First Name: ______________________ Last Name: ______________________

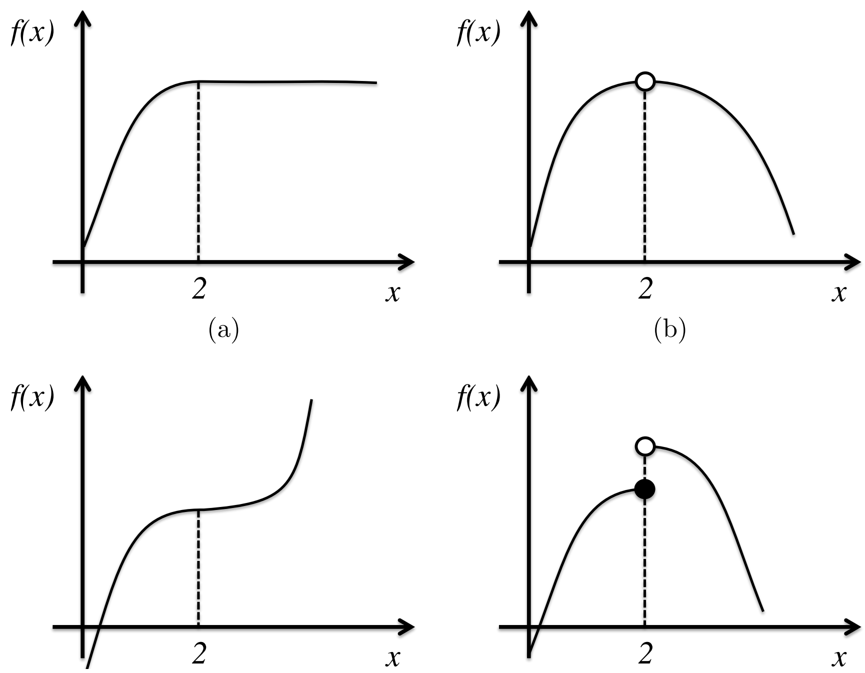
Student-No: ______________________ Grade: ______________________

Short answer questions — you must show your work

1. [6 marks] Each part is worth 2 marks.

   (a) The following statement is false: "If $\lim_{x\to 2} f'(x) = 0$, then $x = 2$ is a local extremum." Draw a graph of a function that illustrates a counterexample.

   Solution: Note that although $\lim_{x\to 2} f'(x) = 0$ exists, f' may not be defined at $x = 2$ and f may be discontinuous at $x = 2$. A couple of counterexamples are illustrated below. Can you think of more?



   (b) The population of a certain bacterial colony triples every day. Find the relative growth rate of this bacterial colony.

   Answer: $\ln 3$

   Solution: Since the population triples every day, it's growing exponentially. Let's assume $P(t) = P(0)e^{kt}$ where $P(0)$ is the population at $t = 0$, k is a constant and t is time in hours. Since population triples every day, we have $P(1) = 3P(0)$.

$$P(t) = P(0)e^{kt}$$
$$P(1) = 3P(0) = P(0)e^{k} \Rightarrow 3 = e^{k} \Rightarrow \ln(3) = k \Rightarrow k = \ln 3$$

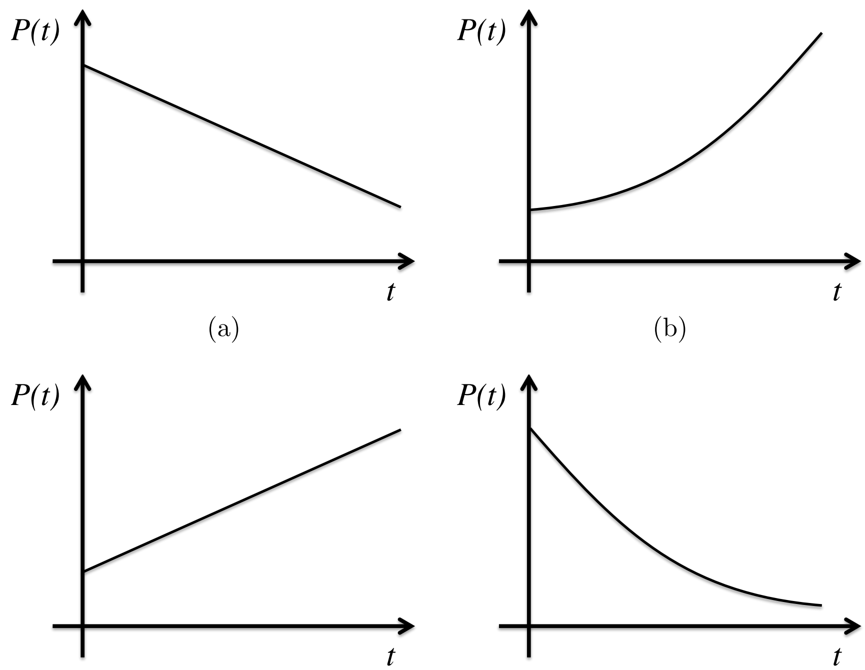
Relative growth rate is defined at $\frac{1}{P(t)}\frac{dP}{dt}$. so we have

$$\frac{dP}{dt} = P(0)ke^{kt}$$
$$\frac{1}{p}\frac{dP}{dt} = \frac{P(0)ke^{kt}}{P(t)} = \frac{P(0)ke^{kt}}{P(0)e^{kt}} = k$$

So the relative growth rate is $k = \ln 3$.

(c) The population of an ant colony, $P(t)$, reduces by 20% every day. Identify all the graphs that may show $P(t)$.

Answer: d



(a) (b) (c) (d)

Solution: The population reduces by 20% every day so choices (a) and (c) are not acceptable.

Since the population reduces by 20% everyday, it is an exponential function of time. So the correct answer is (d).

Long answer question — you must show your work

2. 6 marks A toy train runs on a circular track of radius $R = \sqrt{5}$m described by the equation $x^2 + y^2 = R^2$. The velocity of the train in the x-direction is given by dx/dt and the velocity of the train in the y-direction is given by dy/dt. When the train is at the point $(2, 1)$ on the track, its velocity in x-direction is -0.3m/s. Find the train's velocity in y-direction at that moment.

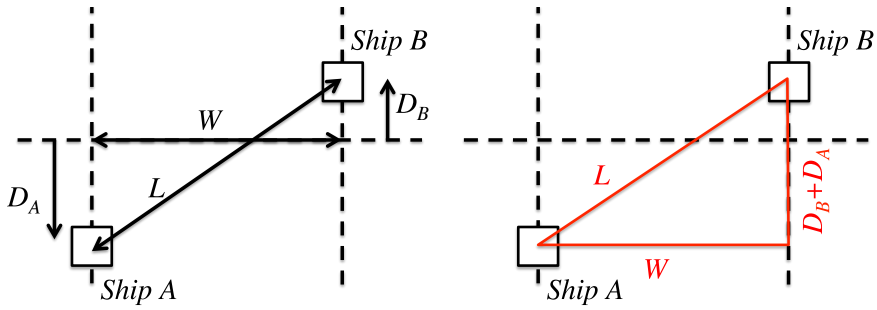
Solution:

$$x^2 + y^2 = 5$$
$$\frac{d}{dt}(x^2 + y^2) = \frac{d5}{dt}$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

we know that $x = 2m,\ y = 1m,\ \dfrac{dx}{dy} = -0.3m/s$. We can substitute these values in the above equation to find $\dfrac{dy}{dt}$.

$$2 \times 2 \times (-0.3) + 2 \times 1 \times \frac{dy}{dt} = 0$$
$$\frac{dy}{dt} = \frac{1.2}{2} = 0.6m/s$$

3. [6 marks] At noon, ship A is 60km east of ship B. Ship A is sailing south at 40km/h and ship B is sailing north at 30km/h. How fast is the distance between the ships changing at 4:00pm?



(a) (b)

Solution:

Step0: See Fig. (a) above.

Step1: We choose noon as our reference time. So t denotes time in hours since noon.

$$W = 60km,$$
$$\frac{dD_A}{dt} = 40km/h, \quad \frac{dD_B}{dt} = 30km/h$$
$$D_A(0) = 0, \quad D_B(0) = 0$$

Step2: We're looking for $\frac{dL}{dt}$.

Step3: We can use Pythagorean theorem to write the following equation that relates D_A, D_B and L. See Fig. (b) above.

$$L^2 = W^2 + (D_A + D_B)^2$$

Step4: W is constant and does not change with time. So we have $\frac{dW}{dt} = 0$.

D_A, D_B and L change with time.

Step5: We now differentiate with respect to t.

$$\frac{dL^2}{dt} = \frac{d}{dt}(W^2 + (D_A + D_B)^2)$$
$$2L\frac{dL}{dt} = 0 + 2(D_A + D_B)\left(\frac{dD_A}{dt} + \frac{dD_B}{dt}\right)$$
$$\frac{dL}{dt} = \frac{D_A + D_B}{L}\left(\frac{dD_A}{dt} + \frac{dD_B}{dt}\right)$$

Step6: We need to find D_A, D_B and L at 4pm. D_A and D_B both have constant growth rates. So they are both growing linearly.

$$D_A(0) = 0, \frac{dD_A}{dt} = 40km/h \Rightarrow D_A = 40t,$$
$$D_B(0) = 0, \frac{dD_B}{dt} = 30km/h \Rightarrow D_B = 30t$$

We can now use these equations to find $D_A(4)$ and $D_B(4)$, then plug these parameters in the euqation given in step3 to find $L(4)$.

$$\Rightarrow D_A(4) = 160km,\ D_B(4) = 120km$$
$$\Rightarrow L = \sqrt{W^2 + (D_A + D_B)^2} = \sqrt{60^2 + (160 + 120)^2}$$

Now putting everything back in equation , we have

$$\frac{dL}{dt} = \frac{D_A + D_B}{L}\left(\frac{dD_A}{dt} + \frac{dD_B}{dt}\right) = \frac{160 + 120}{\sqrt{60^2 + (160 + 120)^2}}(40 + 30)$$