Escaping

Peter Grogono

Imagine that you are standing on the surface of a planet and throwing stones vertically upwards. The harder you throw, the higher they go. Can you throw a stone so hard that it never comes back?

Let

$$\begin{aligned}
m &\stackrel{\text{def}}{=} \text{the mass of the stone} \\
M &\stackrel{\text{def}}{=} \text{the mass of the planet} \\
R &\stackrel{\text{def}}{=} \text{the radius of the planet} \\
V &\stackrel{\text{def}}{=} \text{the initial velocity of the stone} \\
G &\stackrel{\text{def}}{=} \text{the gravitational constant} \\
r(t) &\stackrel{\text{def}}{=} \text{the distance of the stone from the centre of the planet at time } t \\
v(t) &\stackrel{\text{def}}{=} \text{the velocity of the stone at time } t
\end{aligned}$$

According to Newton's law of gravitation, a body at distance r from the planet has an acceleration $\frac{dv}{dt}$ given by

$$m\frac{dv}{dt} = -\frac{GMm}{r^2}. \qquad (1)$$

We assume that the effect of the planet's atmosphere has a negligible effect on the stone. Since $\frac{dv}{dt} = \frac{dr}{dt}\frac{dv}{dr} = v\frac{dv}{dr}$, we can cancel m and rewrite (1) as

$$v\frac{dv}{dr} = -\frac{GM}{r^2}$$

or

$$v\,dv = -\frac{GM}{r^2}\,dr.$$

Integration gives

$$\frac{1}{2}v^2 = \frac{GM}{r} + C. \qquad (2)$$

and, after substituting the initial conditions $v = V$ and $r = R$, we have

$$\frac{1}{2}V^2 = \frac{GM}{R} + C$$

from which we can obtain the value of the integration constant, C:

$$C = \frac{1}{2}V^2 - \frac{GM}{R}.$$

Substituting this value of C into (2) gives

$$\frac{1}{2}v^2 = \frac{GM}{r} + \frac{1}{2}V^2 - \frac{GM}{R}. \qquad (3)$$

Suppose that the stone ascends for a while and then stops. At this time, $v = 0$ and (3) becomes

$$0 = \frac{GM}{r} + \frac{1}{2}V^2 - \frac{GM}{R}$$

which we can rearrange to obtain

$$r = \frac{R}{1 - \dfrac{RV^2}{2GM}}.$$

As v increases, r increases without limit, becoming infinite when

$$V = \sqrt{\frac{2GM}{R}}.$$

This value of V is the *escape velocity* at the planet's surface.

We can obtain the same result by noticing that the stone must have greater kinetic energy ($\frac{1}{2}mV^2$) than its gravitational binding energy, GMm/R. When V is the escape velocity, these two energies must be the same:

$$\frac{1}{2}mV^2 = \frac{GMm}{R}.$$

In MKS units, the gravitational constant is $G = 6.679 \times 10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$ and, for the earth:

$$\begin{aligned} M &= 5.974 \times 10^{24}\ \mathrm{kg} \\ R &= 6.378 \times 10^{6}\ \mathrm{m} \end{aligned}$$

Consequently, the escape velocity of the earth is

$$\begin{aligned} V &= \sqrt{\frac{2 \times 6.679 \times 10^{-11} \times 5.974 \times 10^{24}}{6.378 \times 10^{6}}} \\ &= 11185.6\ \mathrm{m/s} \\ &\approx 40268\ \mathrm{km/h} \end{aligned}$$

Suppose that you are standing on a small asteroid. Could you jump high enough to escape it altogether? A person on the earth can jump perhaps a metre off the ground. This corresponds to a velocity of 4.4 m/s, which we will use as an approximate value for jumping off an asteroid.

Suppose that the radius of the asteroid is R_a and its density, σ, is the same as the earth's: 5.5 m/kg^3. Then its mass is

$$M_a = \frac{4}{3}\pi\sigma R_a^3$$

and we have

$$\begin{aligned} R_aV^2 &= 2GM_a \\ &= \frac{8}{3}\pi G\sigma R_a^3 \end{aligned}$$

which we can rearrange to give

$$R_a = \frac{V}{\sqrt{\frac{8}{3}\pi G\sigma}}.$$

Putting in the numbers,

$$R_a = \frac{4.4}{\sqrt{\frac{8}{3} \times \pi \times 6.679 \times 10^{-11} \times 5.5}} = 79315.2\,\mathrm{m}$$

You may infer that, if you were standing on an asteroid with a radius of 79 km, weighing about 2.09×10^{15} kg, or about 2.3×10^{12} tons, you could probably jump off it into deep space.

It would be interesting to find out how long it takes to achieve a given distance. We can rearrange (3) to give

$$v = \sqrt{V^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right)}$$

and, since $v = \dfrac{dr}{dt}$, we can derive the differential equation

$$\frac{dr}{\sqrt{V^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right)}} = dt.$$

Although the left side is integrable, the result is rather messy and we obtain t as a rather complicated function of r, which is not very useful.