Rocket Propulsion

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A rocket moves forwards by expelling fuel backwards. As it uses fuel, it becomes lighter. What is the maximum speed that a rocket can attain with a given amount of fuel. We assume the rocket starts in empty space, where there is no appreciable gravitational field.

Let

$$\begin{aligned} M_f &\stackrel{\text{def}}{=} \text{mass of fully-fueld rocket} \\ M_e &\stackrel{\text{def}}{=} \text{mass of empty rocket (all fueld used)} \\ U &\stackrel{\text{def}}{=} \text{velocity of expelled fuel relative to rocket} \\ M(t) &\stackrel{\text{def}}{=} \text{mass of rocket as a function of time} \\ v(t) &\stackrel{\text{def}}{=} \text{velocity of rocket as a function of time} \end{aligned}$$

Consider the time interval from t to $t + \Delta t$, where Δt is small. During this interval, the rocket emits a mass of fuel ΔM. At the beginning of the interval, the momentum of the rocket is Mv. During the interval, the rocket's velocity increases to $v + \Delta v$ and the ejected fuel is travelling at velocity $v - U$. At the end of the interval, the momentum is $(M - \Delta M)(v + \Delta v) + (v - U)\Delta M$. Since momentum is conserved,

$$Mv = (M - \Delta M)(v + \Delta v) + (v - U)\Delta M.$$

Neglecting second-order terms, this simplifies to

$$\frac{\Delta v}{\Delta M} = \frac{U}{M}.$$

Suppose that the rocket loses mass at a rate k so that $\Delta M = -k\Delta t$. Then

$$\frac{U}{M} = \frac{\Delta v}{\Delta M} = \frac{\Delta v}{\Delta t}\frac{\Delta t}{\Delta M} = -\frac{1}{k}\frac{\Delta v}{\Delta t}.$$

Letting $\Delta M \to 0$ and $\Delta t \to 0$ and using the fact that $M = M_f - kt$, we have

$$\begin{aligned} dv &= -\frac{kU\,dt}{M} \\ &= -\frac{kU\,dt}{M_f - kt}. \end{aligned}$$

Integration gives

$$\begin{aligned} v &= \int \frac{kU\,dt}{M_f - kt} \\ &= -U\ln(M_f - kt) + C. \end{aligned}$$

Initially, $t = 0$ and $v = 0$, giving

$$0 = -U\ln(M_f) + C$$

and so

$$\begin{aligned} v &= U \ln(M_f) - U \ln(M_f - kt) \\ &= U \ln\left(\frac{M_f}{M_f - kt}\right). \end{aligned}$$

When the fuel runs out, $M_f - kt = M_e$, and the final velocity of the rocket, V, is given by

$$V = U \ln\left(\frac{M_f}{M_e}\right).$$

We note that, if $M_f/M_e = e$, then $\ln(M_f/M_e) = 1$ and $V = U$. Also, as $M_e \to 0$, $V \to \infty$. Thus a rocket can achieve an arbitrarily high final speed (ignoring relativistic effects) if most of its initial mass consists of fuel. In practice, because the logarithm function increases so slowly, the effect is not dramatic:

Proportion of fuel (% of initial mass)	Final velocity (multiple of U)
90.	2.3
99.	4.6
99.9	6.9
99.99	9.2
99.999	11.5