MATH 110
2016, Jan 11-15
Week 2

Webworks everyweek, due mon. 10pm

Written assignments every other week, available on section website, due next week, same day at the beginning of the lecture.

Quizzes every other week (starting next fri.)

Today / this week: related rates
from the ref. book: 2.6

Derivative as a rate of change

Rate of change of area of a circle as the radius changes.

Rate of consumption of gas as the car accelerates

Rate of change of pressure ~~as~~ ~~you~~ ~~as~~ as you dive deeper in water

Example: The radius of a circle is increasing at a rate of 2 cm/s. How fast is the area of the circle increasing when its radius is 10 cm.

Step0: Make a sketch and label variables/parameters

Step1: Identify/write the known information

rate of change of r with t: 2 cm/s $\Rightarrow \frac{dr}{dt} = 2$ cm/s

at the time of interest: $r = 10$ cm

Step2: write the question in a mathematical form:

How fast $\Rightarrow$ rate of change of area with time: $\frac{dA}{dt}$ (unit? $\frac{cm^2}{s}$)

Step3: find an equation connecting ~~the known inform~~ the variables (here A and r)

$A = \pi r^2$

Step 4: Identify ~~the~~ which parameters are changing and which ones are fixed: here A, r and t are all changing.

Step 5: differentiate wrt the desirable independent variable

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Never plug the value of a changing parameter before step 6

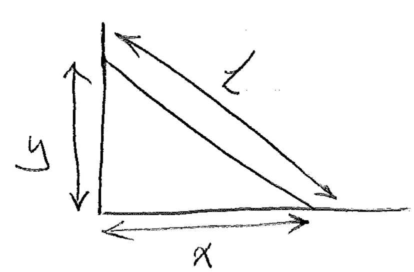
Step 6: put/plug all the known values in the previous step and solve for the desired part

$$\frac{dA}{dt} = 2\pi * 10\,cm * 2\,\frac{cm}{s} = 40\pi\,\frac{cm^2}{s}$$

Example:

A ladder 10m long rests against a vertical wall if the bottom of the ladder slides away from the wall at a rate of 0.5 m/s, how fast is the top end of the ladder sliding down the wall when the bottom of the ladder is 6m from the wall?

Step 0



Step 1: $L = 10m$, at the instant of interest: $x = 6m$

x is increasing at 0.5m/s $\Rightarrow \frac{dx}{dt} = 0.5\ m/s$

Step 3: find an equation that relates y and x

Pythagorean theorem $\Rightarrow x^2+y^2=l^2$

Step 4: x and y are changing w/ time. l doesn't change w/ time.

Question: diff. wrt
a) x b) t c) y d) none of the above e) I don't know

Step 5: diff. wrt t $\Leftarrow$ looking for $\frac{dy}{dt}$

$$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(l^2)$$

remember l does not change w/ time $\Rightarrow \frac{dl}{dt}=0$

Or just use $x^2+y^2=10^2$

$$\frac{d}{dt}(x^2)+\frac{d}{dt}(y^2)=0$$

chain rule / implicite diff.

$$\frac{dx^2}{dx}\cdot\frac{dx}{dt}+\frac{dy^2}{dy}*\frac{dy}{dt}=0$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$\frac{dy}{dt}=\frac{-2x}{2y}\frac{dx}{dt}=\frac{-x}{y}\frac{dx}{dt}$$

Step 6: need to find $\frac{dy}{dt}$ $\Rightarrow$ should find the value of $x, y, \frac{dx}{dt}$

$x=6\,m$, $x^2+y^2=l^2 \Rightarrow y=\sqrt{l^2-x^2}=\sqrt{100-36}=\sqrt{64}=$

Week 2

Wed. Jan. 13, 2016

Today: related rates (ref book 2.6)

derivatives → Related rates → Min / Max

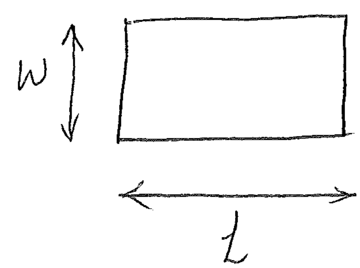
derivatives → A function is completely described if all of its derivatives are known at a point

Example: the width and length of a rectangle are changing at a rate of $-0.2\ \frac{m}{s}$ and $0.1\ m/s$ respectively. How fast is the area of the rectangle changing when it's ~~the~~ width and height are 3m and 4m, respectively.

a) $-0.8\ \frac{m^2}{s}$ b) $0.3\ \frac{m^2}{s}$ c) $1.1\ \frac{m^2}{s}$ d) $-0.5\ \frac{m^2}{s}$

e) I don't know!

Step 0



Step 1: W changing at $-0.2\ m/s$ $\Rightarrow \frac{dW}{dt} = -0.2\ m/s$

L changing at $0.1\ m/s$ $\Rightarrow \frac{dL}{dt} = 0.1\ m/s$

at the instant of interest $W = 3m$, $L = 4m$

Step 2: $\frac{dA}{dt}$? $\Leftarrow$ How fast is the area changing

Step 3: relat A, W, L $\Rightarrow A = LW$

Step 4: L, W and A are all changing w/ time $\Rightarrow \begin{cases} L = L(t) \\ W = W(t) \\ A = A(t) \end{cases}$

Step 5: diff wrt t

$$\frac{d}{dt}\left(l_{(t)} W_{(t)} \right) = \frac{d(A_{(t)})}{dt}$$ product rule

$$\frac{dL}{dt} * W_{(t)} + L_{(t)} * \frac{dW}{dt} = \frac{dA}{dt}$$

Step 6: $\frac{dl}{dt} = -0.2\,\frac{m}{s}$ $\frac{dW}{dt} = 0.1\,\frac{m}{s}$

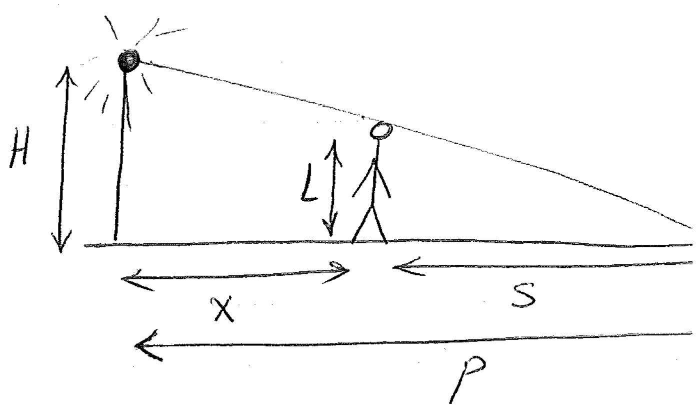
$l = 4m$ $W = 3m$

$$\frac{dA}{dt} = -0.2 * 4 + 0.1 * 3 = \cancel{4} -0.8 + 0.3$$

$$= -0.5\,\frac{m^2}{s}$$

Example: A man who is 1.7m tall is walking at a speed of 2 m/s away from a street light that is 3m above the ground. When the man is 4 m away from the light, at How fast is the tip of his shadow moving? At How fast is the length of his shadow changing?

Step 0: make a sketch:



Step 1. $H = 3\ m$

$L = 1.7\ m$

$$\frac{dx}{dt} = 2\ m/s$$

Step 2: $\frac{dP}{dt} = ?$

Step 3. eq. relating x, P

$$P = x + s$$

$$\frac{s}{P} = \frac{L}{H} \Rightarrow \frac{P - x}{P} = \frac{L}{H}$$

$$1 - \frac{x}{P} = \frac{L}{H}$$

Step 4: ℓ, H : constants

$$s = s(t) \qquad P = P(t)$$
$$x = x(t)$$

Step 5:

$$\frac{d}{dt}\left(1 - \frac{x(t)}{P(t)}\right) = \frac{d}{dt}\left(\frac{\ell}{H}\right) = 0$$

$$\frac{\frac{dx}{dt} * P(t) - x(t) * \frac{dP}{dt}}{(P(t))^2} = 0$$

$$\frac{dP}{dt} = \frac{dx}{dt} * \frac{P(t)}{x(t)}$$

Step 6:

$$\left.\begin{aligned} &x = 4m \\ &\ell = 1.7m,\ H = 3m \\ &\frac{P - x}{P} = \frac{\ell}{H} \end{aligned}\right\} \Rightarrow \frac{P-4}{P} = \frac{1.7}{3} \Rightarrow 1 - \frac{4}{P} = \frac{1.7}{3}$$

$$\Rightarrow \frac{4}{P} = 1 - \frac{1.7}{3} = \frac{1.3}{3}$$

$$P = \frac{12}{1.3}\,m$$

$$\Rightarrow \frac{dP}{dt} = 2\,m/s * \frac{12/1.3\ m}{4\ m} = \frac{6}{1.3}\,m/s$$

At what rate is the length of his shadow changing?

looking for $\frac{ds}{dt}$

$$P = S + X \quad , \quad P = P(t) \ , \ S = S(t) \ , \ X = X(t)$$

$$\frac{dP}{dt} = \frac{d}{dt}(S + X)$$

$$\frac{dP}{dt} = \frac{ds}{dt} + \frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{dP}{dt} - \frac{dx}{dt} = \frac{6}{1.3} - 2$$

$$= \frac{6 - 2.6}{1.3} = \frac{3.4}{1.3} \text{ m/s}$$

Exercise: At what rate is the tip of his shadow moving wrt his distance from the street light?

We are looking for a) $\frac{dP}{dt}$ b) $\frac{ds}{dt}$ c) $\frac{dP}{dx}$ d) $\frac{ds}{dx}$

e) I don't know!

MATH 110

Office hours: Mon. 9-10am, LSK 300
Wed 1:30-2:30, LSK 300

Reminder: $\frac{2*6}{3} = \frac{2}{3}*6 = 2*\frac{6}{3} = 2*6*\frac{1}{3} = \cdots$

$\frac{x\frac{dx}{dt}}{t} = \frac{x}{t}\frac{dx}{dt} = x*\frac{1}{t}*\frac{dx}{dt}$

for this course always keep $\frac{d*}{d\square}$ together; i.e.

don't write $xdx*\frac{1}{t}*\frac{1}{dt}$

$\frac{d*}{d\square}$: rate of change of $*$ as $\square$ changes

if $*$ is given in terms of $\square$, do regular diff.

e.g. $\frac{dx^2}{dx} = 2x$ $\qquad$ $\frac{d(3+\sin(t))}{dt} = \cos(t)$

if $*$ is not given in terms of $\square$ do implicit diff.

e.g. $\frac{dy^2}{dx} = \frac{dy^2}{dy}*\frac{dy}{dx} = 2y\frac{dy}{dx}$

$\frac{d\cos(y)}{dt} = \frac{d(\cos(y))}{dy}*\frac{dy}{dt}$

Also note: $\frac{dy^2}{dx} \neq \frac{(dy)^2}{dx}$

$dx^2 = (x^2)'$ $\qquad$ treat $d\square$ as one unit

Example: Water is flowing into a (right) conical tank at a rate of $3\frac{m^3}{s}$. If the radius of the top of the cone is 4m, the height is 6m and the depth of water is 2m, how fast is the water level rising? The volume of a right cone of radius r and height h is given by $V = \frac{\pi}{3} r^2 h$

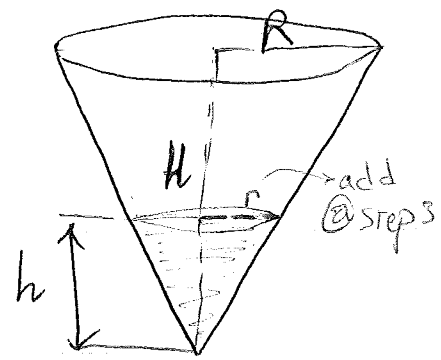
Question: Workout steps 0 to 2. Which of the following would you use to solve the problem?

a) Pythagorean theorem
b) Similar triangles
c) Step 0
d) Step 1
e) Step 2

Step 0:

V: volume of tank

W': volume of water



Step 1: $H = 6m$, $R = 4m$, $h = 2m$ $\qquad \frac{dW}{dt} = 3\frac{m^3}{s}$

Step 2: $\frac{dh}{dt}$

Step 3: h, W, an eq. w/ at least h, w in there ... ?

$$W = \frac{\pi}{3} r^2 h$$

Step 4: W, r, h change w/ time $\Rightarrow$ W(t), r(t), h(t)

(H, R are constants $\Rightarrow \frac{dR}{dt} = 0, \frac{dH}{dt} = 0$)

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Step 5. Diff wrt t

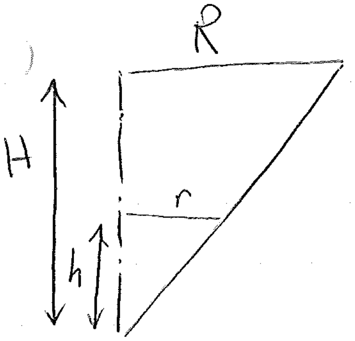
$$\frac{d}{dt}\left(W(t)\right) = \frac{d}{dt}\left(\frac{\pi}{3} r^2_{(t)} h(t)\right)$$

product rule

$$\frac{dW}{dt} = \frac{\pi}{3}\left[\frac{dr^2}{dt} * h(t) + (r(t))^2 \frac{dh}{dt}\right]$$

$$= \frac{\pi}{3}\left[2r\frac{dr}{dt} * h + r^2 * \frac{dh}{dt}\right]$$

Step 6: need the value of r and h and $\frac{dr}{dt}$!



$$\frac{h}{H} = \frac{r}{R}$$ this is always correct for all time

$$r = \frac{R}{H} h \quad (*)$$

I need $\frac{dr}{dt}$ too!

(1) ⟹ I can diff. (*)

Q) ~~eval~~ Find $\frac{d}{dt}\left(\frac{R}{H} h\right)$

a) $\frac{R'H - RH'}{H^2} h'$

b) $\frac{R'H - RH'}{H^2} h + \frac{R}{H} h'$

c) $\frac{R'}{H'} h'$

e) none of

Remember $r = r(t)$, $h = h(t)$

$$\frac{dR}{dt} = 0, \quad \frac{dH}{dt} = 0$$

$$\Rightarrow r(t) = \frac{R}{H} h(t)$$

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{R}{H} h(t)\right) = \frac{R}{H}\frac{dh}{dt}$$

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$$\frac{dW}{dt} = \frac{\pi}{3}\left[2 * \frac{R}{H}h * \frac{R}{H}\frac{dh}{dt} + \left(\frac{R}{H}h\right)^2 \frac{dh}{dt}\right]$$

$$\frac{dW}{dt} = \frac{\pi}{3}\left[2\,\frac{R^2h^2}{H^2} + \frac{R^2}{H^2}h^2\right]\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{3}{\pi} * \frac{1}{\left[2\,\frac{R^2h^2}{H^2} + \frac{R^2}{H^2}h^2\right]}\,\frac{dW}{dt}$$

$$= \frac{3}{\pi} * \frac{1}{\left[2 * \frac{4^2 * 2^2}{6^2} + \frac{4^2}{6^2} * 2^2\right] m^2} * 3\,\frac{m^3}{s}$$

can leave it here

$$= \frac{3}{\pi} * \frac{\cancel{3}}{\cancel{3} * \frac{2^2 * 2^2}{6^2}}\;\frac{m}{s}$$

$$= \frac{3^2 * 6^2}{\pi * 3 * 2^4} = \frac{3 * 3^2}{\pi * 2^2} = \frac{27}{4\pi}$$

Fri class ended here

↳ (Jan 15, 2016)