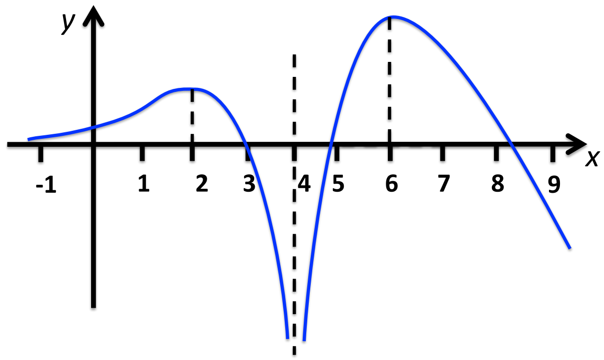
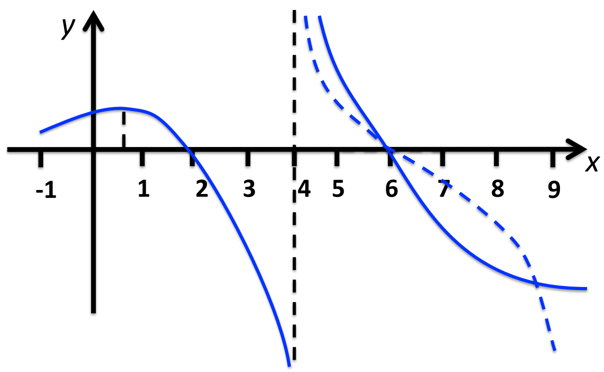
Assignment 3 - Solution

Problem 1.

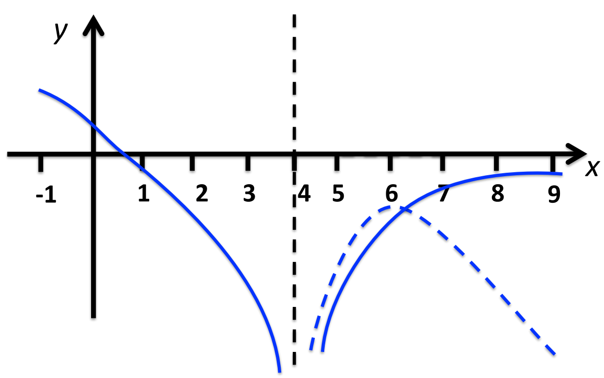
The graph of function f is shown. Sketch the graphs of f' and f''.



$f(x)$



$f'(x)$

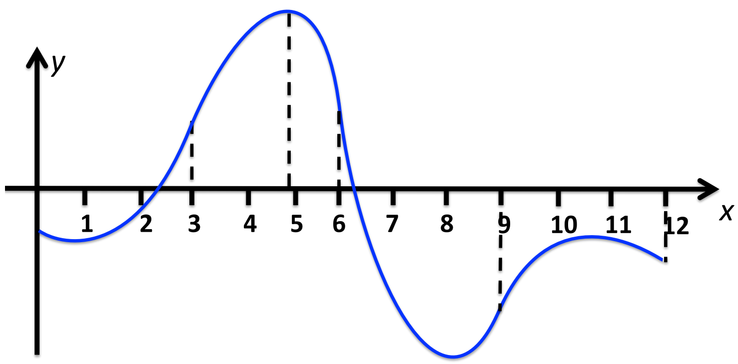


$f'(x)$

Problem 2.

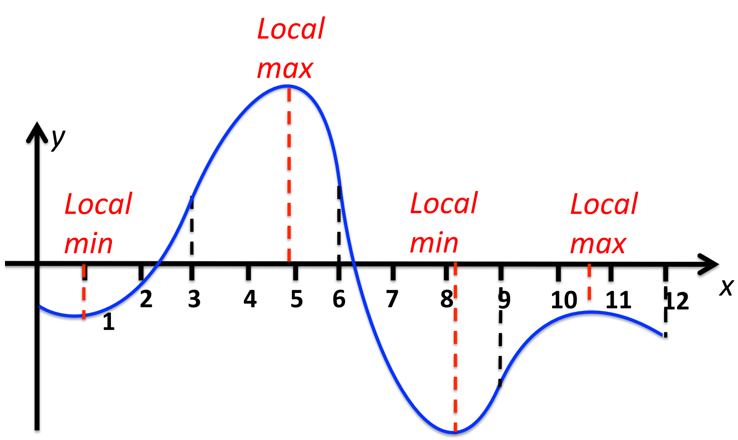
a. Sketch a possible graph of a function f that satisfies the following conditions and is differentiable for all x:

- **i.** Domain of f is $0 < x < 12$
- **ii.** f' is increasing on $(0, 3)$
- **iii.** $f'' < 0$ on $(3, 6)$
- **iv.** f is concave up on $(6, 9)$
- **v.** f has an inflection point at $x = 9$



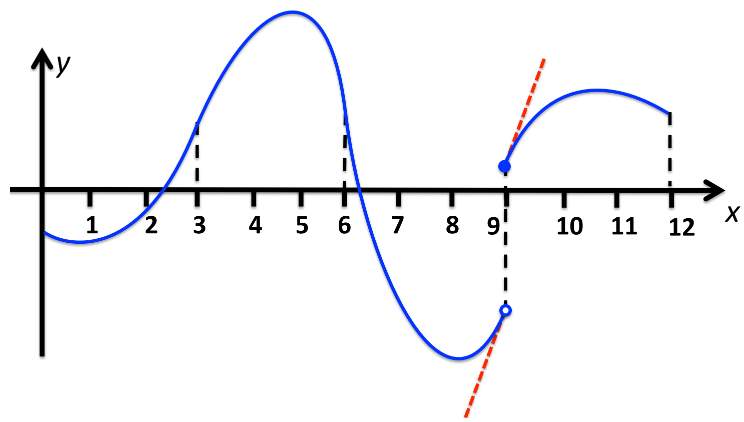
a

b. On your graph, find and label the local minimums and maximums.



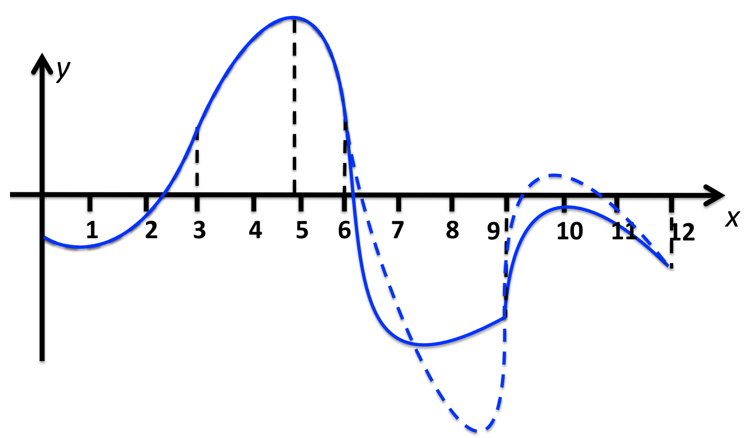
b.

c. Sketch a possible graph of a function f that satisfies the conditions given in part **a** and is differentiable for all x in its domain, except $x = 9$ where f is not continuous and $\lim_{x\to 9^+} f' = \lim_{x\to 9^-} f'$.



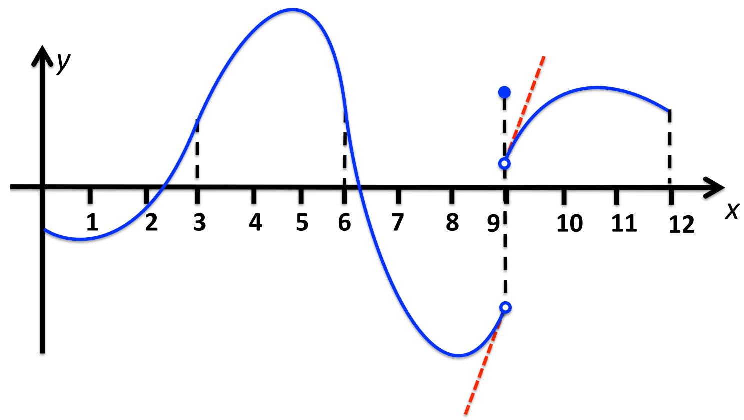
c.

d. Sketch a possible graph of a function f that satisfies the conditions given in part **a** and is differentiable for all x in its domain, except $x = 9$ where f is continuous but not differentiable.



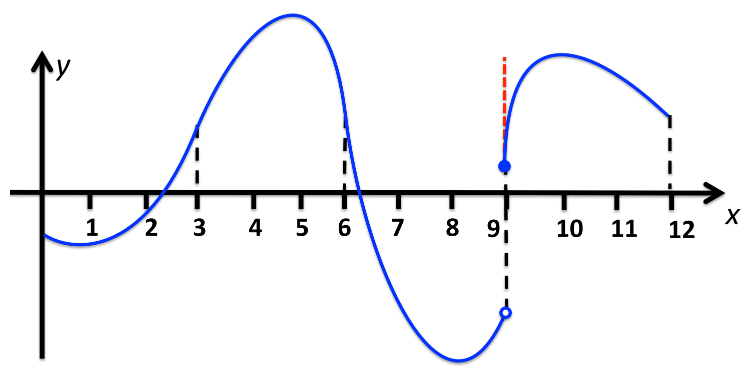
d.

e. Sketch a possible graph of a function f that satisfies the conditions given in part **a** and is differentiable for all x in its domain, except $x = 9$ where f is not continuous but has a maximum and $\lim_{x\to 9^+} f' = \lim_{x\to 9^-} f'$.



e.

f. Sketch a possible graph of a function f that satisfies the conditions given in part **a** and is differentiable for all x in its domain, except $x = 9$ where f is not continuous, $\lim_{x\to 9^+} f'$ is not defined, but $\lim_{x\to 9^-} f'$ is defined.



f.

Problem 3.

Show that if $f(x)$ is increasing and concave up on $[a, b]$, then $f((a+b)/2) < \dfrac{f(a)+f(b)}{2}$. *Hint: Use the mean value theorem on the function f on the intervals $(a, (a+b)/2)$ and $((a+b)/2, b)$.*

Solution:
MVT on $(a, (a+b)/2)$: a number c_1, $a < c < (a+b)/2$ exists such that

$$f'(c_1) = \frac{f((a+b)/2) - f(a)}{(a+b)/2 - a} = \frac{f((a+b)/2) - f(a)}{(b-a)/2} \tag{1}$$

MVT on $((a+b)/2, b)$: a number c_2, $(a+b)/2 < c_2 < b$ exists such that

$$f'(c_2) = \frac{f(b) - f((a+b)/2)}{b - (a+b)/2} = \frac{f(b) - f((a+b)/2)}{(b-a)/2} \tag{2}$$

Since f is concave up, we know that $f''(x) > 0$ on the interval (a, b). This means that $f'(x)$ is increasing on this interval. $c_2 > c_1$ thus implies $f'(c_2) > f'(c)$. Using this inequality together with equations (1) and (2) we find

$$\frac{f(b) - f((a+b)/2)}{(b-a)/2} > \frac{f((a+b)/2) - f(a)}{(b-a)/2}$$

Notice that $b > a$ and thus $(b-a)/2 > 0$. We can therefore multiply both sides of the above inequality by $(a-b)/2$):

$$\begin{aligned} f(b) - f((a+b)/2) &> f((a+b)/2) - f(a) && \text{add } f((a+b)/2) + f(a) \text{ to both sides} \\ f(b) + f(a) &> 2f((a+b)/2) && \text{divide both sides by 2} \\ \frac{f(b)+f(a)}{2} &> f((a+b)/2) \end{aligned}$$

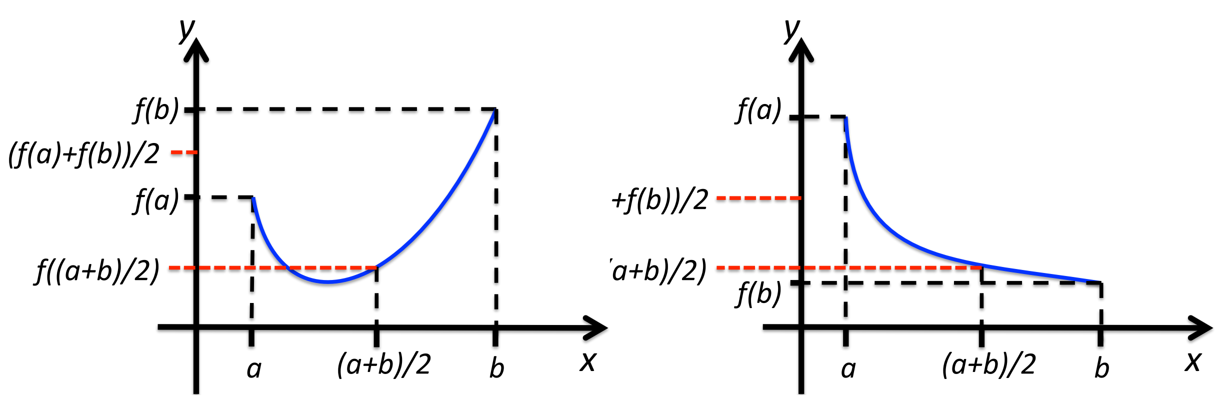


Figure 1: Illustrative examples for problem 3.