)|\hat{\chi}_1(t) - \frac{\hat{\chi}_1^2(t)(y(t)-y_d(t))^2}{\hat{\chi}_1(t)|y(t)-y_d(t)|+\delta_1} \\ \le& -\lambda(y(t)-y_d(t))^2 + b_r(y(t)-y_d(t)) \\ &\cdot\left(\frac{1}{(s+\lambda)^{r-1}}\,\sigma_r(t) - \sigma_1(t)\right) + \delta_1. \end{aligned} \tag{68}$$

By applying Lemma 3, it yields

$$\dot{V}(t) \le -\lambda(y(t)-y_d(t))^2 + \lambda^2|y(t)-y_d(t)| \sum_{i=2}^{r} \lambda^{-i}\sqrt{\frac{2\delta_i}{\lambda}} + \delta_1. \tag{69}$$

Under the condition

$$|y(t)-y_d(t)| > \sum_{i=1}^{r} \lambda^{-i+1}\sqrt{\frac{2\delta_i}{\lambda}}$$

from (69), it can be easily proved that

$$\dot{V}(t) \le -\delta_1. \tag{70}$$

Thus, $V(t)$ will decrease monotonically at a speed faster than δ_1. Therefore, it can be seen that the condition

$$|y(t)-y_d(t)| \le \sum_{i=1}^{r} \lambda^{-i+1}\sqrt{\frac{2\delta_i}{\lambda}}$$

can be satisfied in finite time. Thus, there exists $T > \tau_1$ such that

$$|y(t)-y_d(t)| \le \sum_{i=1}^{r} \lambda^{-i+1}\sqrt{\frac{2\delta_i}{\lambda}} \tag{71}$$

for $t > T$, and $V(t)$ [i.e., $|y(t)|$, $\hat{K}_{11}(t)$ and $\hat{K}_{12}(t)$] is uniformly bounded for $t_0 \le t \le T$. By (21) and (22), it can be seen that $\hat{K}_{11}(t) = \hat{K}_{11}(T)$ and $\hat{K}_{12}(t) = \hat{K}_{12}(T)$ for $t > T$. Thus, it can be concluded that $y(t)$, $\hat{K}_{11}(t)$ and $\hat{K}_{12}(t)$ are uniformly bounded for all $t \ge t_0$. Therefore, by assumption (A5), it can be concluded that the unknown signal $\overline{v}(t)$ is in fact a bounded signal in the closed loop.

Now, the uniform boundedness of $\sigma_1(t)$ is shown. By choosing the monic Hurwitz polynomial

$$f(s) = \frac{1}{b_r}\, b(s)(s+\lambda)^r \tag{72}$$

and (1) can be rewritten as

$$\frac{a(s)}{f(s)}\, y(t) = \frac{b_r}{s+\lambda}\,\overline{u}(t) + \frac{d(s)}{f(s)}\,\overline{v}(t) + \varepsilon_2(t) \tag{73}$$

where $\varepsilon_2(t)$ is an exponentially decaying term which arises from the nonzero initial conditions [see also (7)]. By using the uniform boundedness of $y(t)$ and $\overline{v}(t)$, from (73), it can be concluded that $(1/(s+\lambda))\overline{u}(t)$ is also uniformly bounded by observing that $a(s)/f(s)$ is proper and $d(s)/f(s)$ is strictly proper. Thus, for any positive integer i, it can be concluded that $(1/(s+\lambda)^i)\overline{u}(t)$ are uniformly bounded. Therefore, by (14), it can be seen that $\phi(t)$ is uniformly bounded. Further, $\omega_1(t)$ is uniformly bounded. By the definition of $\hat{\chi}_1(t)$ in (20), it is obvious that $\hat{\chi}_1(t)$ is uniformly bounded. By the definition of $\sigma_1(t)$ in (19), it is obvious that $\sigma_1(t)$ is uniformly bounded. By Lemma 1, it can be seen that $\hat{\sigma}_1(t)$ is also uniformly bounded.

By employing the uniform boundedness of $\sigma_1(t)$ and Lemma 3 (for the case $i=1$), it can be seen that $(1/(s+\lambda)^{r-1})\sigma_r(t)$, i.e., $\overline{u}(t)$, is uniformly bounded. Therefore, $\chi_{21}(y, \overline{u})$ and $\chi_{22}(y, \overline{u})$ are uniformly bounded. By the definition of $\hat{\chi}_2(t)$ in (28), it is obvious that $\hat{\chi}_2(t)$ is uniformly bounded. So, $\sigma_2(t)$ is uniformly bounded. By Lemma 2, it can be seen that $\hat{\sigma}_2(t)$ is also uniformly bounded.

By employing the uniform boundedness of $\sigma_2(t)$ and Lemma 3 (for the case $i=2$), it can be seen that $(1/(s+\lambda)^{r-2})\sigma_r(t)$, i.e., $\dot{\overline{u}}(t) + \lambda\overline{u}(t)$, is uniformly bounded. Thus, by using the uniform boundedness of $\overline{u}(t)$, it can be seen that $\dot{\overline{u}}(t)$ is uniformly bounded. Therefore, $\chi_{31}(y, \dot{\overline{u}})$ and $\chi_{32}(y, \dot{\overline{u}})$ are uniformly bounded. Thus, $\hat{\chi}_3(t)$ is uniformly bounded. So, $\sigma_3(t)$ is uniformly bounded. By Lemma 2, it can be seen that $\hat{\sigma}_3(t)$ is also uniformly bounded.

By forwarding the analysis to the last step, it can be proved that $\sigma_i(t)$ (for $i = 1, \ldots, r$), $\hat{\sigma}_i(t)$ (for $i = 1, \ldots, r-1$) and $\hat{\chi}_i(t)$ (for $i = 1, \ldots, r$) are all uniformly bounded. Therefore, $u(t) = \sigma_r(t)$ is uniformly bounded. The theorem is proved.

REFERENCES

[1] X. Chen and T. Fukuda, "Model reference robust control for SISO systems with unmatched disturbances," *Int. J. Control*, vol. 73, pp. 666–677, 2000.

[2] X. Chen, T. Fukuda, and K. K. Young, "A new nonlinear robust disturbance observer," *Syst. Control Lett.*, vol. 41, pp. 189–199, 2000.

[3] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction and Control*. Englewood Cliffs, NJ: Prentice-Hall, 1984.

[4] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.

[5] I. M. Y. Mareels, "A simple selftuning controller for stably invertible systems," *Syst. Control Lett.*, vol. 4, pp. 5–16, 1984.

[6] D. E. Miller and E. J. Davison, "An adaptive controller which provides an arbitrarily good transient and steady-state response," *IEEE Trans. Automat. Control*, vol. 36, pp. 68–81, 1991.

[7] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[8] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence, and Robustness*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[9] W. E. Schmitendorf and B. R. Barmish, "Robust asymptotic tracking for linear systems with unknown parameters," *Automatica*, vol. 22, pp. 355–360, 1986.

[10] ——, "Guaranteed asymptotic output stability for systems with constant disturbance," *ASME J. Dyn. Syst. Measur. Control*, vol. 109, pp. 186–189, 1987.

[11] J. J. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1991.

[12] V. I. Utkin, *Sliding Modes in Control Optimization*. New York: Springer-Verlag, 1992.

[13] K. D. Young, V. I. Utkin, and U. Ozguner, "A control engineer's guide to sliding mode control," *IEEE Trans. Contr. Syst. Technol.*, vol. 7, pp. 328–324, May 1999.

[14] J. Y. Zhu, S. C. Martindale, and D. M. Dawson, "A new design for model reference robust control using state-space techniques," *Int. J. Control*, vol. 62, pp. 1061–1084, 1995.

[15] A. S. I. Zinober, Ed., *Variable Structure and Lyapunov Control*. London, U.K.: Springer-Verlag, 1994.