① MATH 110
Week 8,
Feb 22-26

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This week: Concavity and the 2^{nd} derivativ test (Ref 3.3, 3.4)
Asymptotes (Ref 3.6, 3.7)

Recap: if f is differentiable at a
- f is CU at a if the graph of f is above the tangent line to f at a for all x close to a (but not equal to a)
- f is CD at a if the graph of f is below the tangent line to f at a for all x close to a (but not equal to a)

$f''(x) > 0$ on interval $I \Rightarrow \begin{cases} f'(x) \text{ is increasing on } I \\ f \text{ is CU on } I \end{cases}$

$f''(x) < 0$ on interval $I \Rightarrow \begin{cases} f'(x) \text{ is decreasing on } I \\ f \text{ is CD on } I \end{cases}$

If $f''(\ ) = 0$, then $f(x)$ may be CU or CD or neither at a.

The 2nd derivative test

$f'(c) = 0$ and $f''(c) < 0 \Rightarrow \begin{cases} f \text{ is CD at } c \\ f \text{ has a local max at } c \end{cases}$

$f'(c) = 0$ and $f''(c) > 0 \Rightarrow \begin{cases} f \text{ is CU at } c \\ f \text{ has a local min at } c \end{cases}$

$f'(c) = 0$ and $f''(c) = 0 \Rightarrow f$ may have a local min max or neither at c

$(a, f(a))$ is an inflection point if concavity of f changes at

Example (from last lecture)

$y = x^4 - 4x^3$
$y' = x^2(4x - 12)$
$y'' = 12x\,(x-2)$

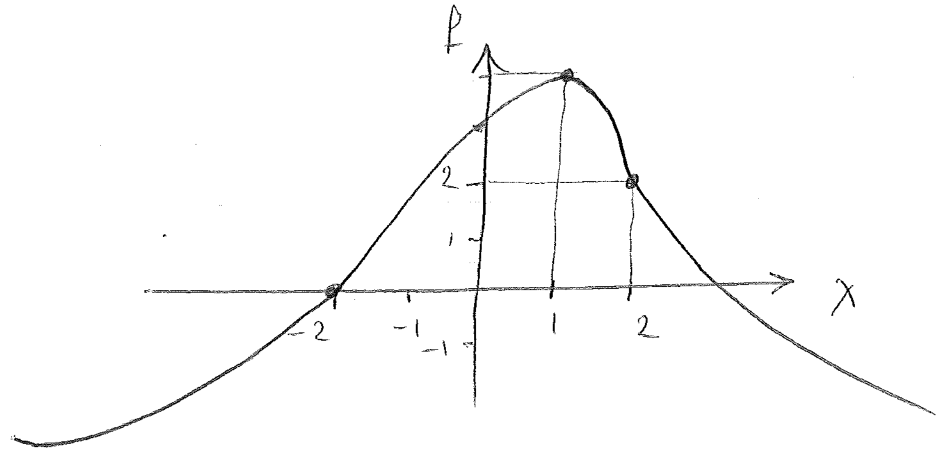
		0		2		3	
y'	$-$	0	$-$		$-$	0	$+$
y''	$+$	0	$-$	0	$+$		$+$
y	$\searrow$ CU	IP	$\searrow$ CD	IP	$\searrow$ CU	MIN	$\nearrow$ CU

Example: Sketch a possible graph of a function f that satisfies the following conditions: i) $f(2) = 2$, $f(1) = 4$, $f(-2) = 0$

ii) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

iii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

		-2		1		2	
f'	+		+		−		−
f''	+		−		−		+
f	CU ↗	0	CD ↗	4	CD ↘	2	CU ↘



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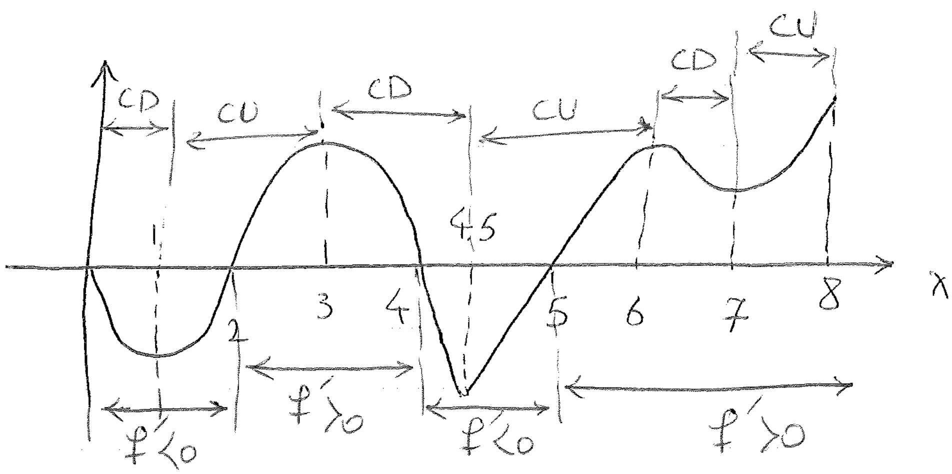
Example: A graph of the 1st derivative f' of a function f is shown.

a) On what intervals is f increasing?

b) At what values of x does f have a local min or max?

c) On what intervals is f concave up or concave down?

d) What are the x coordinates of the inflection points?



f increasing: $(2, 4) \cup (5, 8)$

local min at: $x = 2, 5$.
local max at: $x = 4$

concave up on: $(1, 3) \cup (4.5, 6) \cup (7, 8)$

concave down on: $(0, 1) \cup (3, 4.5) \cup (6, 7)$

inflection points at: $x = 1, 3, 4.5, 6, 7$

Wed lecture started here

Asymptotes / Asymptotic behavior of functions

Free fall → Supersonic skydive (Felix Baumgartner 03/2012)
→ Terminal velocity → $V(t)$ has a horizontal asymptote

Collapse of Tacoma Narrows Bridge → Oscillation amplitude vs. excitation frequency has a vertical asymptote (neglect damping)

Uranium half life → $M(t)$ has a horizontal asymptote.
U 238 → 4.5 billion years
235 → 700 million years

Fish population (halved since 1970's; i.e. in about 45 years)

* limit of $f(x)$ as x becomes arbitrarily large ("Approaches infinity")

$$f(x) = \frac{x-5}{3x+2} = \frac{x\left(1-\frac{5}{x}\right)}{x\left(3+\frac{2}{x}\right)} = \frac{1-\frac{5}{x}}{3+\frac{2}{x}}$$

x	$f(x)$
10	0.156
100	0.3146
1000	0.3314
10^4	0.3331
10^5	0.3333

As x becomes arbitrarily large, $f(x)$ approaches $1/3$

$$\lim_{x\to\infty} f(x) = \frac{1}{3}$$

(Slide)

Example: $f(x) = \dfrac{\sin(x^3)}{x^2+1}$, find $\lim\limits_{x\to\infty} f(x)$

x	$\dfrac{\sin(x^3)}{x^2+1}$
10	$8.1 * 10^{-3}$
100	$-3.5 * 10^{-5}$
1000	$5.5 * 10^{-7}$
10^4	$-6.1 * 10^{-9}$
10^5	$8.6 * 10^{-11}$

Note that $-1 \leq \sin(x^3) \leq 1$. However, (x^2+1) becomes arbitrarily large as $x\to\infty$

$$\lim_{x\to\infty} f(x) = 0$$

Example: a) $\lim\limits_{x\to\infty} \dfrac{x+5}{2x^2-100} = ?$

Exercise: make a table of values of $\dfrac{x+5}{2x^2-100}$ for large values of x

factor the largest power of x

$$\lim_{x\to\infty} \frac{x+5}{2x^2-100} = \lim_{x\to\infty} \frac{x\left(1+\frac{5}{x}\right)}{x^2\left(2-\frac{100}{x^2}\right)} = \lim_{x\to\infty} \frac{1+\frac{5}{x}}{x\left(2-\frac{100}{x^2}\right)} = \lim_{x\to\infty} \frac{1}{2x} = 0$$

b) $\lim\limits_{x\to\infty} \dfrac{5x^3-2x+4}{6x^3+4x^2-10} = ?$

$$\lim_{x\to\infty} \frac{5x^3-2x+4}{6x^3+4x^2-10} = \lim_{x\to\infty} \frac{x^3\left(5-\frac{2}{x^2}+\frac{4}{x^3}\right)}{x^3\left(6+\frac{4}{x}-\frac{10}{x^3}\right)} = \lim_{x\to\infty} \frac{5x^3}{6x^3} = \frac{5}{6}$$

Exercise: Find $\lim\limits_{x\to-\infty} \dfrac{5x^3-2x+4}{6x^3+4x^2-10}$

Definition The lin $y=K$ is a horizontal asymptote of f if $\lim_{x\to\infty} f(x) = K$ or $\lim_{x\to-\infty} f(x) = K$.

Question: How many horizontal asymptotes does $f(x) = x^3 - 2$ have? a) 0 b) 1 c) 2

Example: Find all the horizontal asymptotes of $f(x) = \frac{3x + \sin(x)}{\sqrt{2x^2+3x}}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty} \frac{3x+\sin(x)}{\sqrt{2x^2+3x}} = \lim_{x\to\infty}\left[\frac{3x}{\sqrt{x^2(2+\frac{3}{x})}} + \frac{\sin(x)}{\sqrt{2x^2+3x}}\right]$$

$$= \lim_{x\to\infty} \frac{3x}{\sqrt{x^2}\sqrt{2+\frac{3}{x}}} = \lim_{x\to\infty} \frac{3x}{|x|\sqrt{2}} = \lim \frac{3x}{\sqrt{2}\,x}$$

Note: $x\to+\infty$; therefore $|x| = x$

$$\lim_{x\to\infty} f(x) = \frac{3}{\sqrt{2}}$$

Exercise: find all horizontal asymptotes of $f(x) = \frac{\ln(x)}{\sqrt{x}}$ and $g(x) = \frac{e^x}{x^5}$

What about $\lim_{x\to-\infty} f(x)$?

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3x}{\sqrt{x^2}\sqrt{2+\frac{3}{x}}} = \lim \frac{3x}{|x|\sqrt{2}} = \lim_{x\to-\infty} \frac{3x}{-\sqrt{2}\,x} = \frac{3}{-\sqrt{2}}$$

Note: $x\to-\infty$; therefore $|x| = -x$

So $f(x) = \frac{3x+\sin(x)}{\sqrt{2x^2+3x}}$ has two horizontal asymptotes:

$$y = \frac{3}{\sqrt{2}} \text{ and } y = \frac{3}{-\sqrt{2}}$$

Fri lecture starts here

SLIDES

$f(x) = \dfrac{x-5}{3x+2}$ $\lim_{x \to -2/3} f(x) = ?$

Note in the fig. that $\lim_{x \to \frac{-2}{3}^+} f(x)$ and $\lim_{x \to \frac{-2}{3}^-} f(x)$ are not the same.

$$\lim_{x \to \frac{-2}{3}^+} \frac{x-5}{3x+2} \overset{?}{=} \frac{\frac{-2}{3}-5}{3 \cdot \frac{-2}{3}+2} = \frac{-17/3}{0}$$

$$\left(x \to \frac{-2}{3}^+\right) \Rightarrow x > \frac{-2}{3} \Rightarrow x + \frac{2}{3} > 0 \Rightarrow 3x+2 > 0$$

$\Rightarrow$ as $x \to \frac{-2}{3}^+$, $(3x+2) \to 0^+$

$$\lim_{x \to \frac{-2}{3}^+} \frac{x-5}{3x+2} \overset{?}{=} \frac{-17/3}{0^+}$$

$$\lim_{x \to \frac{-2}{3}^+} \frac{x-5}{3x+2} = -\infty$$

$$\lim_{x \to \frac{-2}{3}^-} \frac{x-5}{3x+2} = \frac{-17/3}{0}$$

$$\left(x \to \frac{-2}{3}^-\right) \Rightarrow x < \frac{-2}{3} \Rightarrow x + \frac{2}{3} < 0 \Rightarrow 3x+2 < 0$$

$\Rightarrow$ as $x \to \frac{-2}{3}^-$, $(3x+2) \to 0^-$

$$\lim_{x \to \frac{-2}{3}^-} \frac{x-5}{3x+2} = \frac{-17/3}{0^-}$$

$$\lim_{x \to \frac{-2}{3}^-} \frac{x-5}{3x+2} = +\infty$$

Reminder:

Definition: let f be a function defined on both sides of a except possibly at a itself. Then $\lim_{x \to a} f(x) = +\infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

let f be defined on both sides of a, except possibly at a itself. Then $\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a but not equal to a.

Definition The vertical line $x = a$ is a vertical asymptote of the graph of f if either or both the one-sided limits $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ is infinite.

Example: Find all the vertical asymptotes of the following functions

a) $f(x) = \dfrac{x^2-3x+2}{x^2-1}$

$x^2-1=0 \Longrightarrow \quad x=\mp 1$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-3x+2}{x^2-1} \overset{?}{=} \frac{1-3+2}{1-1} = \frac{0}{0} \; !$$

$$\lim_{x\to 1} \frac{x^2-3x+2}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x-2)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x-2}{x+1} = \frac{-1}{2}$$

So $x=1$ is NOT a vertical asymptote.

$$\lim_{x\to -1} \frac{x^2-3x+2}{x^2-1} \overset{?}{=} \frac{1+3+2}{1-1} = \frac{6}{0}$$

$\Rightarrow$ $x=-1$ is a vertical asymptote

Note: $$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{\overset{+6}{(x^2-3x+2)}}{\underset{-2}{(x-1)}\underset{0^+}{(x+1)}} = -\infty$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{\overset{+6}{(x^2-3x+2)}}{\underset{-2}{(x-1)}\underset{0^-}{(x+1)}} = +\infty$$

Note

$(x\to -1^+) \Rightarrow [(x+1)\to 0^+]$

$(x\to -1^-) \Rightarrow [(x+1)\to 0^-]$

Q: The function $y = \frac{5e^x + 3}{3e^x - 2}$ has — horizontal and — vertical asymptotes.

a) 0, 0   b) 1, 0   c) 1, 1   d) 2, 1   e) None of the above

Horizontal asymptotes:

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{5e^x+3}{3e^x-2} = \lim_{x\to\infty} \frac{\cancel{e^x}(5+3/e^x)}{\cancel{e^x}(3-2/e^x)} = \lim_{x\to\infty} \frac{5+\frac{3}{e^x}}{3-\frac{2}{e^x}} = \frac{5}{3}$$

So $y = \frac{5}{3}$ is a horizontal asymptote.

$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} \frac{5e^x+3}{3e^x-2} = \frac{3}{-2}$$

Note: $\lim_{x\to-\infty} e^x = 0$

So $y = \frac{-3}{2}$ is a horizontal asymptote.

Vertical asymptotes:

$$3e^x - 2 = 0 \Rightarrow 3e^x = 2 \Rightarrow e^x = \frac{2}{3} \Rightarrow \ln(e^x) = \ln\left(\frac{2}{3}\right)$$

$$\Rightarrow x = \ln\left(\frac{2}{3}\right)$$

$$\lim_{x\to\ln(\frac{2}{3})} \frac{5e^x+3}{3e^x-2} \stackrel{?}{=} \frac{5e^{\ln(\frac{2}{3})}+3}{3e^{\ln\frac{2}{3}}-2} = \frac{5*\frac{2}{3}+3}{3*\frac{2}{3}-2} = \frac{\frac{10}{3}+3}{0}$$

So $x = \ln\left(\frac{2}{3}\right)$ is a vertical asymptote.

Exercise: Assuming $y = \frac{5e^x+3}{3e^x-2}$, find $\lim_{x \to \ln(\frac{2}{3})^+} y$ and $\lim_{x \to \ln(\frac{2}{3})^-} y$.

Q: The function $y = \frac{\ln(x+3)-2}{x^2-10}$ has ___ horizontal and ___ vertical asymptotes.

a) 0, 0 b) 1, 0 c) 1, 1 d) 2, 1 e) None of the above

Note: Domain of y?

Horizontal asymptotes:

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{\ln(x+3)-2}{x^2-10} = \lim_{x \to \infty} \frac{\ln(x+3)\left(1 - \frac{2}{\ln(x+3)}\right)}{x^2\left(1 - \frac{10}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{\ln(x+3)}{x^2} = 0$$ See slides

So y has one horizontal asymptote: $y = 0$

Vertical asymptotes:

$x^2 - 10 = 0 \Rightarrow y = \mp\sqrt{10}$

$$\lim_{x \to \sqrt{10}} y \overset{?}{=} \frac{\ln(\sqrt{10}+3)-2}{0}$$

Note: $\lim_{x \to -\sqrt{10}} y$ is not defined because y is not defined as $x \to -\sqrt{10}^-$ or $x \to -\sqrt{10}^+$. So $x = -\sqrt{10}$ is not a vertical asymptote.

So $x = \sqrt{10}$ is a vertical asymptote. Are there any other vertical asymptotes?

Q: $\lim_{x \to -3^+} y = ?$ a) 2 b) 1 c) ∞ d) $-\infty$ e) None of the above

$$\left(x \to -3^+\right) \Rightarrow \left(x+3 \to (-3+3)^+\right) \Rightarrow \left(x+3 \to 0^+\right)$$

$$\Rightarrow \lim_{x \to -3^+} \ln(x+3) = -\infty$$

$$\lim_{x \to -3^+} y = \lim_{x \to -3^+} \frac{\overset{-\infty}{\ln(x+3)}}{\underset{-1}{x^2-10}} = +\infty$$

$\Rightarrow$ $x = -3$ is a vertical asymptote