Problem 1.

Find all the ctirical points of the function

$$f(x) = \begin{cases} 2x^3 - 3x^2 - 12x & x \leq 0 \\ e^{-11x}(1-x) - 1 & 0 < x \end{cases}$$

Solution:

- $x \leq 0$

  $$\frac{df}{dx} = \frac{d}{dx}(2x^3 - 3x^2 - 12x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$$

  Here $f(x)$ is defined by a polynomial; so the derivative exists everywhere. We need to find the points where $f'(x) = 0$.

  $$6(x^2 - x - 2) = 0 \Rightarrow x^2 - x - 2 = \frac{0}{6} \Rightarrow x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \Rightarrow x = -1 \text{ or } x = 2$$

  $x = -1$ is a critical number. Note that we have assumed $x \leq 0$. So $x = 2$ is not an acceptable solution here.

  We will need to check continuity and differentiability at $x = 0$.

- $0 < x$

  $$\frac{df}{dx} = -11e^{-11x}(1-x) - e^{-11x} = e^{-11x}(11x - 12)$$

  $f' = e^{-11x}(11x - 12)$ is defined everywhere on $(0, \infty)$. So we need to find the points where $f'(x) = 0$

  $$f'(x) = 0 \Rightarrow \begin{cases} e^{-11x} = 0 \text{ This does not have a solution since for all } 0 < x,\ e^{-11x} > 0 \\ \text{OR} \\ 11x - 12 = 0 \Rightarrow x = \dfrac{12}{11} \end{cases}$$

  Note that $x = \frac{12}{11}$ is in the interval $(0,\ \infty)$. So $x = \frac{12}{11}$ is a critical number.

  We can now check continuity and differentiability at $x = 0$.

  $$\lim_{x \to 0^-} f(x) = 0$$
  $$\lim_{x \to 0^+} f(x) = e^0(1-0) - 1 = 1 - 1 = 0$$
  $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$$

So f is continuous at $x = 0$. What about differentiability?

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} (6x^2 - 6x - 12) = -12$$
$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} e^{-11x}(11x - 12) = -12$$

So $f(x)$ is continuous and differentiable at $x = 0$. Therefore, $x = 0$ is not a critical number.

Problem 2.

Show that the function $f(x) = 3^{x-2} - 2(x+1)^2 + 18$ has at least one critical point. Since you don't have a calculator, a table of the function values is provided below. *Note: you do not need to find the critical point.*

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	16.1	10.3	1	-11	-23	-27	1	133	585	2005	6337

Solution: Note that $f(2) = f(6) = 1$.

$f(x)$ is the addition of an exponential function, 3^{x-2}, and a polynomial function, $-2(x+1)^2+18$. Both exponential functions and polynomials are differentiable for all real numbers x. So the function is continuous and differentiable on $[2,\ 6]$. Therefore, using Rolle's Theorem we know there is at least one number c, $2 < c < 6$ so that

$$f'(c) = \frac{f(6) - f(2)}{6 - 2} = \frac{1 - 1}{4} = 0$$

So $f'(x) = 0$ has at least one solution in the interval $(2, 6)$. So f has at least one critical point on that interval.