Week 5 - The power rule

Oct. 13 & 15, 2015

We know that if n is a nonnegative integer then $\frac{d}{dx}(x^n) = nx^{n-1}$

- Find $\frac{d}{dx}(x^m)$ given that m is a negative integer.

  Solution: We can use the quotient rule. Let's assume that $m = -n$ where n is a positive integer, then we have

$$\frac{d}{dx}(x^m) = \frac{d}{dx}(x^{-n}) = \frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{\frac{d(1)}{dx}x^n - \frac{dx^n}{dx}(1)}{(x^n)^2} = \frac{0 \times x^n - nx^{n-1} \times 1}{x^{2n}}$$
$$= \frac{-nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1} = mx^{m-1}$$

  So we have found that $\frac{d}{dx}(x^m) = mx^{m-1}$ when m is a negative integer.

- Find $\frac{d}{dx}\left(x^{m/n}\right)$ given that m and n are integers.

  Solution: So far we know that $\frac{d}{dx}(x^m) = mx^{m-1}$ where $m \neq 0$ is an integer. Let's see if we can use this to find $\frac{d}{dx}\left(x^{m/n}\right)$.

$$\begin{aligned}
&f(x) = x^{m/n}, && \text{raise both sides to the power } n\\
&f(x)^n = x^m, && \text{differentiate with respect to } x\\
&\frac{d}{dx}(f(x)^n) = \frac{dx^m}{dx}, && \text{use the chain rule}\\
&nf(x)^{n-1}\frac{df}{dx} = mx^{m-1}, && \text{isolate } df/dx\\
&\frac{df}{dx} = \frac{m}{n}\frac{x^{m-1}}{f(x)^{n-1}}, && \text{plug the definition of } f(x)\\
&\frac{df}{dx} = \frac{m}{n}\frac{x^{m-1}}{\left(x^{m/n}\right)^{n-1}}, && \text{simplify!}
\end{aligned}$$

$$\frac{df}{dx} = \frac{m}{n}x^{\left(m-1-\frac{m(n-1)}{n}\right)} = \frac{m}{n}x^{\frac{mn-n-mn+m}{n}} = \frac{m}{n}x^{\frac{mn-n-mn+m}{n}} = \frac{m}{n}x^{(m/n)-1}$$

  So we have shown that $\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{m/n-1}$

- Assuming that $\frac{d}{dx}\ln x = \frac{1}{x}$, show that for all real values of $n \neq 0$ we have $\frac{d}{dx}(x^n) = nx^{n-1}$.

Solution:

$f(x) = x^n,$	take the natural logarithm of the equation
$\Rightarrow \ln(f(x)) = \ln(x^n) = n\ln(x),$	take the derivative with respect to x
$\Rightarrow \dfrac{d}{dx}(\ln(f(x))) = \dfrac{d}{dx}(n\ln(x)),$	use the chain rule
$\Rightarrow \dfrac{1}{f(x)}\dfrac{df}{dx} = \dfrac{n}{x},$	isolate df/dx
$\Rightarrow \dfrac{df}{dx} = \dfrac{nf(x)}{x},$	plug the definition of $f(x)$
$\Rightarrow \dfrac{df}{dx} = \dfrac{nx^n}{x} = nx^{n-1}$	

So we have shown that for all real numbers $n \neq 0$, $\dfrac{d}{dx}(x^n) = nx^{n-1}$.