Week 6 - On growth rates and continuous compounding

Oct. 20 & 22, 2015

1. Growth rates

We talked about exponential and linear growth. Examples of **exponential growth** were

a. Growth of a bacterial colony: Suppose that a bacterial colony has a population of 10 bacteria on day 0. Each bacterium divides into two bacteria every day. The following table thus gives the population, P, for days $t = 0$ to 4.

t	0	1	2	3	4
$P(t)$	10	20	40	80	160

This illustrates that $P(t) = 10 \times 2^t$.

b. Compound interest: Suppose I have deposited \$500 in a savings account with an annual interest rate of 5%. Assume that I don't make any additional withdrawals/deposits and the interest rate remains constant. Then the balance, S, after $t = 0$ to 4 years would be

t	0	1	2	3	4
$S(t)$	500	$500\left(1+\frac{5}{100}\right) = 525$	$500\left(1+\frac{5}{100}\right)^2 \approx 551$	$500\left(1+\frac{5}{100}\right)^3 \approx 579$	608

So we have $S(t) = 500\left(1+\frac{5}{100}\right)^t$.

We also talked about **linear growth**. Our example on linear growth was about a bamboo:

c. Suppose I buy a bamboo that is $40cm$ long and grows $6.5cm$ per day. The length of the bamboo, L, after $t = 0$ to 4 days then would be

t	0	1	2	3	4
$L(t)$	40	$40 + 6.5 \times 1 = 46.5$	$40 + 6.5 \times 2 = 53$	$40 + 6.5 \times 3 = 59.5$	$40 + 6.5 \times 4 = 66$

So we have $L(t) = 40 + 6.5t$.

Now let's look at the growth rates:

a. $P(t) = 10 \times 2^t$ where $P(t)$ is the population of a bacterial colony after t days. Remember that growth rate is defined as dP/dt. So we have

$$\frac{dP}{dt} = \frac{d}{dt}\left(10 \times 2^t\right) = 10\frac{d}{dt}\left(2^t\right) = 10 \times \ln(2) \times 2^t$$

Since dP/dt is positive, $P(t)$ increases with time. Also note that $P(t) = 10 \times 2^t$, therefore we have

$$\frac{dP}{dt} = \ln(2) \times 10 \times 2^t = \ln(2)P(t)$$

Since $P(t)$ increases with time, the growth rate $dP/dt = \ln(2)P(t)$ increases with t as well; e.g. for $t = 0$ to 4 the growth rate is

t	0	1	2	3	4
$P(t)$	10	20	40	80	160
$\frac{dP}{dt}$	$10 \times \ln(2) \approx 6.9$	$10 \times \ln(2) \times 2 \approx 13.9$	27.7	55.5	110.9

b. $S(t) = 500\left(1 + \frac{5}{100}\right)^t$ where $S(t)$ is the account balance after t years. Remember that growth rate is defined as dS/dt. So we have

$$\frac{dS}{dt} = \frac{d}{dt}\left(500\left(1 + \frac{5}{100}\right)^t\right) = 500\frac{d}{dt}\left(\left(1 + \frac{5}{100}\right)^t\right) = 500 \times \ln\left(1 + \frac{5}{100}\right) \times \left(1 + \frac{5}{100}\right)^t$$

Note that dS/dt is positive. So $S(t)$ increases with t. Since $S(t) = 500\left(1 + \frac{5}{100}\right)^t$ we also have

$$\frac{dS}{dt} = \ln\left(1 + \frac{5}{100}\right) \times 500 \times \left(1 + \frac{5}{100}\right)^t = \ln\left(1 + \frac{5}{100}\right)S(t)$$

Since $S(t)$ increases with time, this illustrates that growth rate $dS/dt = \ln\left(1 + \frac{5}{100}\right)S(t)$ also increases with t; e.g. for $t = 0$ to 4 we have

t	0	1	2	3	4
$S(t)$	500	525	551	579	608
$\frac{dS}{dt}$	$500 \times \ln\left(1 + \frac{5}{100}\right) \approx 24.4$	$500 \times \ln\left(1 + \frac{5}{100}\right) \times \left(1 + \frac{5}{100}\right) \approx 25.6$	26.9	28.2	29.7

c. $L(t) = 40 + 6.5t$ where $L(t)$ is the length of the bamboo after t days.

$$\frac{dL}{dt} = \frac{d}{dt}(40 + 6.5t) = 6.5$$

Since dL/dt is positive, $L(t)$ increases with time. However, the growth rate, $dL/dt = 6.5$, does not change with time; e.g. for $t = 0$ to 4 we have

t	0	1	2	3	4
$L(t)$	40	46.5	$40 + 53$	59.5	66
$\frac{dL}{dt}$	6.5	6.5	6.5	6.5	6.5

Constant growth rate is the characteristic feature of quantities that grow linearly.

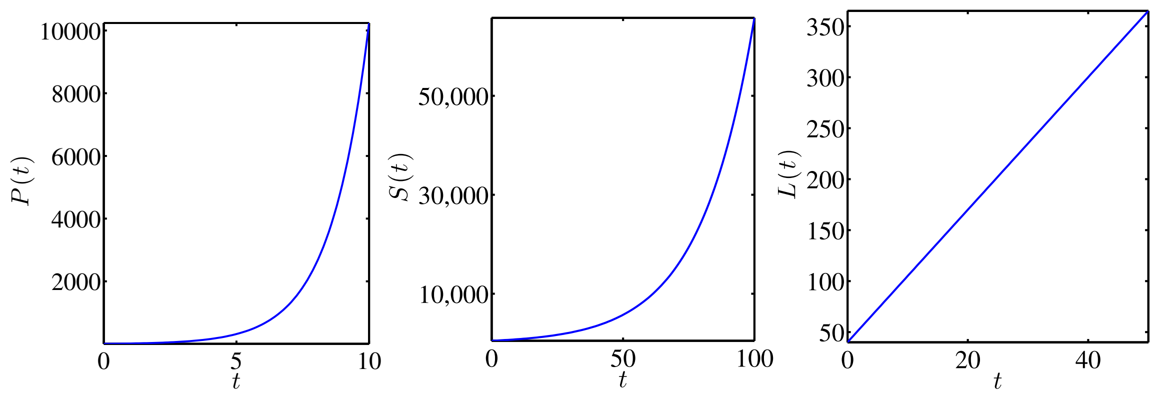

Figure 1: Graphs of $P(t)$, $S(t)$ and $L(t)$.

Graphs of $P(t), S(t)$ and $L(t)$ are provided in Fig. 1. Notice how $P(t)$ and $S(t)$ grow progressively faster with time; i.e. the slope of the tangent line to these curves, that is dP/dt and dS/dt, increase with t.

Lastly, let's consider the relative growth rates. That is the percentage change per unit time in the population of the bacterial colony, or in the account balance etc.

a. $P(t) = 10 \times 2^t$

$\dfrac{dP}{dt} = \ln(2) \times 10 \times 2^t = \ln(2)P(2)$

So when $t = 0$, the population of the colony is increasing at a rate of 6.9 bacteria per day. Since the population at $t = 0$ is $P(0) = 10$ we can say that the population is growing at a rate of $6.9/10 \times 100 = 69\%$ per day or at a relative rate of 0.69 per day. That is to say that the relative growth rate on day t is $\dfrac{1}{P}\dfrac{dP}{dt}$. e.g. for days $t = 0$ to $t = 4$

t	0	1	2	3	4
$P(t)$	10	20	40	80	160
$\dfrac{dP}{dt}$	$10 \times \ln(2) \approx 6.9$	$10 \times \ln(2) \times 2 \approx 13.9$	27.7	55.5	110.9
$\dfrac{1}{P}\dfrac{dP}{dt}$	$\frac{6.9}{10} = 0.69$	$\frac{13.9}{20} = 0.69$	$\frac{27.7}{40} = 0.69$	$\frac{55.5}{80} = 0.69$	$\frac{110.9}{160} = 0.69$

Notice that the relative growth rate of the population, $\dfrac{1}{P}\dfrac{dP}{dt} = \ln(2) \approx 0.69$, does not change with time. This is the hallmark of quantities that grow exponentially.

b. $S(t) = 500\left(1 + \dfrac{5}{100}\right)^t$,

$$\frac{dS}{dt} = 500 \times \ln\left(1 + \frac{5}{100}\right) \times \left(1 + \frac{5}{100}\right)^t = \ln\left(1 + \frac{5}{100}\right) S(t)$$

So when $t = 0$, the account balance grows at \$24.4 per year. Since the account balance at $t =$ is $S(0) = 500$, this means that the account balance is growing at a rate of $\frac{24.4}{500} \times 100 = 4.88\%$ per year or at a relative rate of 0.0488 per year. So for days $t = 0$ to 4 we have

t	0	1	2	3	4
$S(t)$	500	525	551	579	608
$\frac{dS}{dt}$	24.4	25.6	26.9	28.2	29.7
$\frac{1}{S}\frac{dS}{dt}$	$\frac{24.4}{500} = 0.0488$	$\frac{25.6}{525} = 0.0488$	$\frac{26.9}{551} = 0.0488$	$\frac{28.2}{579} = 0.0488$	$\frac{29.7}{608} = 0.0488$

Notice that the relative growth rate of the population, $\frac{1}{S}\frac{dS}{dt} = \ln\left(1 + \frac{5}{100}\right) \approx$ 0.0.0488, does not change with time. This is expected of quantities that grow exponentially.

c. $L(t) = 40 + 6.5t$, $\frac{dL}{dt} = 6.5$

So when $t = 0$, the bamboo grows at $6.5cm$ per day. Since the length of the bamboo at $t =$ is $L(0) = 40cm$, this means that the length of the bamboo is growing at a rate of $\frac{6.5}{40} \times 100 = 16.25\%$ per day or at a relative rate of 0.1625 per day. So for days $t = 0$ to 4 we have

t	0	1	2	3	4
$L(t)$	40	46.5	53	59.5	66
$\frac{dL}{dt}$	6.5	6.5	6.5	6.5	6.5
$\frac{1}{L}\frac{dL}{dt}$	$\frac{6.5}{40} \approx 0.16$	$\frac{6.5}{46.5} \approx 0.14$	$\frac{6.5}{53} \approx 0.12$	$\frac{6.5}{59.5} \approx 0.11$	$\frac{6.5}{66} \approx 0.1$

Note that the relative growth rate is $\frac{1}{L}\frac{dL}{dt} = \frac{6.5}{L(t)}$ which is the growth rate divided by the current value of the quantity L. Since $L(t)$ increases with time and the growth rate is constant, the relative growth rate decreases with time.

2. Continuous compounding

Assume that I have a savings account with an annual interest rate of r. If instead of receiving the interest rate annually, we receive the interest rate split evenly over n payments, after a year the account balance becomes $P\left(1 + \frac{r}{n}\right)^n$ where P is the principal value at the beginning of the year. I mentioned in the class that as $n \to \infty$ we have

$$\lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^n = P\,e^r$$

Similarly, the balance after t years would be $P\left(1+\frac{r}{n}\right)^{nt}$. Again as $n \to \infty$ we have

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = P \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = P\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right)^{t} = P\,e^{rt}$$