WebWork assignment 2.13 is due on Sun. 10pm

① MATH 110
Last week!

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Recap: polynomial approximation of f centered at $x=a$

Zeroth order approx. : $P_0(x) = f(a)$

first order approx. : $P_1(x) = f(a) + f'(a)(x-a)$: linear approx

second order approx: $P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$: quadratic approx.

Today : higher order approximations / Taylor polynomials.

Wed. : Antiderivatives

Fri. : Review / Optimization

Example: $f(x) = 3x^2 + 2x - 5$.

linear approximation centered at $x=1$: $P_1(x) = 8x - 8$

quadratic approximation centered at $x=1$:

$P_2(x) = f(a) + f'(x-a) + \frac{f''(a)}{2}(x-a)^2$

$f'(x) = 6x + 2 \Rightarrow f'(1) = 8$

$f''(x) = 6 \Rightarrow f''(1) = 6$

$f(1) = 0$

$\Rightarrow P_2(x) = 0 + 8(x-1) + \frac{6}{2}(x-1)^2$

$P_2(x) = 8x - 8 + 3x^2 - 6x + 3$
$= 3x^2 + 2x - 5$

Question (30 seconds): Find the quadratic approx. of

$f(x) = 3x^2 + 2x - 5$ centered at $x = 100$

a) 0

b) $8x - 8$

c) $3x^2 + 2x - 5$

d) $3x^2 + 2x$

e) None of the above

(Note: $f'(x) = 6x$, $f''(x) = 6$, $f'''(x) = 0$)

With quadratic approximations we have

$$f(a) = P_2(a), \quad f'(a) = P_2'(a) \quad \text{and} \quad f''(a) = P_2''(a)$$

How can I improve my approximation even further?

e.g. define $p(x)$ such that $f'''(a) = p'''(x)$

Reminder: $f' = \frac{df}{dx}$ $\qquad f'' = \frac{d}{dx}(f') = \frac{d^2f}{dx^2}$

$$f''' = \frac{d}{dx}(f'') = \frac{d}{dx}\left(\frac{d}{dx}(f')\right) = \frac{d^3f}{dx^3}$$

...

$$f^{(n)}(x) = \frac{d^nf}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}f}{dx^{n-1}}\right) = \cdots$$

Examples: Find the 4^{th} derivative of the following func.

a) $f(x) = 2x^3 + 3x^2 + 1$

$$f'(x) = \frac{d}{dx}\left(2x^3 + 3x^2 + 1\right) = 6x^2 + 6x + 0$$

$$f''(x) = \frac{d}{dx}(f') = \frac{d}{dx}\left(6x^2 + 6x\right) = 12x + 6$$

$$f'''(x) = \frac{d^3 f}{dx^3} = \frac{d}{dx}(f'') = \frac{d}{dx}\left(12x + 6\right) = 12$$

$$f''''(x) = \frac{d^4 f}{dx^4} = \frac{d}{dx}(f''') = \frac{d}{dx}(12) = 0$$

$$\boxed{f(x) = 2x^3 + 3x^2 + 1 \Rightarrow f''''(x) = 0}$$

b) $f(x) = \sin(x)$

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}\left(\sin(x)\right) = \cos(x)$$

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}(f') = \frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

$$f'''(x) = \frac{d^3 f}{dx^3} = \frac{d}{dx}(f'') = \frac{d}{dx}\left(-\sin(x)\right) = -\cos(x)$$

$$f''''(x) = \frac{d^4 f}{dx^4} = \frac{d}{dx}(f''') = \frac{d}{dx}\left(-\cos(x)\right) = \sin(x)$$

$$\boxed{f(x) = \sin(x) \Rightarrow f''''(x) = \sin(x)}$$

Question : Find $\dfrac{d^{54} e^{x}}{dx^{54}}$

a) $54 e^{x}$ b) e^{54x} c) e^{x}

d) e^{54+x} e) none of the above

So .... How should I defin $P_3(x)$? $P_3(x) = P_2(x) + \dfrac{f'''(a)}{3*2}(x-a)^3$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3*2}(x-a)^3$$

Note: $\dfrac{dP_3}{dx} = f'(a) + f''(a)(x-a) + \dfrac{f'''(a)}{2}(x-a)^2$

$$P_3'(a) = f'(a)$$

$$\frac{d^2P_3}{dx^2} = \frac{d}{dx}(P_3') = f''(a) + f'''(a)(x-a)$$

$$P_3''(a) = f''(a)$$

$$\frac{d^3P_3}{dx^3} = \frac{d}{dx}(P_3'') = f'''(a)$$

$$P_3'''(a) = f'''(a)$$

What about $P_4(x)$?

$$P_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3*2}(x-a)^3 + \frac{f''''(a)}{4*3*2}(x-a)^4$$

$$P_5(x) = P_4(x) + \frac{f^{(5)}(a)}{5*4*3*2}(x-a)^5$$

Note: $f^{(5)}(x) = \frac{d^5 f}{dx^5}$

I can define $P_6, P_7, \dots$ in a similar way.

$\boxed{\text{Definition}}$ Taylor polynomials

Let f be a function with $f', f'', \dots, f^{(n)}$ defined at $x=a$. The n^{th}-order Taylor polynomial for f with its center at $x=a$, denoted P_n, has the property that it matches f in value, slope, and all derivatives up to the n^{th} derivative at a. The n^{th} order Taylor polynomial centered at a is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Reminder: $n! = 1 * 2 * 3 * \dots * n$

e.g. $5! = 1 * 2 * 3 * 4 * 5$

Note: $0! = 1$

Math Insight Applet (try $f(x) = \sin(2x)$
$f(x) = 3x^2 + 2x - 5$
$f(x) = e^x$)

Example: Find the 4^{th}-order Taylor polynomial for $f(x) = e^x$ with its center at $x = 0$.

$$f(x) = e^x \Rightarrow f'(x) = f''(x) = f'''(x) = \dots = e^x$$

$$P_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3*2}(x-a)^3 + \frac{f''''(a)}{4*3*2}(x-a)^4$$

$$P_4(x) = e^0 + e^0(x-0) + \frac{e^0}{2}(x-0)^2 + \frac{e^0}{3*2}(x-0)^3 + \frac{e^0}{4*3*2}(x-0)^4$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3*2} + \frac{x^4}{4*3*2}$$

Example: Find the 4^{th} order approximation of $f(x) = \frac{1}{x}$ centered at $x = 1$.

$$P_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3*2}(x-1)^3 + \frac{f''''(1)}{4*3*2}(x-1)^4$$

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow \frac{df}{dx} = f' = -x^{-2} \Rightarrow \frac{d^2f}{dx^2} = f'' = -1*(-2)\,x^{-3} = 2x^{-3}$$

$$\Rightarrow \frac{d^3f}{dx^3} = f''' = 2*(-3)*x^{-4} = -6x^{-4}$$

$$\Rightarrow \frac{d^4f}{dx^4} = f'''' = -6*(-4)\,x^{-5} = 24x^{-6}$$

$$f(1) = 1 \;,\quad f'(1) = \frac{-1}{1^2} = -1 \;,\quad f'' = 2(1)^{-3} = 2 \;,\quad f'''(1) = -6(1)^{-4} = -6$$

$$f''''(1) = 24(1)^{-5} = 24$$

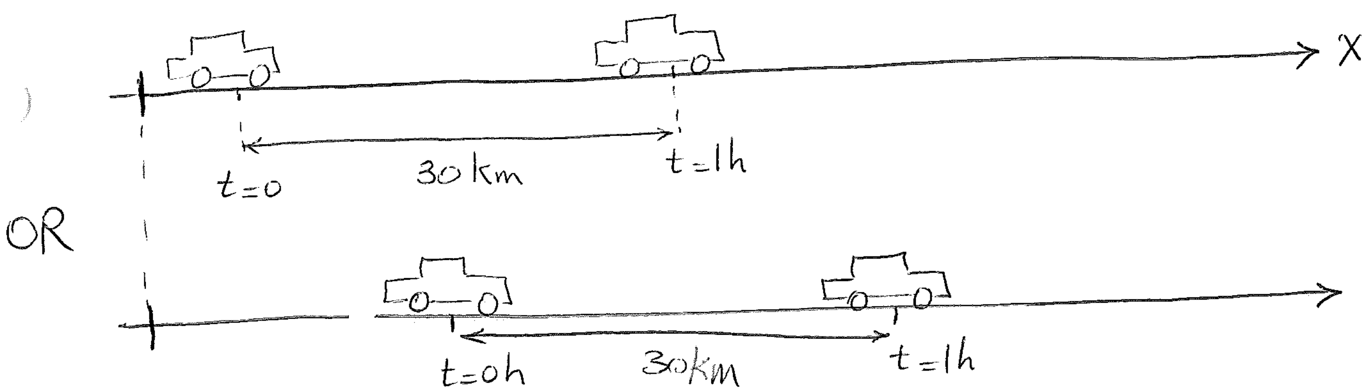
$$P_4(x) = 1 - 1*(x-1) + \frac{2}{2}(x-1)^2 + \frac{-6}{3*2}(x-1)^3 + \frac{24}{4*3*2}(x-1)^4$$

$$P_4(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$$

Antiderivatives:

So far we've been finding the rate of change of a given function f. What if the rate of change is given and we want to find the function?

e.g. the velocity of a car on a straight road is $30\ \frac{km}{h}$. What can we say about its position?



X_c : position of the car

$$X_c = X_c(0) + 30 * t$$

Definition A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

in the previous example

$f_{(t)} = 30 \frac{km}{h} = F'_{(t)} \Rightarrow F_{(t)} = 30t + C$

$f_{(t)}$ → velocity of the car

$F_{(t)}$ → position of the car

C → position of the car at $t=0$ (constant)

$F_{(t)} = 30t + C$ i.e. $G_{(t)} = 30t + 3$ and $H_{(t)} = 30t + 100$

are both antiderivatives of $f_{(t)} = 30$.

$\boxed{\text{Theorem}}$ If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F_{(x)} + C$ where C is an arbitrary constant.

Example: Find f if $f'_{(x)} = e^x + 2x + 6x^5 + 3$ and $f_{(0)} = 2$

Step 1: find the general antiderivative:

$$f'_{(x)} = e^x + 2x + 6x^5 + 3$$

$$f_{(x)} = e^x + x^2 + x^6 + 3x + C$$

$f_{(0)} = 2 \Rightarrow e^0 + 0 + 0 + 0 + C = 2 \Rightarrow 1 + C = 2 \Rightarrow C = 1$

$f_{(x)} = e^x + x^2 + x^6 + 3x + 1$

Example: Find $f(x)$ if $f''(x) = \frac{1}{3}\cos(2\pi x)$, $f(0)=0$, $f(3)=3$

Reminder: $f'' = \frac{df'}{dx}$, $f' = \frac{df}{dx}$

$$f''(x) = \frac{1}{3}\cos(2\pi x)$$

$$f'(x) = \frac{1}{3*2\pi} * \sin(2\pi x) + C$$

$$f(x) = \frac{-1}{3*(2\pi)^2} * \cos(2\pi x) + Cx + D$$

CHECK

$$\frac{d}{dx}\left(\frac{\sin(2\pi x)}{3*2\pi} + C\right) = \frac{2\pi\cos(2\pi x)}{3*2\pi}$$

C, D are

The general antiderivative: $f(x) = \frac{-1}{3*(2\pi)^2}\cos(2\pi x) + Cx + D$

$$f(0)=0 \Rightarrow f(0) = \frac{-1}{3*(2\pi)^2}\cos(0) + 0 + D = 0 \Rightarrow D = \frac{\cos(0)}{3*(2\pi)^2} = \frac{1}{3*(2\pi)^2}$$

$$f(3)=1 \Rightarrow f(3) = \frac{-1}{3*(2\pi)^2}\overset{1}{\cos(6\pi)} + 3C + \frac{1}{3*(2\pi)^2} = 3$$

$$f(3) = \frac{-1}{3*(2\pi)^2} + 3C + \frac{1}{3*(2\pi)^2} = 3$$

$$\Rightarrow 3C = 3 \Rightarrow C = 1$$

$$f(x) = \frac{-1}{3*(2\pi)^2}\cos(2\pi x) + x + \frac{1}{3*(2\pi)^2}$$

Exercise: Find the (general) antiderivative of the following functions:

a) $f(x) = e^{3x-2}$ b) $f(x) = x\cos(3x^2+5)$

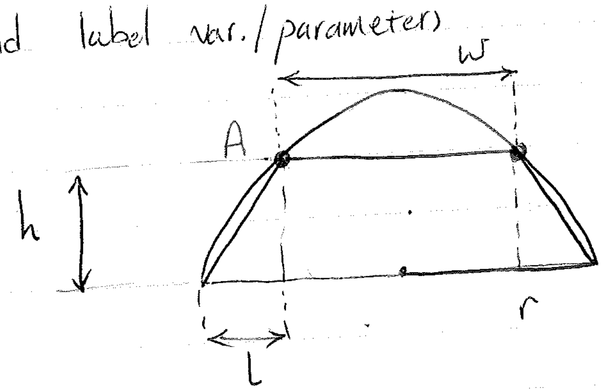
c) $f(x) = 0$ d) $f(x) = xe^x + e^x$

Office hours: Tuesday April 12, 9:30 - 10:30 am

Review of optimization

Example: Find the area of the largest trapezoid that can be inscribed in a circle of radius r and whose base is a diameter of the circle.

Step 0: Make a sketch and label var./parameters



Step 1: Identify and write the known information: ?

Step 2.1 Identify the var. to be optimized : A: area of the trapezoid

Step 2.2 Identify independent variables with respect to which you should optimize. What is changing as I move point A on the circle? (see the fig. on the previous page)

h, W, L and θ are changing. But, once either one is fixed, the other ones cannot be changed anymore. So they are not independent. We should pick one of them as the independent variable. Ideally, we want to pick the one that makes our life easier.... Which one is more helpful in finding A? Step 3 should help us figure this out!

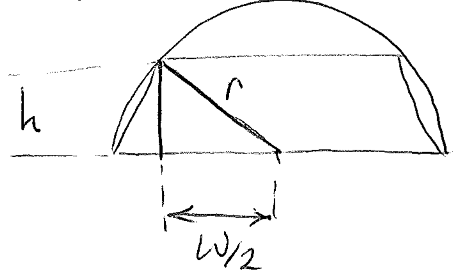
Step 3: equation for A

$$A = \frac{1}{2} h * (W + 2r)$$

So if I can find an equation that relates h and W, I can pick one of them as the independent variable.

(Step 1.
Pythagorean theorem

$$h^2 + \left(\frac{W}{2}\right)^2 = r^2$$
)



So ... I pick h as my independent variable.

Step 4 : Identify which variables change w/ h:

r is constant $\Rightarrow \frac{dr}{dh} = 0$

w changes with h

Step 5 : diff. A with respect to h and find the critical numbers

$$A = \frac{1}{2} h \, (w + 2r)$$

$$\frac{dA}{dh} = \frac{d}{d}\left(\frac{1}{2} h \, (w+2r)\right) = \frac{1}{2} \frac{d}{dh}\left(h \, (w+2r)\right)$$ product rule

$$\frac{dA}{dh} = \frac{1}{2} \frac{dh}{dh} * (w+2r) + \frac{1}{2} h * \frac{d}{dh}(w+2r)$$

$$\frac{dA}{dh} = \frac{1}{2}(w+2r) + \frac{1}{2} h \frac{dw}{dh}$$

$h^2 + \left(\frac{w}{2}\right)^2 = r^2$, looking for $\frac{dw}{dh}$ $\Rightarrow$ diff with respect to h

$$\frac{dh^2}{dh} + \frac{d}{dh}\left(\frac{w^2}{4}\right) = \frac{dr^2}{dh} \Rightarrow 2h + \frac{1}{4} * 2w \frac{dw}{dh} = 0$$

$$\Rightarrow \frac{w}{2} \frac{dw}{dh} = -2h \Rightarrow \frac{dw}{dh} = \frac{-4h}{w}$$

plug $\frac{dw}{dh}$ in equation ✱ in previous page

$$\frac{dA}{dh} = \frac{1}{2}(w+2r) + \frac{1}{2}h * \frac{-4h}{w} = \frac{w+2r}{2} - \frac{4h^2}{2w}$$

$$\frac{dA}{dh} = \frac{w^2+2rw-4h^2}{2w}$$

Critical numbers?

- end points? what is the domain? $0 \leq h \leq r$

$\Rightarrow h=0, \; h=r$

- $\frac{dA}{dh}$ is not defined $\Rightarrow w=0 \Rightarrow h = \sqrt{r^2 - \left(\frac{0}{2}\right)^2} = r$

- $\frac{dA}{dh} = 0 \Rightarrow w^2 + 2rw - 4h^2 = 0$

I have one equation here w/ 2 unknowns: h, w.
Have to use another equation as well to be able to solve for h and w

$$h^2 + \left(\frac{w}{2}\right)^2 = r^2$$

$$\left\{\begin{array}{l} w^2+2rw-4h^2=0 \\ h^2+\left(\frac{w}{2}\right)^2 = r^2 \Rightarrow h^2 = r^2 - \frac{w^2}{4} \end{array}\right\} \Rightarrow w^2+2rw-4\left(r^2-\frac{w^2}{4}\right)=0$$

$$\Rightarrow w^2 + 2rw - 4r^2 + w^2 = 0$$

$$\Rightarrow 2w^2 + 2rw - 4r^2 = 0$$

$$\Rightarrow w^2 + rw - 2r^2 = 0 \Rightarrow w = \frac{-r \mp \sqrt{r^2 + 8r^2}}{2} = \frac{-r \mp \sqrt{9r^2}}{2}$$

$$\Rightarrow w = \frac{-r \mp 3r}{2} = \begin{cases} -2r & \times \text{ w should be positive} \\ r & \Rightarrow h = \sqrt{r^2 - \frac{r^2}{4}} = \frac{\sqrt{3}}{2} r \end{cases}$$

Critical numbers?

- endpoints $\Rightarrow$ what is the domain? $0 \leq h \leq r$

$$\Rightarrow \boxed{h=0}, \boxed{h=r}$$

- $\frac{dA}{dh}$ is not defined $\Rightarrow w = 0 \Rightarrow \boxed{h = r}$

- $\frac{dA}{dh} = 0 \Rightarrow \boxed{h = \frac{\sqrt{3}}{2} r}$

We're looking for the abs. max. So we can just evaluate A at all the critical numbers. $A = \frac{1}{2} h (w + 2r)$

$h = 0 \Rightarrow A = 0$

$h = r \Rightarrow w = 0 \Rightarrow A = \frac{1}{2} r * 2r = r^2$

$h = \frac{\sqrt{3}}{2} r \Rightarrow w = r \Rightarrow A = \frac{1}{2} * \frac{\sqrt{3}}{2} r * (r + 2r) = \frac{3\sqrt{3}}{4} r^2 \approx 1.3 r^2 > r^2$

A has its abs. max at $h = \frac{\sqrt{3}}{2} r$ where $A = \frac{3\sqrt{3}}{4} r^2$

Example: Find the 4^{th} order Taylor polynomial of $f(x) = \sin(x-1)$ centered at: $x = 1$.

$$P_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3*2}(x-1)^3 + \frac{f''''(1)}{4*3*2}(x-1)^4$$

$$f(1) = \sin(1-1) = \sin(0) = 0 \Rightarrow \boxed{f(0) = 0}$$

$u = x-1$

$$f'(x) = \frac{d}{dx}\left(\sin(x-1)\right) = \frac{d\sin(u)}{du} * \frac{du}{dx} = \cos(u) * \frac{d(x-1)}{dx}$$

$$f'(x) = \cos(x-1) * 1 = \cos(x-1)$$

$$f'(1) = \cos(1-1) = \cos(0) = 1 \Rightarrow \boxed{f'(1) = 1}$$

$u = x-1$

$$f''(x) = \frac{df'}{dx} = \frac{d}{dx}\left(\cos(x-1)\right) = \frac{d\cos(u)}{du} * \frac{du}{dx} = -\sin(u)\frac{d(x-1)}{dx}$$

$$f''(x) = -\sin(x-1) * 1 = -\sin(x-1)$$

$$f''(1) = -\sin(1-1) = -\sin(0) = 0 \Rightarrow \boxed{f''(1) = 0}$$

$$f'''(x) = \frac{d}{dx}(f'') = \frac{d}{dx}\left(-\sin(x-1)\right) = \frac{d(-\sin(u))}{du} * \frac{du}{dx} = -\cos u \frac{d(x-1)}{dx}$$

$$f'''(x) = -\cos(x-1) * 1 = -\cos(x-1)$$

$$f'''(1) = -\cos(1-1) = -\cos(0) = -1 \Rightarrow \boxed{f'''(1) = -1}$$

$$f''''(x) = \frac{d}{dx}\left(-\cos(x-1)\right) = \frac{d(-\cos(u))}{du} * \frac{du}{dx} = -(-\sin(u))\frac{d(x-1)}{dx}$$

$$f''''(x) = +\sin(x-1) * 1 = \sin(x-1) \Rightarrow \boxed{f''''(1) = \sin(0) = 0}$$

$$P_4(x) = 0 + 1*(x-1) + \frac{0}{2}*(x-1)^2 + \frac{-1}{3*2}(x-1)^3 + \frac{0}{4*3*2}(x-1)^4$$

$$P_4(x) = (x-1) - \frac{(x-1)^3}{6}$$

Example: Find $f(x)$ if $f''(x) = \frac{3}{x^4}$ and $f(1) = \frac{3}{2}$ and $f(\frac{1}{2}) = 3$.

$f''(x) = \frac{3}{x^4}$ Reminder: $\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{-3}{x^4}$

$\Rightarrow f'(x) = a * \frac{1}{x^3} + C \Rightarrow$ I need to find a such that $\frac{df'}{dx} = f'' = \frac{3}{x^4}$

$\Rightarrow \frac{df'}{dx} = \frac{d}{dx}\left(\frac{a}{x^3} + C\right) = \frac{-3a}{x^4} = \frac{3}{x^4} \Rightarrow -3a = 3 \Rightarrow \boxed{a = -1}$

$\Rightarrow f'(x) = \frac{-1}{x^3} + C$ Reminder: $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}$

$f(x) = \frac{b}{x^2} + Cx + D \Rightarrow$ I need to find b such that

$\frac{df}{dx} = f' = \frac{-1}{x^3} + C \Rightarrow \frac{df}{dx} = \frac{d}{dx}\left(\frac{b}{x^2} + Cx + D\right) = \frac{-2b}{x^3} + C = \frac{-1}{x^3} + C$

$\Rightarrow \frac{-2b}{x^3} = \frac{-1}{x^3} \Rightarrow -2b = -1 \Rightarrow b = \frac{1}{2}$

$\Rightarrow f = \frac{1}{2x^2} + Cx + D$: most general antiderivative

$$f(1) = \frac{1}{2} + C + D = \frac{3}{2}$$

$$f\left(\frac{1}{2}\right) = \frac{1}{2*\left(\frac{1}{2}\right)^2} + \frac{C}{2} + D = 3 \Rightarrow \frac{1}{\left(2*\frac{1}{4}\right)_{1/2}}^{2} + \frac{C}{2} + D = 3$$

$$\Rightarrow \begin{cases} \frac{1}{2} + C + D = \frac{3}{2} \Rightarrow D = \frac{3}{2} - C - \frac{1}{2} \quad * \\ 2 + \frac{1}{2}C + D = 3 \quad ** \end{cases}$$

plug $*$ in $**$ $\Rightarrow$ $2 + \frac{1}{2}C + \frac{3}{2} - C - \frac{1}{2} = 3$

$$2 + \frac{3}{2} - \frac{1}{2} - 3 = C - \frac{C}{2} = \frac{C}{2}$$

$$0 = \frac{C}{2} \Rightarrow \boxed{C = 0}$$

$$\Rightarrow D = \frac{3}{2} - 0 - \frac{1}{2} = 1 \Rightarrow \boxed{D = 1}$$

$$f(x) = \frac{1}{2x^2} + 0 * x + 1 = \frac{1}{2x^2} + 1$$