Last week: Optimization

Q: How confident do you feel about your skills in solving optimization problems?

a) not confident at all

b) kind of confident

c) Confident

d) Quite confident.

This week: Linear approximation, Taylor polynomials.

tangent line approx.
linearization of a func.

Slides 1-7

Note that the closer x is to 1, i.e. the smaller $|x-1|$, the better the estimate we get using the tangent line at $x=1$.

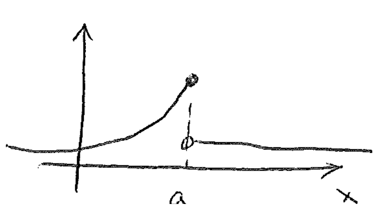
$\boxed{\text{Theorem}}$ linear approximation of f at a:

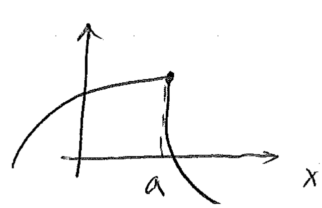
Suppose f is differentiable on an interval I containing the point a. The linear approximation of f at a is the linear function

$$L(x) = f(a) + f'(a)(x-a) \quad \text{for } x \in I$$

Note that $L(x)$ is the equation of the tangent line to $f(x)$ at $x=a$.

Why differentiable? because if there's a corner or discontinuity at $x=a$, the function f never looks like a "line" regardless of how far we zoom in.





Example : Use linear approximation of $f(x)=\sqrt{x+3}$ at $x=22$ to give an estimate of $f(x)$ at $x=23$ and $x=100$

$f(22)=\sqrt{22+3}=\sqrt{25}=5$

$L(x)=f(22)+f'(22)(x-22)$

$f'(x)=\frac{d}{dx}\left(\sqrt{x+3}\right)=\frac{1}{2\sqrt{x+3}} \Rightarrow f'(22)=\frac{1}{2\sqrt{22+3}}=0.1$

$\Rightarrow L(x)=5+0.1(x-22)$

$L(23)=5+0.1(23-22)=5+0.1\times 1=5.1 \approx$

$f(23)=\sqrt{26}\approx 5.099$

* This means that when $x=22$, if we change x by 1 unit (to 23) $f(x)$ approximately changes by $0.1\times 1=0.1$ the exact change in $f(x)$ is $(\sqrt{26}-5)$

* Error in our estimate: $|f(23)-L(23)|=\sqrt{26}-5.1$

$L(100)=5+0.1(100-22)=12.8$ $\qquad f(100)=\sqrt{103}\approx 10.149$

When $x=22$, if we change x by 78 units (to 100), $f(x)$ approximat changes by $f'(22)\times 78=7.8$. The exact change in f is $\sqrt{103}-5\approx 5.149$

Error in our linear approximation : $|f(100) - L(100)| = |\sqrt{103} - 12.8|$

≈ 2.651

Slides 8-9

Example: Use linear approximation to estimate $\frac{1}{\sqrt{112}}$

Strategy: I should first define a function $f(x)$ based on which I will build my linear approximation. I will then pick the number a where I will find the equation of the tangent line to $f(x)$ and use it to find a linear approximation of $\frac{1}{\sqrt{112}}$.

$\boxed{\text{Question}}$: Which one is a suitable function?

a) x b) $\frac{1}{x}$ c) $\sqrt{x}$ d) $\frac{1}{\sqrt{x}}$ e) I don't know!

$\boxed{\text{Question}}$ What is a suitable choice for number a?

a) $\frac{1}{\sqrt{112}}$ b) $\sqrt{112}$ c) 112 d) 100

e) none of the above

If I pick $f(x) = x$ or $f(x) = \frac{1}{x}$ it doesn't help me much in evaluating the square root ...

If I pick $f(x) = \frac{1}{\sqrt{x}}$, the $f(112) = \frac{1}{\sqrt{112}}$. So basically I'm looking for a linear approx. of $f(112)$.

$$f(x) = \frac{1}{\sqrt{x}} \Rightarrow f'(x) = \frac{-1}{2\sqrt{x^3}}$$

I can now choose $a = 100 = 10^2$ or $a = 121 = 11^2$.

Assuming $a = 100$

$$L(x) = f(100) + f'(100)(x - 100)$$

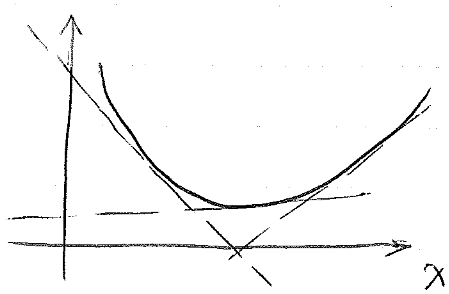
$$L(112) = \frac{1}{\sqrt{100}} + \frac{-1}{2\sqrt{100^3}}(112-100) = \frac{1}{10} - \frac{1}{2 * (10^6)^{1/2}} * 12$$

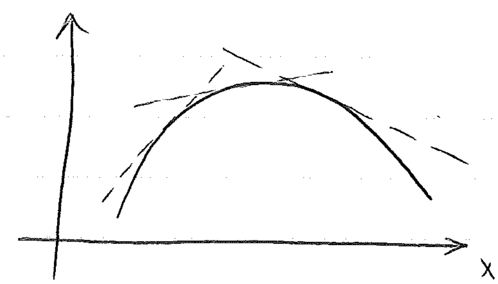
$$= \frac{1}{10} - \frac{12}{2 * 10^3} = \frac{1}{10} - \frac{6}{1000} = 0.1 - 0.006 = 0.094$$

$\boxed{\text{Question}}$:

Can you tell, without using a calculator, if $L(112) = 0.094$ is an overestimate or an underestimate of $f(112)$?

$$\left(f(112) = \frac{1}{\sqrt{112}} \approx 0.0945 \right)$$





tangents are below the curve where it's concave up. $\Rightarrow$ We underestimate using linear approximation.

tangents are above the curve where it's concave down $\Rightarrow$ we overestimate using linear approximation

Finish Slide 9, slides 10–14

Exercise: Use a linear approximation of $\cos(y) = x$ to give an estimat of y at $x = 0.6$. Assume that $0 < y < \pi$.

MATH 110
March 30 - April 1

Quiz next week, Wednesday, on Optimization

Last time $x, a \in I$, f differentiable on I $\Big\}$ Linear approx of $f(x)$ about $x=a$: $L(x) = f(a) + f'(a)(x-a)$

f is CU on $I \Rightarrow$ Linear approx. gives an underestimat
f is CD on $I \Rightarrow$ Linear approx gives an overestimate of value of f

So far we have worked with linear approx. where

$$f(a) = L(a)$$
$$f'(a) = L'(a)$$

What if I want a "better" approximation, $p(x)$? One that satisfies

$$f''(a) = p''(a)$$
$$f'(a) = p'(a)$$
$$f(a) = p(a)$$

Zeroth order approximation: $P_0(x) = f(a)$
First order approximation: $P_1(x) = f(a) + f'(a)(x-a)$
second order approximation: $P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$$P_2'(x) = f'(a) + \frac{f''(a)}{\cancel{2}} * \cancel{2} * (x-a) = f'(a) + f''(a)(x-a)$$

$$P_2''(x) = f''(a)$$

The 2^{nd} order approx. is also called a quadratic approx.

Example: Find the quadratic approximation of $f(x) = e^x$ centered a $x = 0$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$a = 0$

$$f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow f''(x) = \frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx}\left(e^x\right) = e^x$$

$$f(0) = e^0 = 1 \Rightarrow f'(0) = e^0 = 1 \Rightarrow f''(0) = e^0 = 1$$

$$P_2(x) = 1 + 1 * (x-0) + \frac{1}{2} * (x-0)^2 = 1 + x + \frac{x^2}{2}$$

Note: $P_1(x) = f(0) + f'(0)(x-0) = 1 + x$

$P_0(x) = f(0) = 1$

(Math Insight applet)

Question : Find the quadratic approximation of $f(x) = \sin(2x)$ centered at $x = 0$.

a) $2x$
b) $-2x$
c) $2x + 2x^2$
d) $-2x - 2x^2$
e) None of the above

$f(x) = \sin(2x)$ $\qquad P_2(x) = f(a) + f'(a) + \frac{f''}{2}(x-a)^2$ $\qquad a = 0$

$f'(x) = \frac{df}{dx} = \frac{d\sin(2x)}{dx} = \frac{d\sin u}{du} * \frac{du}{dx}$, $\qquad u = 2x$

$f'(x) = \cos(u) * \frac{d(2x)}{dx} = \cos(2x) * 2 \implies f'(0) = 2\cos(0) = 2$

$f''(x) = \frac{df'}{dx} = \frac{d}{dx}\left(2\cos(2x)\right) = 2\frac{d}{dx}\left(\cos(2x)\right) = 2\frac{d\cos u}{du} * \frac{du}{dx}$, $\quad u = 2x$

$f''(x) = 2 * (-\sin u) * \frac{d(2x)}{dx} = -2\sin(2x) * 2 = -4\sin(2x)$

$f''(0) = -4\sin(0) = 0$

$f(0) = \sin(0) = 0$

$P_2(x) = 0 + 2(x-0) + \frac{0}{2}(x-0)^2 = 2x$

Note: $\quad P_1(x) = f(0) + f'(0)(x-0) = 2x$

$\qquad P_0(x) = f(0) = 0$

Question: Find the linear approximation of $f(x) = 3x^2 + 2x - 5$ centred at $x = 1$

a) 0 b) $8x - 8$ c) $8x - 7$ d) $3x - 7$

e) None of the above

$$\left.\begin{array}{l} f(x) = 3x^2 + 2x - 5 \Rightarrow f(1) = 0 \\ f'(x) = 6x + 2 \Rightarrow f'(1) = 8 \end{array}\right\} \Rightarrow P_1(x) = 0 + 8(x-1) = 8x - 8$$

Question: Find the quadratic approximation of $f(x) = 3x^2 + 2x - 5$ centered at $x = 1$.

a) 0 b) $8x - 8$ c) $3x^2 + 2x - 5$ d) $3x^2 + 2x$

e) None of the above

$$f''(x) = \frac{d}{dx}(6x+2) = 6 \Rightarrow f''(1) = 6 \qquad P_2(x) = P_1(x) + \frac{f''(1)}{2}(x-1)^2 \Rightarrow$$

$$P_2(x) = 8x - 8 + \frac{f''(6)}{2}(x-1)^2 = 8x - 8 + \frac{6}{2}(x^2 - 2x + 1) = 8x - 8 + 3x^2 - 6x +$$

$$P_2(x) = 3x^2 + 2x - 5$$

Question (30 seconds): Find the quadratic approx. of $f(x) = 3x^2+2x-5$ centered at $x=100$

a) 0 b) $8x-8$ c) $3x^2+2x-5$ d) $3x^2+2x$

e) None of the above

With quadratic approximations we have

$$f(a) = P_2(a), \quad f'(a) = P_2'(a) \quad \text{and} \quad f''(a) = P_2''(a)$$

How can I improve my approximation even further?

e.g. define $p(x)$ such that $f'''(a) = p'''(x)$