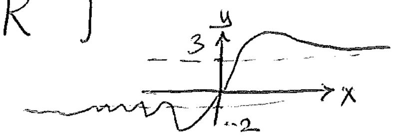
Ida Karim fazli

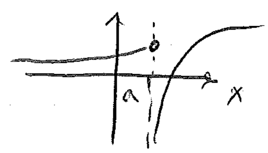
Last week: Vertical and horizontal asymptotes

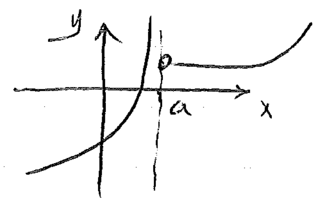
$$\left.\begin{array}{l}\lim\limits_{x\to\infty} f(x) = K \\ \text{OR} \\ \lim\limits_{x\to-\infty} f(x) = K\end{array}\right\} \Rightarrow y = K \text{ is a horizontal asymptote } f$$

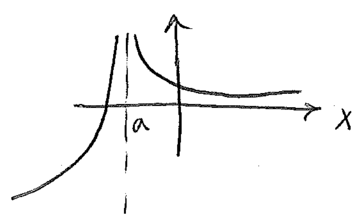


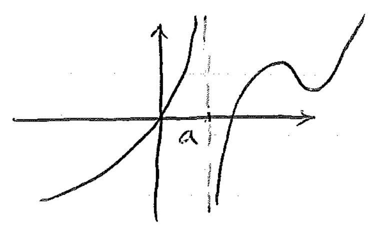
$y=3$ and $y=-2$ are both horizontal asympto

$$\left.\begin{array}{l}\lim\limits_{x\to a^+} f(x) = \mp\infty \\ \text{OR} \\ \lim\limits_{x\to a^-} f(x) = \mp\infty\end{array}\right\} \Rightarrow x = a \text{ is a vertical asymptote o } f(x$$









This week: L'Hopital's rule (Ref book: 3.7)
Curve sketching (Stewart 4.5,

SLIDES

Examples

a) $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} \stackrel{?}{=} \dfrac{0}{0}$ Use L'Hopital's rule

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{(\sin(x))'}{(x)'} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$$

Do Not Use Quotient Rule

b) $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x^2} \stackrel{?}{=} \dfrac{0}{0}$ Use L'Hopitals rule

$$\lim_{x \to 0} \frac{\sin(x)}{x^2} = \lim_{x \to 0} \frac{(\sin(x))'}{(x^2)'} = \lim_{x \to 0} \frac{\cos(x)}{2x} \stackrel{?}{=} \frac{1}{0} = \infty$$

Question in slides

$$\lim_{x \to 0^+} \frac{\sin(x)}{x^2} = +\infty = \lim_{x \to 0^+} \frac{\cos(x)}{2x}$$

$$\lim_{x \to 0^-} \frac{\sin(x)}{x^2} = -\infty = \lim_{x \to 0^-} \frac{\cos(x)}{2x}$$

c) $\lim_{x \to 0} \dfrac{x^3}{1-\cos(x)} \overset{?}{=} \dfrac{0}{0}$ Use L'Hopital's rule

$$\lim_{x \to 0} \frac{x^3}{1-\cos(x)} = \lim_{x \to 0} \frac{(x^3)'}{(1-\cos(x))'} = \lim_{x \to 0} \frac{3x^2}{\sin(x)} \overset{?}{=} \frac{0}{0}$$

Use L'Hopital's rule agai

$$\lim_{x \to 0} \frac{x^3}{1-\cos(x)} = \lim_{x \to 0} \frac{3x^2}{\sin(x)} = \lim_{x \to 0} \frac{(3x^2)'}{(\sin(x))'} = \lim_{x \to 0} \frac{6x}{\cos(x)} = \frac{0}{1}$$

$$\lim_{x \to 0} \frac{x^3}{1-\cos(x)} = 0$$

2. Find $\lim_{x \to \infty} \dfrac{x^5}{e^x}$

$\boxed{\text{Choices}}$

a) $5!$ b) 5

c) 0 d) $+\infty$

e) None of the above

$\lim_{x \to \infty} \dfrac{x^5}{e^x} \overset{?}{=} \dfrac{\infty}{\infty}$

Use L'Hopital's Rule $\Rightarrow$ $\lim_{x \to \infty} \dfrac{x^5}{e^x} = \lim_{x \to \infty} \dfrac{5x^4}{e^x} \overset{?}{=} \dfrac{\infty}{\infty}$ Use L'Hopital rule aga

$$\lim_{x \to \infty} \frac{x^5}{e^x} = \lim_{x \to \infty} \frac{5x^4}{e^x} = \lim \frac{20x^3}{e^x} \overset{?}{=} \frac{\infty}{\infty}$$ Use L'Hopital's rule

$$\lim_{x\to\infty} \frac{x^5}{e^x} = \lim_{x\to\infty} \frac{5x^4}{e^x} = \lim_{x\to\infty} \frac{5*4x^3}{e^x} = \lim_{x\to\infty} \frac{5*4*3x^2}{e^x} \overset{?}{=} \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{5*4*3*2x}{e^x} \overset{?}{=} \frac{\infty}{\infty}$$

What is $\lim_{x\to\infty} \frac{x^{100}}{e^x}$

$$= \lim_{x\to\infty} \frac{5*4*3*2*1}{e^x} = 0 \Rightarrow$$ e^x grows faster than x^5 as $x \to \infty$

Exercise: Find $\lim_{x\to\infty} \frac{x^n}{e^x}$ assuming that $n>0$.   SLIDE

Q: Find $\lim_{x\to\infty}$ ____

$\boxed{\text{Choices}}$

a) $+\infty$ b) $-\infty$

c) 1   d) 0

e) none of the above

$$\lim_{x\to\infty} \frac{x}{\ln(x)} \overset{?}{=} \frac{\infty}{\infty}$$ Use L'Hopital's rule

$$= \lim_{x\to\infty} \frac{(x)'}{(\ln(x))'} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} \frac{x}{1} = \infty$$

$\Rightarrow$ x grows faster than ln

Exercise: Find $\lim_{x\to\infty} \frac{\ln(x)}{x^n}$ assuming $n>0$.

Exercises: a) $\lim_{x\to\infty} \frac{e^x + x^5}{x^6+3}$ b) $\lim_{x\to\infty} \frac{x\ln x}{x^2+1}$

*c) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$ d) $\lim_{x\to -\infty} \frac{x^3+2x}{3x^3-5x^2+1}$

Ida Karimfazli

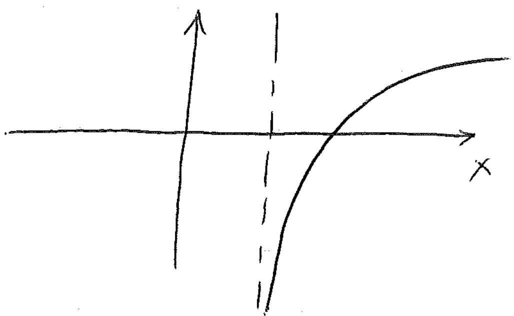
Feb 29 - March 4
Week 9

Last time: L'Hopital's rule

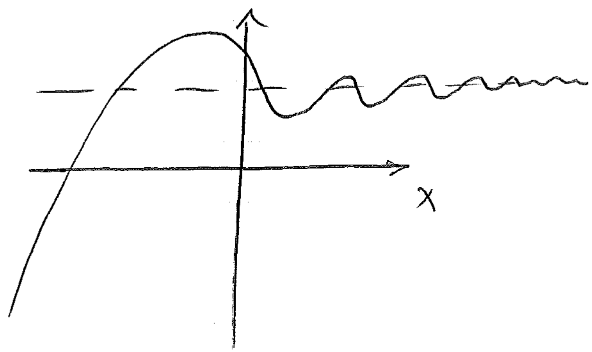
- $a \in I$
- f and g differentiable on I
- $g'(x) \neq 0$ on I except maybe at $x=a$

$\Bigg\}$, $\lim_{x \to a} \frac{f(x)}{g(x)} = $ "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" $\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

- Asymptotes



Vertical asymptote



Horizontal asymptote

Today (and fri): Curve sketching

Q: find $\lim_{x\to\infty} \frac{1}{x^2-4}$

a) 0 b) $\frac{1}{\infty}$ c) $+\infty$ d) none of the above e) I don't know

$$\lim_{x\to\infty} \frac{1}{x^2-4} \overset{?}{=} \frac{1}{\infty} \longrightarrow a$$

large positive

$$\lim_{x\to\infty} \frac{1}{x^2-4} = 0$$

Q: $\lim_{x\to -2^+} \frac{1}{x^2-4} = ?$

a) $\frac{1}{0}$ b) $+\infty$ c) $-\infty$ d) none of the above e) I don't know

$$\lim_{x\to -2^+} \frac{1}{x^2-4} \overset{?}{=} \frac{1}{0} \overset{?}{=} \pm\infty$$

$(x \to -2^+) \Rightarrow$ e.g. $x = -1.9$

$$\frac{1}{(-1.9)^2-4} = \frac{1}{3.61-4} = \frac{1}{-0.39} < 0$$

$$\lim_{x\to -2^+} \frac{1^+}{(x^2-4)_-} = -\infty$$

$$\lim_{x\to -2^-} \frac{1}{x^2-4} \overset{?}{=} \frac{1}{0} \overset{?}{=} \mp\infty$$

$(x \to -2^-) \Rightarrow$ e.g. $x = -2.1$

$$\frac{1}{(-2.1)^2-4} = \frac{1}{4.41-4} = \frac{1}{0.41} > 0$$

$$\lim_{x\to -2^-} \frac{1^+}{(x^2-4)^+} = +\infty$$

Example: Sketch a graph of the function $f(x) = \frac{1}{x^2-4}$

Step 1. Domain:

Step 2. Intercepts? with x-axis; find the points on the curve where $y=0$

with y-axis, find the point on the curve where $x=0$

Step 3. Symmetry: $f(x) = f(-x)$ for all x in the domain of f
$\Rightarrow f$ is an even function and the curve is symmetric about the y-axis (e.g. $(x) = |x|$, $(x) = \cos(x)$ or $(x) = x^2$ ...)

$f(x) = -f(-x)$. for all x in the domain of f
$\Rightarrow f$ is an odd function and the curve is symmetric about the origin. (e.g. $(x) = x^3$, $(x) = \sin($

Step 4: Asymptotes: find the horizontal and vertical asymptotes of f

Step 5: Intervals where f increases or decreases
$f' > 0 \Rightarrow f$ increases $\quad f' < 0 \Rightarrow f$ decreases

Step 6: local max and min values

Step 7: Concavity and points of inflection

Step 8: Sketch it!

Step1. $x \neq \mp 2$

Step2. $\begin{cases} y=0 \Rightarrow \dfrac{1}{x^2-4}=0 \Rightarrow 1 = 0 * (x^2-4) & \text{does not have a solution} \\ x=0 \rightarrow f_{(0)} = \dfrac{1}{0-4} = \dfrac{-1}{4} \end{cases}$

Step3. $\left.\begin{array}{l} f_{(x)} = \dfrac{1}{x^2-4} \\ f_{(-x)} = \dfrac{1}{(-x)^2-4} = \dfrac{1}{x^2-4} \end{array}\right\} \Rightarrow f_{(x)} = f_{(-x)} \Rightarrow f$ is an even func

i.e. the curve is symmetric about the y-axis

Step4. $\lim_{x \to \infty} f_{(x)} = 0$ $\quad \lim_{x \to -\infty} f_{(x)} = 0 \Rightarrow y=0$ is a horizontal asympto

$\lim_{x \to 2^+} f_{(x)} = -\infty$ $\quad \lim_{x \to -2^-} f_{(x)} = +\infty$

f is an even func.

$\lim_{x \to 2^-} f_{(x)} = -\infty$ $\quad \lim_{x \to 2^+} f_{(x)} = +\infty \Rightarrow x=-2$ and $x=2$ ar vertical asymptotes

Step 5: $\dfrac{df}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{x^2-4}\right) \overset{\text{chain rule}}{=} \dfrac{d}{du}\left(\dfrac{1}{u}\right) * \dfrac{du}{dx} = \dfrac{-1}{u^2}\dfrac{d}{dx}(x^2-4$

$u = x^2-4$

$$\frac{df}{dx} = \frac{-1}{(x^2-4)^2} * 2x = \frac{-2x}{(x^2-4)^2}$$

Now find critica
points to find inter
where f' is

$-2x = 0 \Rightarrow x = 0$

$(x^2-4)^2 = 0 \Rightarrow x^2-4=0 \Rightarrow x^2 = 4 \Rightarrow x = \mp 2$

		-2		0		2	
$-2x$	$+$		$+$	0	$-$		$-$
$(x^2-4)^2$	$+$		$+$		$+$		$+$
f'	$+$		$+$		$-$		$-$

	-3	-2	-1	0	1	2	3
f'	$\frac{+6}{5^2}$		$\frac{2}{+3}$		$\frac{-1}{+3^2}$		$\frac{-6}{5^2}$
	$+$		$+$	0	$-$		$-$

Step 6: $\left.\begin{array}{l} x = 0 \\ f(0) = \frac{-1}{4} \end{array}\right\}$ local max

Step 7: $f' = \dfrac{-2x}{(x^2-4)^2}$ $\qquad f'' = \dfrac{d^2f}{dx^2} = \dfrac{d}{dx}\left(\dfrac{-2x}{(x^2-4)^2}\right)$

$$= \frac{-2(x^2-4)^2 - (-2x)\,2 * 2x\,(x^2-4)}{(x^2-4)^4}$$

$$f'' = \frac{(x^2-4)\left[-2(x^2-4)+8x^2\right]}{(x^2-4)^4} = \frac{-2x^2+8+8x^2}{(x^2-4)^3} = \frac{6x^2+8}{(x^2-4)^3}$$

$6x^2+8 \geq 8 > 0$ i.e. $6x^2+8=0$ does not have a solution

$(x^2-4)^3=0 \Rightarrow x^2-4=0 \Rightarrow x=\mp 2$

	-3	-2		2	
$6x^2+8$	$+$		$+$		$+$
$(x^2-4)^3$	$+$		$-$		$+$
f''	$+$		$-$		$+$

	-3	-2	0	2	
f''	$+$		$-$		$+$

Note that $x=2$ and $x=-2$ are not inflection points because $x=2$, $x=-2$ are not in the domain of f