Sky Hooks

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A wire of the right length could hang in space, its weight balanced by centrifugal force. Let

$$
\begin{aligned}
\rho &= \text{density of wire in kilograms per metre} \\
R &= \text{earth radius} \\
\omega &= \text{angular velocity of the earth} \\
g &= \text{gravitational acceleration at earth's surface} \\
T(x) &= \text{tension in the wire at distance } x \text{ from earth's centre}
\end{aligned}
$$

Consider a small segment of the wire, length dx, at distance x from earth's centre. The forces acting on this segment are:

$$
\begin{aligned}
\text{Tension pulling downwards} &= -T(x) \\
\text{Tension pulling upwards} &= T(x + dx) \\
\text{Weight pulling downwards} &= \rho g dx \left(\frac{R}{x}\right)^2 \\
\text{Centrifugal force pulling upwards} &= \rho \omega^2 x dx
\end{aligned}
$$

Equating these gives

$$\rho \omega^2 x dx + T(x + dx) = \rho g dx \left(\frac{R}{x}\right)^2 + T(x)$$

or

$$\frac{dT}{dx} = \rho \left(\frac{gR^2}{x^2} - \omega^2 x\right) \tag{1}$$

Integrating (1) gives

$$T = \rho \left(\frac{gR^2}{x} + \frac{1}{2}\omega^2 x^2\right) + C \tag{2}$$

where C is a constant of integration.

Suppose that the high end of the cable is at a distance kR from the earth's centre. At this height, the tension must be zero and (2) becomes

$$0 = \rho \left(\frac{gR^2}{kR} + \frac{1}{2}\omega^2 k^2 R^2\right) + C$$

and therefore

$$C = \rho \left(\frac{gR}{k} + \frac{1}{2}\omega^2 k^2 R^2\right)$$

and

$$T = \rho \left(\frac{gR}{k} - \frac{gR^2}{x} + \frac{1}{2}\omega^2 \left(k^2 R^2 - x^2\right)\right) \tag{3}$$

The smallest possible value of k is achieved when the cable just hangs free: that is, $T = 0$ when $x = R$. Substituting $x = R$ in (3) gives

$$gR - gRk + \frac{1}{2}\omega^2 R^2 k^3 - \frac{1}{2}\omega^2 R^2 k = 0$$

which is a cubic equation for k:

$$\frac{1}{2}\omega^2 R^2 k^3 - \left(\frac{1}{2}\omega^2 R^2 + gR\right) k + gR = 0 \qquad (4)$$

With $\alpha = 2g/\omega^2 R$, (4) becomes

$$k^3 - (1 + \alpha)k + \alpha = 0$$

Clearly, $k_1 = 1$ is one root. The other roots are

$$k_2, k_3 = \frac{-1 \pm \sqrt{4\alpha + 1}}{2}$$

Since $\alpha > 0$, both roots are real, and we want the larger one. The distance of the top of the cable from the earth's centre is

$$\begin{aligned} kR &= \frac{\sqrt{4\alpha + 1} - 1}{2} \\ &= \frac{\sqrt{8g/\omega^2 R + 1} - 1}{2} \end{aligned}$$

Putting some numbers in (MKS units):

$$\begin{aligned} g &= 9.81 \\ R &= 6.4 \times 10^6 \\ \omega &= 7.27221 \times 10^{-5} \\ \alpha &= 579.678 \\ k &= 23.5817 \\ kR &= 1.50923 \times 10^8 \\ \text{synch} &= 4.23544 \times 10^7 \\ \text{Max tension} &= 48662 \end{aligned}$$