Practice problem on asymptotes

Find the horizontal and vertical asymptotes of $y = \dfrac{(\sqrt{x^2+x+3}-3)(2x+1)}{x^2-4}$

Horizontal asymptotes: to find the horizontal asymptotes we should examine $\lim_{x\to\infty} y$ and $\lim_{x\to-\infty} y$

- examining $\lim_{x\to\infty} y$

$$
\begin{aligned}
\lim_{x\to\infty} y &= \lim_{x\to\infty} \frac{(\sqrt{x^2+x+3}-3)(2x+1)}{x^2-4} \\
&= \lim_{x\to\infty} \frac{\left(\sqrt{x^2\left(1+\frac{1}{x}+\frac{3}{x^2}\right)}-3\right)(2x+1)}{x^2-4} \\
&= \lim_{x\to\infty} \frac{\left(\sqrt{x^2}\sqrt{\left(1+\frac{1}{x}+\frac{3}{x^2}\right)}-3\right)x\left(2+\frac{1}{x}\right)}{x^2\left(1-\frac{4}{x^2}\right)} \qquad \text{Note: } \sqrt{x^2}=|x| \\
&= \lim_{x\to\infty} \frac{\left(|x|\sqrt{\left(1+\frac{1}{x}+\frac{3}{x^2}\right)}-3\right)x\left(2+\frac{1}{x}\right)}{x^2\left(1-\frac{4}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{|x|\left(\sqrt{\left(1+\frac{1}{x}+\frac{3}{x^2}\right)}-\frac{3}{|x|}\right)x\left(2+\frac{1}{x}\right)}{x^2\left(1-\frac{4}{x^2}\right)} \qquad \text{Note: } (x\to+\infty)\Rightarrow(|x|=x) \\
&= \lim_{x\to\infty} \frac{x^2\left(\sqrt{1+\frac{1}{x}+\frac{3}{x^2}}-\frac{3}{x}\right)\left(2+\frac{1}{x}\right)}{x^2\left(1-\frac{4}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{\left(\sqrt{1+\frac{1}{x}+\frac{3}{x^2}}-\frac{3}{x}\right)\left(2+\frac{1}{x}\right)}{\left(1-\frac{4}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{\sqrt{1}\times 2}{1} = 2
\end{aligned}
$$

So $y = 2$ is a horizontal asymptote.

- examining $\lim_{x\to-\infty} y$

  In finding this limit, the first few steps are very similar to what we did to find $\lim_{x\to\infty} y$. These steps are omitted here.

$$\begin{aligned}
\lim_{x\to-\infty} y &= \lim_{x\to-\infty} \frac{(\sqrt{x^2+x+3}-3)(2x+1)}{x^2-4} \\
&= \lim_{x\to\infty} \frac{|x|\left(\sqrt{\left(1+\frac{1}{x}+\frac{3}{x^2}\right)}-\frac{3}{|x|}\right)x\left(2+\frac{1}{x}\right)}{x^2\left(1-\frac{4}{x^2}\right)} \qquad \text{Note: } (x\to-\infty)\Rightarrow(|x|=-x) \\
&= \lim_{x\to\infty} \frac{-x^2\left(\sqrt{1+\frac{1}{x}+\frac{3}{x^2}}-\frac{3}{x}\right)\left(2+\frac{1}{x}\right)}{x^2\left(1-\frac{4}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{-\left(\sqrt{1+\frac{1}{x}+\frac{3}{x^2}}-\frac{3}{x}\right)\left(2+\frac{1}{x}\right)}{\left(1-\frac{4}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{-\sqrt{1}\times 2}{1} = -2
\end{aligned}$$

  So $y=-2$ is also a horizontal asymptote.

To find vertical asymptotes, we notice that the function is the result of multiplication/division of polynomial and square root functions none of which have vertical asymptotes of their own. So the only possible vertical asymptotes may be found where the denominator of the function becomes zero:

$$x^2-4=0 \Rightarrow x=\pm 2$$

- vertical asymptote at $x=-2$?

  To ensure $x=-2$ is a vertical asymptote we need $\lim_{x\to 2^+} y=\pm\infty$ or $\lim_{x\to 2^-} =\pm\infty$.

$$\lim_{x\to-2} y = \lim_{x\to-2} \frac{(\sqrt{x^2+x+3}-3)(2x+1)}{x^2-4}$$

  If I substutute $x=-2$, I find

$$y=\frac{(\sqrt{4-2+3}-3)(-4+1)}{0}=\frac{(\sqrt{5}-3)\times(-3)}{0}$$

  I have found a number devided by 0. So the limits $\lim_{x\to-2^+} y$ and $\lim_{x\to-2^-} y$ are $+\infty$ or $-\infty$. But I don't need to figure that out! We have done enough to ensure $x=-2$ is a vertical asymptote.

- vertical asymptote at $x = 2$?

$$\lim_{x\to 2} y = \lim_{x\to 2} \frac{(\sqrt{x^2+x+3}-3)(2x+1)}{x^2-4}$$

If I substutute $x = 2$, I find

$$y = \frac{(\sqrt{4+2+3}-3)(4+1)}{0} = \frac{(\sqrt{9}-3)\times(5)}{0} = \frac{0}{0}$$

We have found zero divided by zero. So I should factor and simplify to find the limit.

$$\begin{aligned}
\lim_{x\to 2} y &= \lim_{x\to 2} \frac{(\sqrt{x^2+x+3}-3)(2x+1)}{x^2-4} \\
&= \lim_{x\to 2} \frac{(\sqrt{x^2+x+3}-3)(2x+1)(\sqrt{x^2+x+3}+3)}{(x^2-4)(\sqrt{x^2+x+3}+3)} \\
&= \lim_{x\to 2} \frac{(x^2+x+3-9)(2x+1)}{(x^2-4)(\sqrt{x^2+x+3}+3)} \\
&= \lim_{x\to 2} \frac{(x^2+x-6)(2x+1)}{(x^2-4)(\sqrt{x^2+x+3}+3)} \\
&= \lim_{x\to 2} \frac{(x+3)(x-2)(2x+1)}{(x-2)(x+2)(\sqrt{x^2+x+3}+3)} \\
&= \lim_{x\to 2} \frac{(x+3)(2x+1)}{(x+2)(\sqrt{x^2+x+3}+3)} \\
&= \frac{(2+3)(4+1)}{(2+2)(\sqrt{4+2+3}+3)} \\
&= \frac{25}{24}
\end{aligned}$$

So $x = 2$ **is not** a vertical asymptote.