First Name: ____________________ Last Name: ____________________

Student-No: ____________________ Grade: ____________________

## Short answer questions — you must show your work

1. 10 marks Each part is worth 2 marks.

   (a) The cost function is given by $C(q) = 2^{\sqrt{q}+5}$. Find the average cost of producing 16 units.

   Answer: $\overline{C}(16) = 32$

   **Solution:**

   $$C(q) = 2^{\sqrt{q}+5}$$
   $$\overline{C}(q) = \frac{C(q)}{q} = \frac{2^{\sqrt{q}+5}}{q}$$
   $$\overline{C}(16) = \frac{2^{\sqrt{16}+5}}{16} = \frac{2^{4+5}}{2^4} = 2^5 = 32$$

   (b) The position of a ship that sails on a straight line is given by $S(t) = 30t + 5\sin(\pi t/2)$ where $S$ is position in kilometers and $t$ is time in days. Find the acceleration of the ship on the third day.

   Answer: $a(3) = \frac{5\pi^2}{4}$

   **Solution:**

   Position: $S(t) = 30t + 5\sin(\pi t/2)$

   Velocity: $V(t) = \frac{dS}{dt} = \frac{d}{dt}(30t + 5\sin(\pi t/2)) = 30 + 5 \times \frac{\pi}{2}\cos(\pi t/2)$

   Acceleration: $a(t) = \frac{dV}{dt} = \frac{d}{dt}(30 + 5 \times \frac{\pi}{2}\cos(\pi t/2)) = -5 \times \left(\frac{\pi}{2}\right)^2 \sin(\pi t/2)$

   $a(3) = -5 \times \left(\frac{\pi}{2}\right)^2 \sin(\pi 3/2) = -5 \times \left(\frac{\pi}{2}\right)^2 \times (-1) = \frac{5\pi^2}{4}$

   (c) Use logarithmic differentiation to find the derivative of the function $f(x) = (\sin(x-1))^{x^2}$.

   Answer: $\frac{df}{dx} = \big(2x\ln(\sin(x-1)) + x^2\cot(x-1)\big)\,(\sin(x-1))^{x^2}$

**Solution:**

$$\ln(f(x)) = \ln\left(\sin(x-1)\right)^{x^2} \qquad \text{take the logarithm}$$
$$\ln(f(x)) = x^2 \ln\left(\sin(x-1)\right) \qquad \text{differentiate}$$
$$\frac{d(\ln(f(x)))}{dx} = \frac{d}{dx}\left(x^2 \ln(\sin(x-1))\right) \qquad \text{use product rule}$$
$$\frac{d(\ln(f(x))}{dx} = \frac{dx^2}{dx}\ln(\sin(x-1)) + x^2\frac{d}{dx}\ln(\sin(x-1)) \qquad \text{use power rule \& chain rule}$$
$$\frac{d\ln(f)}{df}\frac{df}{dx} = 2x\ln(\sin(x-1)) + x^2\frac{d(\ln(u))}{du}\frac{du}{dx} \qquad u = \sin(x-1)$$
$$\frac{1}{f}\frac{df}{dx} = 2x\ln(\sin(x-1)) + x^2\frac{1}{u}\frac{d\sin(x-1)}{dx} \qquad \text{use chain rule again}$$
$$\frac{1}{f}\frac{df}{dx} = 2x\ln(\sin(x-1)) + x^2\frac{1}{u}\frac{d\sin(v)}{dv}\frac{dv}{dx} \qquad v = x-1$$
$$\frac{1}{f}\frac{df}{dx} = 2x\ln(\sin(x-1)) + x^2\frac{1}{u}\cos(v)\frac{d(x-1)}{dx}$$
$$\frac{1}{f}\frac{df}{dx} = 2x\ln(\sin(x-1)) + x^2\frac{1}{\sin(x-1)}\cos(x-1)\times 1$$
$$\frac{1}{f}\frac{df}{dx} = 2x\ln(\sin(x-1)) + \frac{x^2\cos(x-1)}{\sin(x-1)}$$
$$\frac{df}{dx} = \left(2x\ln(\sin(x-1)) + x^2\cot(x-1)\right)\left(\sin(x-1)\right)^{x^2}$$

(d) Find the equation of the tangent line to $x^3y + e^{x^2-1} = \dfrac{2y^2}{x}$ at the point $(1,1)$

Answer: $y - 1 = \dfrac{7}{3}(x-1)$

**Solution:**

$$\frac{d}{dx}(x^3y + e^{x^2-1}) = \frac{d}{dx}\left(\frac{2y^2}{x}\right)$$
$$\frac{dx^3}{dx}y + x^3\frac{dy}{dx} + \frac{de^u}{du}\frac{du}{dx} = \frac{\frac{d(2y^2)}{dx}x - 2y^2\frac{dx}{dx}}{x^2} \qquad u = x^2-1$$
$$3x^2y + x^3\frac{dy}{dx} + e^u\frac{d(x^2-1)}{dx} = \frac{2x\frac{dy^2}{dy}\frac{dy}{dx} - 2y^2}{x^2}$$
$$3x^2y + x^3\frac{dy}{dx} + e^{x^2-1}2x = \frac{2x2y\frac{dy}{dx} - 2y^2}{x^2}$$
$$3x^2y + x^3\frac{dy}{dx} + e^{x^2-1}2x = \frac{4xy\frac{dy}{dx} - 2y^2}{x^2}$$

At the point $(1, 1)$

$$\begin{aligned}
&3 + \frac{dy}{dx} + 2 = 4\frac{dy}{dx} - 2 \\
&7 = 3\frac{dy}{dx} \\
&\frac{dy}{dx} = \frac{7}{3}
\end{aligned}$$

So an equation of the tangent to the curve at $(1, 1)$ is $y - 1 = \frac{7}{3}(x - 1)$.

(e) Compute $\frac{d^3}{dx^3}\left(\frac{1}{2x}\right)$.

Answer: $-3x^{-4}$

**Solution:**

$$\begin{aligned}
\frac{df}{dx} &= \frac{d}{dx}\left(\frac{1}{2x}\right) = \frac{1}{2}\frac{dx^{-1}}{dx} = \frac{-1}{2}x^{-2} \\
\frac{d^2f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{-1}{2}x^{-2}\right) = \frac{-1}{2}\frac{dx^{-2}}{dx} = x^{-3} \\
\frac{d^3f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right) = \frac{dx^{-3}}{dx} = -3x^{-4}
\end{aligned}$$

## Long answer question — you must show your work

2. 4 marks The demand equation is given by $p = \left(\frac{1}{2} - \frac{q}{200}\right)^2$ where $p$ is the unit price and $q$ is the demand quantity.

(a) Estimate the revenue made by selling the $10^{th}$ unit.

Answer: $MC(9) = \left(\frac{1}{2} - \frac{9}{200}\right)^2 + \left(\frac{1}{2} - \frac{9}{200}\right)\left(\frac{-18}{200}\right)$

**Solution:**

Note that the revenue made selling the 10th unit is $R(10) - R(9) \approx MR(9)$.

$$R(q) = q\left(\frac{1}{2} - \frac{q}{200}\right)^2$$

$$\frac{dR}{dq} = \frac{dq}{dq}\left(\frac{1}{2} - \frac{q}{200}\right)^2 + q\frac{d}{dq}\left(\frac{1}{2} - \frac{q}{200}\right)^2$$

$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)^2 + q\frac{du^2}{du}\frac{du}{dx}, \qquad u = \frac{1}{2} - \frac{q}{200}$$

$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)^2 + q2u\left(\frac{-1}{200}\right)$$

$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)^2 + q2u\left(\frac{-1}{200}\right)$$

$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)^2 + \left(\frac{1}{2} - \frac{q}{200}\right)\left(\frac{-2q}{200}\right)$$

$$MR(9) = R'(9) = \left(\frac{1}{2} - \frac{9}{200}\right)^2 + \left(\frac{1}{2} - \frac{9}{200}\right)\left(\frac{-18}{200}\right)$$

(b) Find the demand quantity at which revenue is maximized.

Answer: $q = 100/3$

**Solution:** From prevoious part we have $dR/dq$.

$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)^2 + \left(\frac{1}{2} - \frac{q}{200}\right)\left(\frac{-2q}{200}\right)$$

$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)\left(\frac{1}{2} - \frac{q}{200} - \frac{2q}{200}\right)$$

$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)\left(\frac{1}{2} - \frac{3q}{200}\right)$$

$$\frac{dR}{dq} = 0 \Rightarrow q = 100 \text{ or } q = \frac{100}{3}$$

$$R(100) = 0, \ R(100/3) = \frac{100}{3}\left(\frac{1}{2} - \frac{100}{600}\right)^2 > 0$$

So revenue is maximized at $q = 100/3$.

3. [6 marks] Find the point (in the first quadrant) on the lemniscate $2(x^2 + y^2)^2 = 9(x^2 - y^2)$ where the tangent is horizontal.

**Solution:**

$$\frac{d}{dx}(2(x^2+y^2)^2) = \frac{d}{dx}(9(x^2-y^2)) \tag{1}$$

$$4(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 18x - 18y\frac{dy}{dx} \tag{2}$$

We're looking for horizontal tangent lines, i.e. points on the curve where $\frac{dy}{dx} = 0$.

So if the tangent line at point $(a, b)$ on the curve is horizontal, I can plug $x = a, y = b$ and $\frac{dy}{dx} = 0$ in equation (2) to get:

$$4(a^2+b^2)2a = 18a$$
$$a^2+b^2 = \frac{9}{4}$$

Also since $(a, b)$ is a point on the curve, we have $2(a^2+b^2)^2 = 9(a^2-b^2)$. So I have two equations that I can solve for $a$ and $b$.

$$\begin{cases} a^2+b^2 = \frac{9}{4} \\ 2(a^2+b^2)^2 = 9(a^2-b^2) \end{cases}$$

Using the first equation I have $b^2 = \frac{9}{4} - a^2$. I can plug this in the second equation to find $a$:

$$2(a^2+b^2)^2 = 9(a^2-b^2) \Rightarrow 2(a^2+\frac{9}{4}-a^2)^2 = 9(a^2-\frac{9}{4}+a^2)$$
$$\Rightarrow 2\left(\frac{9}{4}\right)^2 = 18a^2 \Rightarrow a^2 = \frac{1}{9}\left(\frac{9}{4}\right)^2$$

Since we're looking for a point in the first quadrant, we have $a > 0$ and $b > 0$. Therefore,

$$a = \sqrt{\frac{1}{9}\left(\frac{9}{4}\right)^2} = \frac{1}{3}\left(\frac{9}{4}\right) = \frac{3}{4}$$
$$b = \sqrt{\frac{9}{4} - a^2} = \sqrt{\frac{9}{4} - \frac{9}{16}} = \sqrt{\frac{36}{16} - \frac{9}{16}} = \sqrt{\frac{27}{16}}$$

So the final answer is $a = 3/4$ and $b = \sqrt{27/16}$.