problem also led to a very intuitive proof of the famous RH stability criterion.

APPENDIX
PROOF OF LEMMA 2

Denote $M_n = \mathrm{diag}(1, K_2, K_2K_3, \ldots, K_2K_3 \ldots K_n)$. It is easy to verify, using (1), that

$$M_nA_nM_n^{-1} = \begin{pmatrix} 0 & 1 & & & & & & \\ -K_1K_2 & 0 & 1 & & & & & \\ & -K_2K_3 & 0 & 1 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & -K_{n-2}K_{n-1} & 0 & 1 \\ & & & & & -K_{n-1}K_n & -K_n \end{pmatrix}.$$

Reference [5, Th. 5] implies that the first column of the RH array of the polynomial $\det(sI - M_nA_nM_n^{-1})$ is: $(1, K_n, K_nK_{n-1}, K_nK_{n-1}K_{n-2}, \ldots, K_nK_{n-1}K_{n-2} \ldots K_1)$. But,

$$\begin{aligned} &\det(sI - M_nA_nM_n^{-1}) \\ &= \det(M_n(sI - A_n)M_n^{-1}) \\ &= \det(M_n)\,\det(sI - A_n)\,\det(M_n^{-1}) \\ &= \det(sI - A_n) = P^n(s) \end{aligned}$$

and the proof is completed. □

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Robust Adaptive Control of a Class of Nonlinear Systems with Unknown Backlash-Like Hysteresis

Chun-Yi Su, Yury Stepanenko, Jaroslav Svoboda, and T. P. Leung

***Abstract*—This paper deals with adaptive control of a class of nonlinear dynamic systems preceded by unknown backlash-like hysteresis nonlinearities, where the hysteresis is modeled by a differential equation. By exploiting solution properties of the differential equation and combining those properties with adaptive control techniques, a robust adaptive control algorithm is developed without constructing a hysteresis inverse. The new control law ensures global stability of the adaptive system and achieves both stabilization and tracking to within a desired precision. Simulations performed on a nonlinear system illustrate and clarify the approach.**

***Index Terms*—Adaptive control, cascade systems, hysteresis, nonlinear systems, robust control.**

I. INTRODUCTION

Hysteresis is a property of a wide range of physical systems and devices, such as electro-magnetic fields, mechanical actuators, and electronic relay circuits. Control of a system is typically challenging in the presence of hysteresis nonlinearities. They are nondifferentiable nonlinearities and severely limit system performance in such manners as giving rise to undesirable inaccuracies or oscillations, which can even lead to instability [14]. The development of control techniques to mitigate effects of unknown hystereses has been studied for decades and has recently re-attracted significant attention [1], [2], [4], [10], [12]–[15]. Much of this renewed interest is a consequence of its importance in present day applications. Interest in studying dynamic systems with hysteresis is also motivated by their role as nonlinear systems with *hard nonlinearities* for which traditional control methods are insufficient and so requiring development of new approaches [4].

To address such a challenge, it is important to find a model describing the nonlinear behavior and to utilize this model for controller design. Various models have been proposed to describe the hysteresis [7], for example, the Preisach model [8], Krasnosel'skii–Pokrovkii hysteron [6], Ishlinskii hysteresis operator [6], and the Duhem hysteresis operator [7]. The most familiar and simple model perhaps is the one for a backlash hysteresis described by two parallel lines connected via horizontal line segments. This being said, it is important to mention that modeling a general type of hysteresis is in itself still a research topic and the reader may refer to [7] for a recent review.

Apart from the latter backlash hysteresis model, the above models are very complicated and it is still unclear how to fuse them together with the controller design. Focusing on backlash hysteresis, several adaptive control schemes have recently been proposed (this includes [14], [15], [1], and [12]) to deal with unknown backlash hysteresis. A common feature of those schemes is that they rely on the construction

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of an inverse hysteresis to mitigate the effects of the hysteresis. These results, especially [14] and [15], provide a theoretic framework which can serve as a base for future research.

Inspired by the above research, this paper defines a dynamic hysteresis model to pattern a backlash-like hysteresis. Rather than constructing an inverse hysteresis nonlinearity to mitigate the effects of the hysteresis, we propose a new approach for controller synthesis by using the properties of the hysteresis model. A robust adaptive controller is developed specifically for a class of nonlinear systems preceeded by an unknown backlash-like hysteresis. The new control law ensures global stability of the adaptive system and achieves both stabilization and strict tracking precision. Simulations performed on a nonlinear system illustrate and clarify the approach. We should mention that the proposed method can be thought of as a preliminary step to the fusion of complicated general hysteresis models with controller design.

II. Problem Statement

The controlled system consists of a nonlinear plant preceded by a backlash-like hysteresis actuator, that is, the hysteresis is present as an input of the nonlinear plant. It is a challenging task of major practical interests to develop a control scheme for unknown backlash-like hysteresis. The development of such a control scheme will now be pursued.

A backlash-like hysteresis nonlinearity can be denoted as an operator

$$w(t) = P(v(t)) \tag{1}$$

with $v(t)$ as input and $w(t)$ as output. The operator $P(v(t))$ will be discussed in detail in the forthcoming section. The nonlinear dynamic system being preceded by the above hysteresis is described in the canonical form

$$x^{(n)}(t) + \sum_{i=1}^{r} a_i Y_i\Big(x(t), \dot{x}(t), \ldots, x^{(n-1)}(t)\Big) = bw(t) \tag{2}$$

where Y_i are known continuous, linear, or nonlinear functions. Parameters a_i and control gain b are unknown but constant. It is a common assumption that the sign of b is known. From this point onward, without losing generality, we shall assume $b > 0$. It should be noted that more general classes of nonlinear systems can be transformed into this structure [5].

The control objective is to design a control law for $\mathbf{v(t)}$ in (1), to force the plant state vector, $\mathbf{x} = [x, \dot{x}, \ldots, x^{(n-1)}]^T$, to follow a specified desired trajectory, $\mathbf{x}_d = [x_d, \dot{x}_d, \ldots, x_d^{(n-1)}]^T$, i.e., $\mathbf{x} \rightarrow \mathbf{x}_d$ as $t \rightarrow \infty$.

III. Backlash-Like Hysteresis Model and Its Properties

Traditionally, a backlash hysteresis nonlinearity can be described by

$$w(t) = P(v(t)) = \begin{cases} c(v(t) - B), & \text{if } \dot{v}(t) > 0 \text{ and } w(t) = c(v(t) - B) \\ c(v(t) + B), & \text{if } \dot{v}(t) < 0 \text{ and } w(t) = c(v(t) + B) \\ w(t_-), & \text{otherwise} \end{cases} \tag{3}$$

where $c > 0$ is the slope of the lines and $B > 0$ is the backlash distance. This model is itself discontinuous and may not be amenable to controller design for the nonlinear systems (2).

Instead of using the above model, in this paper we define a continuous-time dynamic model to describe a class of backlash-like hysteresis, as given by

$$\frac{dw}{dt} = \alpha \left| \frac{dv}{dt} \right| (cv - w) + B_1 \frac{dv}{dt} \tag{4}$$

where α, c, and B_1 are constants, satisfying $c > B_1$.

Remark: Other dynamic models for hystereses exist in the literaure [3]. Generally, modeling hysteresis noninearities is still a research topic and the reader may refer to [7] for a recent review.

We shall now examine the solution properties of the dynamic model (4) and explain the corresponding switching mechanism, which is crucial for design of the controller. Equation (4) can be solved explicitly for v piecewise monotone

$$w(t) = cv(t) + d(v) \tag{5}$$

with

$$d(v) = [w_o - cv_o]e^{-\alpha(v - v_o)\,\mathrm{sgn}\,\dot{v}} + e^{-\alpha v\,\mathrm{sgn}\,\dot{v}} \int_{v_o}^{v} [B_1 - c] e^{\alpha\zeta(\mathrm{sgn}\,\dot{v})}\, d\zeta$$

for $\dot{v}$ constant and $w(v_o) = w_0$. Analyzing (5), we see that it is composed of a line with the slope $\mathbf{c}$, together with a term $d(v)$. For $d(v)$, it can be easily shown that if $w(v; v_o, w_o)$ is the solution of (5) with initial values (v_o, w_o), then, if $\dot{v} > 0$ ($\dot{v} < 0$) and $v \rightarrow +\infty$ $(-\infty)$, one has

$$\lim_{v\to\infty} d(v) = \lim_{v\to\infty} [w(v; v_o, w_o) - f(v)] = -\frac{c - B_1}{\alpha} \tag{6}$$

$$\left(\lim_{v\to-\infty} d(v) = \lim_{v\to-\infty} [w(v; v_o, w_o) - f(v)] = \frac{c - B_1}{\alpha} \right). \tag{7}$$

It should be noted that the above convergence is exponential at the rate of α. Solution (5) and properties (6) and (7) show that $w(t)$ eventually satisfies the first and second conditions of (3). Furthermore, setting $\dot{v} = 0$ results in $\dot{w} = 0$ which satisfies the last condition of (3). This implies that the dynamic equation (4) can be used to model a class of backlash-like hystereses and is an approximation of backlash hysteresis (3).

Let us use an example for specified initial data to show the switching mechanism for the dynamic model (4) when $\dot{v}$ changes direction. We note that when $\dot{v} > 0$ on $w(0) = 0$ and $v(0) = 0$, (5) gives

$$w(t) = cv(t) - \frac{c - B_1}{\alpha}\left(1 - e^{-\alpha v(t)}\right) \quad \text{for } v(t) \geq 0 \text{ and } \dot{v} > 0. \tag{8}$$

Let v_s be a positive value of v and consider now a specimen such that v is increasing along the initial curve (8) until a time t_s at which v reaches the level v_s. Suppose now that from the time t_s, the signal v is decreased. In this case, w is given by

$$w(t) = cv(t) + \frac{c - B_1}{\alpha}\left[1 - \left(2e^{-\alpha v_s} - e^{-2\alpha v_s}\right) e^{\alpha v(t)}\right] \quad \text{for } \dot{v} < 0 \tag{9}$$

where $v < v_s$. Equations (8) and (9) indeed show that w switches exponentially from the line $cv(t) - ((c - B_1)/\alpha)$ to $cv(t) + ((c - B_1)/\alpha)$ to generate backlash-like hysteresis curves.

To confirm the above analysis, the solutions of (4) can be obtained by numerical integration with v as the independent variable. Fig. 1 shows that model (4) indeed generates backlash-like hysteresis curves, which confirms the above analysis. The details are described in the section of simulation studies. It should be mentioned that the parameter α determines the rate at which $w(t)$ switches between $-((c - B_1)/\alpha)$ and $((c - B_1)/\alpha)$. The larger the parameter α is, the faster the transition in $w(t)$ is going to be. However, the backlash distance is determined by $((c - B_1)/\alpha)$ and the parameter must satisfy $c > B_1$. Therefore, the parameter α cannot be chosen freely. A compromise should be made in choosing a suitable parameter set $\{\alpha, c, B_1\}$ to model the required shape of backlash-like hysteresis. If the values of the backlash slope

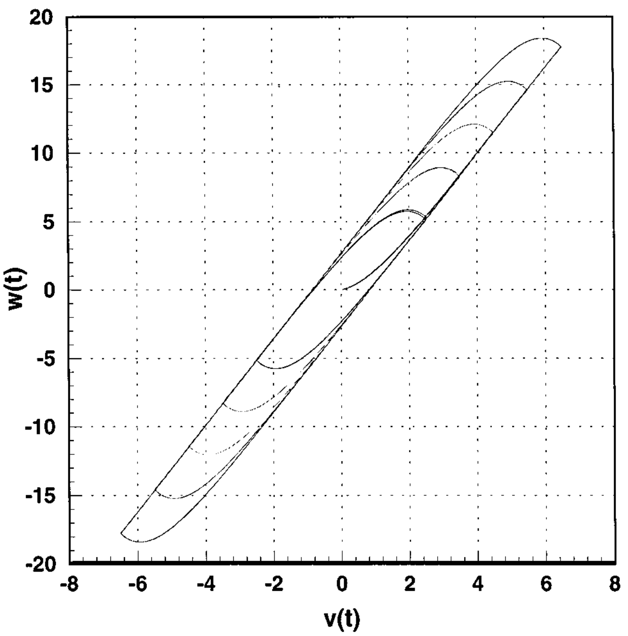

Fig. 1. Hysteresis curves given by (4) or (31) with $\alpha = 1$, $c = 3.1635$, and $B_1 = 0.345$ for $v(t) = k \sin(2.3t)$ with k = 2.5, 3.5, 4.5, 5.5, and 6.5.

and distance are not known implicitly, then adaptations will be used to estimate them. This topic will be clarified shortly.

IV. Adaptive Controller Design

From the solution structure (5) of the model (4) we see that the signal $w(t)$ is expressed as a linear function of input signal $v(t)$ plus a bounded term. In this case, the currently available robust adaptive control techniques can be utilized for the controller design. In this section, we shall propose an adaptive controller for plants of the form described by (2), preceded by the hysteresis described in (4). The proposed controller will lead to global stability and yields tracking to within a desired precision.

Using the solution expression (5), (2) becomes

$$\begin{aligned} x^{(n)}(t) + \sum_{i=1}^{r} a_i Y_i(x(t), \dot{x}(t), \ldots, x^{(n-1)}(t)) \\ = bcv(t) + bd(v(t)) \end{aligned} \tag{10}$$

which results in a linear relation to the input signal $v(t)$. It is very important to note that (6) or (7) imply that there exists a uniform bound ρ such that

$$\|d(v)\| \leq \rho. \tag{11}$$

For the development of a control law, the following assumptions regarding the plant and hysteresis are made.

A1) There exist known constants $b_{\min}$ and $b_{\max}$ such that the control gain b in (2) satisfies $b \in [b_{\min}, b_{\max}]$.

A2) There exist known constants $c_{\min}$ and $c_{\max}$ such that the slope c in (3) satisfies $c \in [c_{\min}, c_{\max}]$.

A3) Define $\theta \triangleq [(a_1/bc), \ldots, (a_r/bc)]^T \in R^r$, then

$$\theta \in \Omega_\theta \triangleq \{\theta \colon \theta_{i\min} \leq \theta_i \leq \theta_{i\max}, \quad \forall i \in \{1, r\}\}$$

where $\theta_{i\min}$ and $\theta_{i\max}$ are some known real numbers.

A4) The bound ρ for the relation $\|d(v)\| \leq \rho$ is known.

A5) The desired trajectory, $\mathbf{x}_d = [x_d, \dot{x}_d, \ldots, x_d^{(n-1)}]^T$ is continuous and available. Furthermore, $[\mathbf{x}_d^T, x_d^{(n)}]^T \in \Omega_d \subset R^{n+1}$ with Ω_d a compact set.

Remark: Assumption A1) is common for the nonlinear system described by (2) [11]. Assumption A2) assumes the slope range of a backlash hysteresis nonlinearity, which is reasonable. In Assumption A3), a new parameter vector θ has been defined for the convenience of further development. Basically, Assumption A3) implies that the ranges of the plant parameters, $a_i, i = 1 \ldots r$, are known in advance. This is a reasonable assumption concerning the prior knowledge of the system. Assumption A4) requires knowledge in regards to the upper bound of the hysteresis loop, which is again quite reasonable and practical. Assumption A5) poses a restriction on the types of reference signals which may be used.

In presenting the developed robust adaptive control law, the following definitions are required:

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_d \qquad \tilde{\theta} = \hat{\theta} - \theta \qquad \tilde{\phi} = \hat{\phi} - \phi \tag{12}$$

where

$\tilde{\mathbf{x}}$ represents the tracking error vector,
$\hat{\theta}$ is an estimate of θ as defined in Assumption A2), and
$\hat{\phi}$ is an estimate of ϕ, which is defined as $\phi \triangleq (bc)^{-1}$.

A filtered tracking error is defined as

$$s(t) = \left(\frac{d}{dt} + \lambda\right)^{(n-1)} \tilde{x}(t) \qquad \text{with } \lambda > 0 \tag{13}$$

which can be rewritten as $s(t) = \Lambda^T \tilde{\mathbf{x}}(t)$ with $\Lambda^T = [\lambda^{(n-1)}, (n-1)\lambda^{(n-2)}, \ldots, 1]$.

Remark: It has been shown in [11] that the definition (13) has the following properties: i) the equation $s(t) = 0$ defines a time-varying hyperplane in R^n on which the tracking error vector $\tilde{\mathbf{x}}(t)$ decays exponentially to zero; ii) if $\tilde{\mathbf{x}}(0) = 0$ and $|s(t)| \leq \epsilon$, where ϵ is a constant, then $\tilde{\mathbf{x}}(t) \in \Omega_\epsilon \triangleq \{\tilde{\mathbf{x}}(t) \,|\, |\tilde{\mathbf{x}}_i| \leq 2^{i-1}\lambda^{i-n}\epsilon, i = 1, \ldots, n\}$ for $\forall t \geq 0$; and iii) if $\tilde{\mathbf{x}}(0) \neq 0$ and $|s(t)| \leq \epsilon$, then $\tilde{\mathbf{x}}(t)$ will converge to Ω_ϵ within a time-constant $(n-1)/\lambda$.

Rather than driving the adaptive law with the filtered error $s(t)$, we prefer to introduce a tuning error, s_ϵ, as follows:

$$s_\epsilon = s - \epsilon \operatorname{sat}\left(\frac{s}{\epsilon}\right) \tag{14}$$

where ϵ is an arbitrary positive constant and $\operatorname{sat}(\cdot)$ is the saturation function.

Remark: The tuning error s_ϵ disappears when the filtered error s is less than ϵ. This shall be the equivalent of creating an adaptation deadband.

Given the plant and hysteresis models subject to the assumption described above, the following control and adaptation laws are presented:

$$v(t) = -k_d s(t) + \hat{\phi} u_{fd}(t) + Y^T(\mathbf{x})\hat{\theta} - k^* \operatorname{sat}\left(\frac{s}{\epsilon}\right) \tag{15}$$

$$u_{fd}(t) = x_d^{(n)}(t) - \Lambda_v^T \tilde{\mathbf{x}}(t) \tag{16}$$

$$\dot{\hat{\theta}} = \operatorname{Proj}\left(\hat{\theta}, -\gamma Y(\mathbf{x}) s_\epsilon\right) \tag{17}$$

$$\dot{\hat{\phi}} = \operatorname{Proj}\left(\hat{\phi}, -\eta u_{fd} s_\epsilon\right) \tag{18}$$

where $Y \triangleq [Y_1, \ldots, Y_r]^T \in R^r$; $\Lambda_v^T = [0, \lambda^{(n-1)}, (n-1)\lambda^{(n-2)}, \ldots, (n-1)\lambda]$; k^* is a control gain, satisfying $k^* \geq \rho/c_{\min}$, whereby, ρ is defined in (11). In addition, the parameters γ and η are positive constants determining the rates of adaptations, and $\mathrm{Proj}(\cdot,\cdot)$ is a projection operator, which is formulated as follows:

$$\{\mathrm{Proj}(\hat{\theta}, -\gamma Y s_\epsilon)\}_i = \begin{cases} 0, & \text{if } \hat{\theta}_i = \theta_{i\max} \text{ and } \gamma(Ys_\epsilon)_i < 0 \\ -\gamma(Ys_\epsilon)_i, & \text{if } [\theta_{i\min} < \hat{\theta}_i < \theta_{i\max}] \\ & \text{or } [\hat{\theta}_i = \theta_{i\max} \text{ and } \gamma(Ys_\epsilon)_i \geq 0] \\ & \text{or } [\hat{\theta}_i = \theta_{i\min} \text{ and } \gamma(Ys_\epsilon)_i \leq 0] \\ 0, & \text{if } \hat{\theta}_i = \theta_{i\min} \text{ and } \gamma(Ys_\epsilon)_i > 0 \end{cases} \tag{19}$$

$$\mathrm{Proj}(\hat{\phi}, -\eta u_{fd} s_\epsilon) = \begin{cases} 0, & \text{if } \hat{\phi}_i = \phi_{\max} \text{ and } \eta u_{fd} s_\epsilon < 0 \\ -\eta u_{fd} s_\epsilon, & \text{if } [\phi_{\min} < \hat{\phi} < \phi_{\max}] \\ & \text{or } [\hat{\phi} = \phi_{\max} \text{ and } \eta u_{fd} s_\epsilon \geq 0] \\ & \text{or } [\hat{\phi} = \phi_{\min} \text{ and } \eta u_{fd} s_\epsilon \leq 0] \\ 0, & \text{if } \hat{\phi} = \phi_{\min} \text{ and } \eta u_{fd} s_\epsilon > 0. \end{cases} \tag{20}$$

Remarks:

1) In the above control law, two projection operators have been introduced. It can be easily proved that the projection operator for $\hat{\theta}$ has the following properties: i) $\hat{\theta}(t) \in \Omega_\theta$ if $\hat{\theta}(0) \in \Omega_\theta$; ii) $\|\mathrm{Proj}(p,y)\| \leq \|y\|$; and iii) $-(p-p^*)^T \Lambda\, \mathrm{Proj}(p,y) \geq -(p-p^*)^T \Lambda y$, where Λ is a positive definite symmetric matrix. Note that those three properties are also valid for the projection operator defined for $\hat{\phi}$. The omission of these equations are in the interest of space saving.
2) The projection operators require knowledge of the parameters $\theta_{i\min}$ and $\theta_{i\max}$. These represent the upper and lower bounds of θ_i, respectively. Assumption A3) is fundental to this end. However, it should be noted that these parameters are only used to specify the range of parameter changes for the projection operator. With regards to this paper, such a range is not restricted as long as the estimated parameters are bounded (required for the stability proof); hence, one can always choose suitable $\theta_{i\min}$ and $\theta_{i\max}$, although such a choice may be conservative.
3) The term $k^*\,\mathrm{sat}(s/\epsilon)$ actually represents the compensation component for the bounded function $d(v)$. It should be noted that if ϵ is chosen too small, such that the linear region of function $\mathrm{sat}(s/\epsilon)$ is excessively "thin," the controller runs the risk of exciting high frequency dynamics. As $\epsilon \to 0$, the function $\mathrm{sat}(s/\epsilon)$ eventually becomes discontinuous. In such a case, the controller becomes a typical variable structure control scheme [16], which may lead to chattering phenomena. This suggests that a tradeoff must be made between the value of ϵ and the trajectory-following requirements.

The stability of the closed-loop system described by (2), (4), and (15)–(20) is established in the following theorem.

Theorem: For the plant in (2) with the hysteresis (4) at the input subject to Assumptions A1)–A5), the robust adaptive controller specified by (15)–(20) ensures that if $\hat{\theta}(t_0) \in \Omega_\theta$ and $\hat{\phi}(t_0) \in \Omega_\phi$, all the closed-loop signals are bounded and the state vector $\mathbf{x}(t)$ converges to $\Omega_\epsilon = \{\mathbf{x}(t) \| \tilde{\mathbf{x}}_i| \leq 2^{i-1}\lambda^{i-n}\epsilon, i = 1, \ldots, n\}$ for $\forall t \geq t_0$.

Proof: Using the expression (10), the time derivative of the filtered error (13) can be written as:

$$\dot{s}(t) = -u_{fd}(t) - \sum_{i=1}^{r} a_i Y_i(\mathbf{x}(t)) + bcv(t) + bd(v). \tag{21}$$

Using the control law (15)–(20), the above equation can be rewritten as

$$\begin{aligned}\dot{s}(t) = &-u_{fd}(t) - \sum_{i=1}^{r} a_i Y_i(\mathbf{x}(t)) \\ &+ bc\left[-k_d s(t) + \hat{\phi} u_{fd}(t) + Y^T(\mathbf{x})\hat{\theta} - k^*\,\mathrm{sat}\left(\frac{s}{\epsilon}\right)\right] \\ &+ bd(v).\end{aligned} \tag{22}$$

To establish global boundedness, we define a Lyapunov function candidate

$$V(t) = \frac{1}{2}\left[\frac{1}{bc} s_\epsilon^2 + \frac{1}{\gamma}\left(\hat{\theta}-\theta\right)^T\left(\hat{\theta}-\theta\right) + \frac{1}{\eta}\left(\hat{\phi}-\phi\right)^2\right]. \tag{23}$$

Since the discontinuity at $|s| = \epsilon$ is of the first kind and since $s_\epsilon = 0$ when $|s| \leq \epsilon$, it follows that the derivative $\dot{V}$ exists for all s, which is

$$\dot{V}(t) = 0 \qquad \text{when } |s| \leq \epsilon. \tag{24}$$

When $|s| > \epsilon$, using (22) and the fact $s_\epsilon \dot{s}_\epsilon = s_\epsilon \dot{s}$, one has

$$\begin{aligned}\dot{V}(t) &= \frac{1}{bc} s_\epsilon \dot{s} + \frac{1}{\gamma}\left(\hat{\theta}-\theta\right)^T \dot{\hat{\theta}} + \frac{1}{\eta}\left(\hat{\phi}-\phi\right)\dot{\hat{\phi}} \\ &= -k_d s_\epsilon s + s_\epsilon\left[\hat{\phi} u_{fd}(t) + Y^T(\mathbf{x})\hat{\theta} - k^*\,\mathrm{sat}\left(\frac{s}{\epsilon}\right)\right] \\ &\quad + \frac{1}{bc} s_\epsilon\left[-u_{fd}(t) - \sum_{i=1}^{r} a_i Y_i(\mathbf{x}(t)) + bd(v)\right] \\ &\quad + \frac{1}{\gamma}\left(\hat{\theta}-\theta\right)^T \dot{\hat{\theta}} + \frac{1}{\eta}\left(\hat{\phi}-\phi\right)\dot{\hat{\phi}} \\ &= -k_d s_\epsilon s + s_\epsilon\left[\hat{\phi} u_{fd}(t) + Y^T(\mathbf{x})\hat{\theta} - k^*\,\mathrm{sat}\left(\frac{s}{\epsilon}\right)\right] \\ &\quad + s_\epsilon\left[-\phi u_{fd}(t) - Y^T\theta + d(v)/c\right] \\ &\quad + \frac{1}{\gamma}\left(\hat{\theta}-\theta\right)^T \dot{\hat{\theta}} + \frac{1}{\eta}\left(\hat{\phi}-\phi\right)\dot{\hat{\phi}}.\end{aligned} \tag{25}$$

The above equation can be simplified, by the choice of s_ϵ, as

$$\begin{aligned}\dot{V}(t) \leq &-k_d s_\epsilon^2 + s_\epsilon\left[\hat{\phi} u_{fd}(t) + Y^T(\mathbf{x})\hat{\theta} - k^*\,\mathrm{sat}\left(\frac{s}{\epsilon}\right)\right] \\ &+ s_\epsilon\left[-\phi u_{fd}(t) - Y^T\theta + d(v)/c\right] \\ &+ \frac{1}{\gamma}\left(\hat{\theta}-\theta\right)^T \dot{\hat{\theta}} + \frac{1}{\eta}\left(\hat{\phi}-\phi\right)\dot{\hat{\phi}}.\end{aligned} \tag{26}$$

By using adaptive laws (17), (18), and the properties

$$\frac{1}{\gamma}(\hat{\theta}-\theta)^T\,\mathrm{Proj}(\hat{\theta}, -\gamma Y s_\epsilon) \leq -(\hat{\theta}-\theta)^T Y s_\epsilon$$

and

$$\frac{1}{\eta}(\hat{\phi}-\phi)\,\mathrm{Proj}(\hat{\phi}, -\eta u_{fd} s_\epsilon) \leq -(\hat{\phi}-\phi) u_{fd} s_\epsilon$$

one obtains

$$\begin{aligned}\dot{V}(t) \leq &-k_d s_\epsilon^2 + s_\epsilon\left[\hat{\phi} u_{fd}(t) + Y^T(\mathbf{x})\hat{\theta} - k^*\,\mathrm{sat}\left(\frac{s}{\epsilon}\right)\right] \\ &+ s_\epsilon\left[-\phi u_{fd}(t) - Y^T\theta + d(v)/c\right] - \left(\hat{\theta}-\theta\right)^T Y s_\epsilon \\ &- \left(\hat{\phi}-\phi\right) u_{fd} s_\epsilon \\ = &-k_d s_\epsilon^2 - k^* s_\epsilon\,\mathrm{sat}\left(\frac{s}{\epsilon}\right) + \frac{d(v)}{c} s_\epsilon.\end{aligned} \tag{27}$$

Since $|s_\epsilon| = s_\epsilon \operatorname{sat}(s/\epsilon)$ for $|s| > \epsilon$, the above becomes

$$\begin{aligned}\dot{V}(t) &\le -k_d s_\epsilon^2 - k^*|s_\epsilon| + \frac{d(v)}{c}\, s_\epsilon \\ &\le -k_d s_\epsilon^2 - k^*|s_\epsilon| + \frac{\rho}{c_{\min}}\, |s_\epsilon| \\ &\le -k_d s_\epsilon^2 \qquad \forall |s| > \epsilon. \end{aligned} \tag{28}$$

Equations (24) and (28) imply that V is a Lyapunov function which leads to global boundedness of s_ϵ, $(\hat{\theta} - \theta)$, and $(\hat{\phi} - \phi)$. From the definition of s_ϵ, $s(t)$ is bounded. It is easily shown that if $\tilde{\mathbf{x}}(0)$ is bounded, then $\tilde{\mathbf{x}}(t)$ is also bounded for all t, and since $\mathbf{x}_d(t)$ is bounded by design, $\mathbf{x}(t)$ must also be bounded. To complete the proof and establish asymptotic convergence of the tracking error, it is necessary to show that $s_\epsilon \to 0$ as $t \to \infty$. This is accomplished by applying Barbalat's lemma [9] to the continuous, nonnegative function

$$V_1(t) = V(t) - \int_0^t \left(\dot{V}(\tau) + k_d s_\epsilon^2(\tau)\right) d\tau$$

with

$$\dot{V}_1(t) = -k_d s_\epsilon^2(t). \tag{29}$$

It can easily be shown that every term in (22) is bounded, hence $\dot{s}$, and $\dot{s}_\epsilon$ are bounded. This implies that $\dot{V}_1(t)$ is a uniformly continuous function of time. Since V_1 is bounded below by 0, and $\dot{V}_1(t) \le 0$ for all t, use of Barbalat's lemma proves that $\dot{V}_1(t) \to 0$. Therefore, from (29) it can be demonstrated that $s_\epsilon(t) \to 0$ as $t \to \infty$. The remark following (13) indicates that $\tilde{\mathbf{x}}(t)$ will converge to Ω_ϵ. □□□

Remark: It is important to note that the backlash-like hysteresis model described by (4) can be extended for the general hysteresis nonlinearities. However, the goal of this paper is to show the controller design strategy using a dynamic hysteresis model in a simple setting that reveals its essential features. This is the motivation for simply using backlash-like hysteresis model.

V. Simulation Studies

In this section, we illustrate the above methodology on a simple nonlinear system described as

$$\dot{x} = a\,\frac{1 - e^{-x(t)}}{1 + e^{-x(t)}} + bw(t) \tag{30}$$

where $w(t)$ represents an output of hysteresis. The actual parameter values are $b = 1$ and $a = 1$. Without control, i.e., $w(t) = 0$, (30) is unstable, because $\dot{x} = (1 - e^{-x(t)}/1 + e^{-x(t)}) > 0$ for $x > 0$, and $\dot{x} = (1 - e^{-x(t)}/1 + e^{-x(t)}) < 0$ for $x < 0$. The objective is to control the system state x to follow a desired trajectory x_d, which will be specified later.

The backlash-like hysteresis is described by

$$\frac{dw}{dt} = \alpha \left|\frac{dv}{dt}\right| [cv - w] + \frac{dv}{dt}\, B_1 \tag{31}$$

with parameters $\alpha = 1$, $c = 3.1635$, and $B_1 = 0.345$. Using input signal $v(t) = k\sin(2.3t)$ with $k = 2.5, 3.5, 4.5, 5.5, 6.5$, the responses of this dynamic equation with the initial condition $w(0) = 0$ are shown in Fig. 1. We should mention that when using a variety of values for both initial values $w(0)$ and frequencies, simulation studies show hysteresis shapes similar to those in Fig. 1. This confirms again that the dynamic model (31) can be used to describe the backlash-like hysteresis. It also shows that the required shape of backlash hysteresis is dependent solely on the selection of a suitable parameter set $\{\alpha, c, B_1\}$.

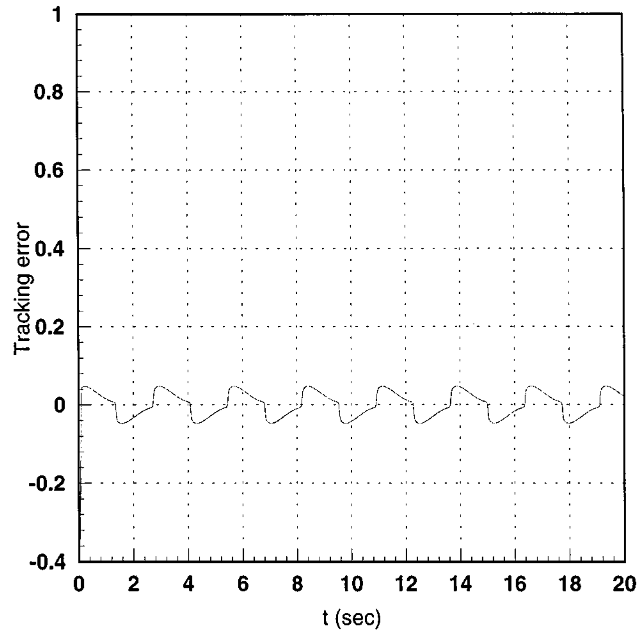

Fig. 2. Tracking error of the state with backlash hysteresis.

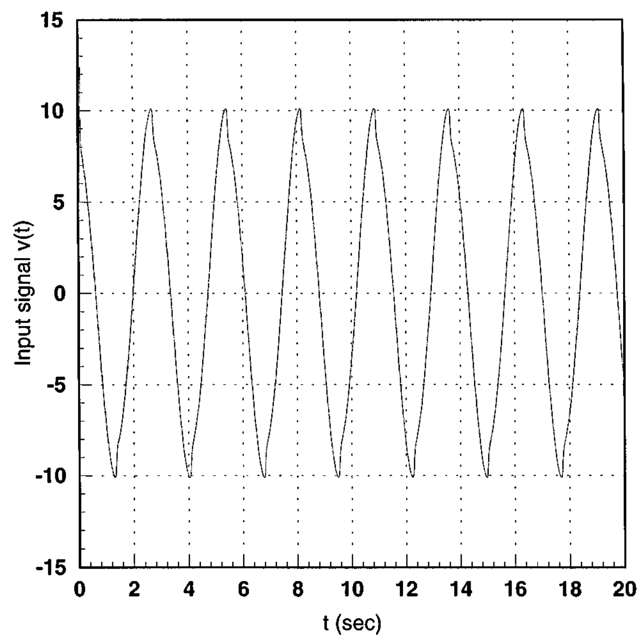

Fig. 3. Control signal $v(t)$ acting as the input of backlash hysteresis.

In the simulations, the robust adaptive control law (15)–(20) was used, taking $k_d = 10$. Since the backlash distance is around 2.5, we can choose the upper bound ρ in (11) as $\rho = 4$ and we also choose $c_{\min} = 3$, which results in $k^* = 4/3$. In the adaptation laws, we choose $\gamma = 0.5$ and $\eta = 0.5$ and the initial parameters $\theta = 1.2/3$ and $\phi = 0.8/3$. The initial state is chosen as $x(0) = 1.05$ and sample time is 0.005. In the simulation the initial value, $v(0)$, is required, which is selected as $v(0) = 0$.

Choosing the desired trajectory $x_d(t) = 12.5\sin(2.3t)$, simulation results are shown in Figs. 2–4. Fig. 2 shows the tracking error for the desired trajectory and Fig. 3 shows the input control signal $v(t)$. The signal $w(t)$ is shown in Fig. 4. We see from Fig. 2 that the proposed robust controller clearly demonstrates excellent tracking performance.

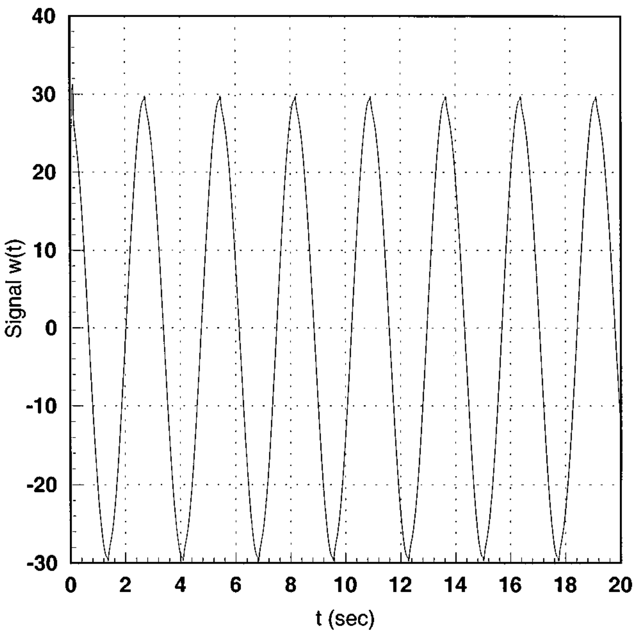

Fig. 4. Signal $w(t)$ acting as the output of backlash hysteresis.

We should mention that it is desirable to compare the control performance with and without considering the effects of hysteresis. Unfortunately, this comparison is not possible in this case as the control law (15)–(20) is designed for the entire cascade system.

VI. Conclusion

In this paper, a robust adaptive control architecture is proposed for a class of continuous-time nonlinear dynamic systems preceded by a backlash-like hysteresis, where the backlash-like hysteresis is modeled by a dynamic equation. By showing the properties of the hysteresis model, a robust adaptive control scheme is developed without constructing the hysteresis inverse. The new adaptive control law ensures global stability of the adaptive system and achieves both stabilization and tracking with excellent precision. Simulations performed on a simple nonlinear system illustrate and clarify the approach.

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Extremum Seeking for Limit Cycle Minimization

Hsin-Hsiung Wang and Miroslav Krstić

***Abstract*—In many physical problems, equilibrium stabilization is not possible and the controlled system is in a limit cycle. If the size of the limit cycle depends on some of the control parameters, then a reasonable objective would be to tune this parameter to minimize the size of the limit cycle. In this paper, we propose a method for achieving this. This method is an extension of our earlier result [13] on extremum seeking for equilibria. We illustrate the method with a Van der Pol oscillator example and present analysis for it using averaging and singular perturbations.**

***Index Terms*—Averaging, extremum seeking, limit cycles, singular perturbations.**

I. Introduction

Limit cycles occur in numerous areas of application. In particular, systems exist in which feedback control can only reduce the size of the limit cycle, but cannot completely eliminate it. The inability to remove the limit cycle and achieve equilibrium stabilization may be associated with actuator constraints, like magnitude and rate saturation. In this situation, the best control requirement is to enforce a stable, "smallest" limit cycle.

The method of "extremum seeking" has traditionally been used for searching for a minimum or a maximum of an *equilibrium map*. This method was an intensely studied topic between the 1940's and 1970's [2]–[5], [15], [19]–[21]. The most frequently cited references include the works by Kazakevich *et al.* [6]–[10], the survey by Sternby [24], and the book of Astrom and Wittenmark [1, Section 13.3]. Pioneering work on stability analysis based on averaging in an example of an extremum-seeking system dates back to Meerkov [16]–[18]. The first stability analysis for a problem with a *general nonlinear dynamical plant*

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