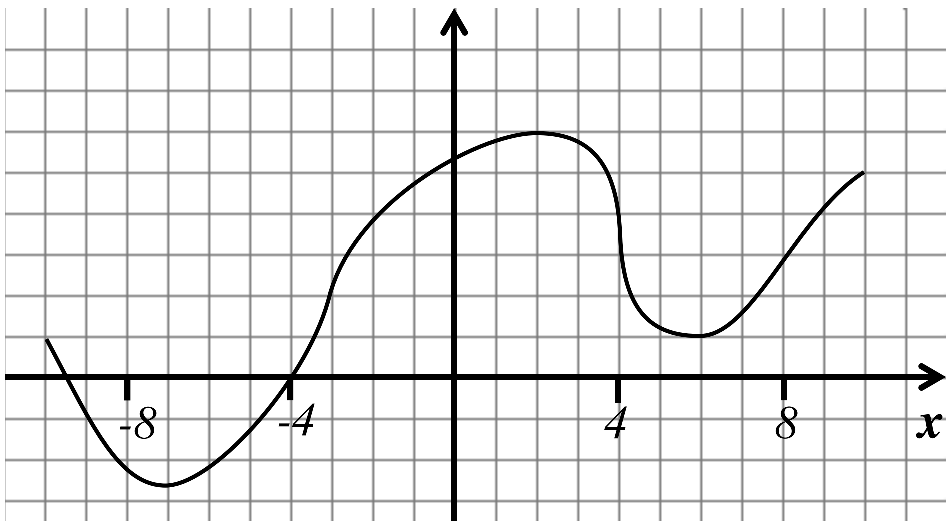
First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Grade: ______________________

Short answer questions — you must show your work

1. [8 marks] Below is a graph of $f'(x)$, the derivative of $f(x)$. The domain of the function is $(-10, 10)$.



(a) Where is $f(x)$ increasing?

Answer: $(-10, -9.5) \cup (-4, 10)$

(b) Determine the x coordinate of all the inflection points of f. If there are no inflection points, write *none* in the answer box.

Answer: $-7, 2, 6$

(c) Where is $f(x)$ concave up?

Answer: $(-7, 2) \cup (6, 10)$

(d) Determine the x coordinate of all the points where f has a local maximum. If there are no local maximums, write *none* in the answer box.

Answer: -9.5

Long answer question — you must show your work

2. [12 marks] Consider the function $f(x) = \dfrac{2x-3}{x-1}$.

(a) Identify the domain of the function.

Solution: Domain: $(-\infty, 1) \cup (1, \infty)$

(b) Identify intervals where f is increasing and the intervals where f is decreasing.

Solution:

$$\frac{df}{dx} = \frac{d}{dx}\left(\frac{2x-3}{x-1}\right) = \frac{2x-2-2x+3}{(x-1)^2} = \frac{1}{(x-1)^2}$$

Note that $(x-1)^2 > 0$ everywhere in the domain of f. So $f(x)$ is increasing on $(-\infty, 1) \cup (1, \infty)$.

(c) Identify intervals where f is concave up and the intervals where f is concave down.

Solution:

$$\frac{d^2 f}{dx^2} = \frac{df'}{dx} = \frac{d}{dx}\left(\frac{1}{(x-1)^2}\right) = \frac{-2}{(x-1)^3}$$

Note that if $x-1<0$, then $(x-1)^3 < 0$. Similarly, if $x-1>0$, then $(x-1)^3 > 0$. Therefore, we have

$$\begin{cases} f'' > 0, & x < 1 \\ f'' < 0, & x > 1 \end{cases}$$

So $f(x)$ is concave up when $x < 1$ and it's concave down when $x > 1$.

(d) List the x coordinate of all the critical number and inflection points. Write down *none* if there are no such points.

Critical numbers:	*none*
Inflection points:	*none*

Solution:
critical numbers:

- $f'(x) = 0 \Rightarrow \dfrac{1}{(x-1)^2} = 0$ but this does not have a solution.
- $f'(x)$ does not exist. $\Rightarrow x = 1$. However, $x = 1$ is not in the domain. So $x = 1$ is not a critical number. So f has no critical numbers in its domain.

Inflection points:

These are points in the domain of f where concavity changes (from CU to CD or from CD to CU). Concavity changes from CU to CD at $x = 1$. However, since $x = 1$ is not in the domain of the function, this is not an inflection point.

(e) Find all the horizontal and vertical asymptotes of f. Write down *none* if there are no horizontal/vertical asymptotes.

Horizontal asymptotes:	$y = 2$
Vertical asymptotes:	$x = 1$

Solution: **Horizontal asymptotes:**

$$\lim_{x\to\infty} \frac{2x-3}{x-1} = \lim_{x\to\infty} \frac{x\left(2-\frac{3}{x}\right)}{x\left(1-\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{2-\frac{3}{x}}{1-\frac{1}{x}} = 2$$

$$\lim_{x\to-\infty} \frac{2x-3}{x-1} = \lim_{x\to-\infty} \frac{x\left(2-\frac{3}{x}\right)}{x\left(1-\frac{1}{x}\right)} = \lim_{x\to-\infty} \frac{2-\frac{3}{x}}{1-\frac{1}{x}} = 2$$

So there's only 1 horizontal asymptote: $y = 2$.

Vertical asymptotes:

$$x - 1 = 0 \Rightarrow x = 1$$
$$\lim_{x\to 1} \frac{2x-3}{x-1} \stackrel{?}{=} \frac{-1}{0}$$

At this point we know enough to claim that $x = 1$ is a vertical asymptote. If we want to study the limits more precisely we have

$$(x \to 1^+) \Rightarrow x > 1 \Rightarrow x - 1 > 0, 2x - 3 \approx -1 \Rightarrow \frac{2x-3}{x-1} < 0$$
$$\Rightarrow \lim_{x\to 1^+} \frac{2x-3}{x-1} = -\infty$$
$$(x \to 1^-) \Rightarrow x < 1 \Rightarrow x - 1 < 0, 2x - 3 \approx -1 \Rightarrow \frac{2x-3}{x-1} > 0$$
$$\Rightarrow \lim_{x\to 1^-} \frac{2x-3}{x-1} = \infty$$

(f) Using what you've found above, graph $f(x)$.

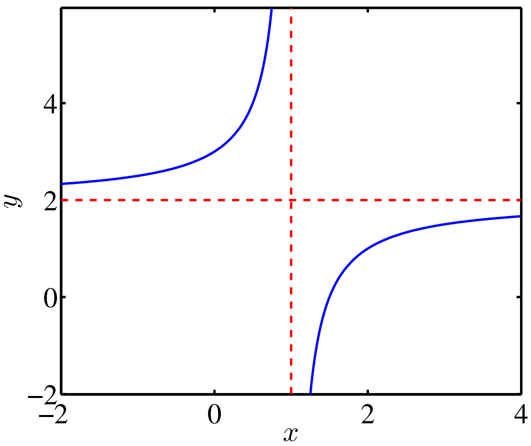


Figure 1: Graph of $f(x)$. Red dashed lines show the horizontal and vertical asymptotes.

(g) Using what you've found above, graph $g(x) = \dfrac{|2x-3|}{x-1}$.

Solution:

$$2x - 3 = 0 \Rightarrow x = 3/2$$

$$|2x-3| = \begin{cases} 2x-3, & x \geq 3/2 \\ -2x+3, & x < 3/2 \end{cases}$$

$$g(x) = \begin{cases} \dfrac{2x-3}{x-1} = f(x), & x \geq 3/2 \\ \dfrac{-(2x-3)}{x-1} = -f(x), & x < 3/2 \end{cases}$$

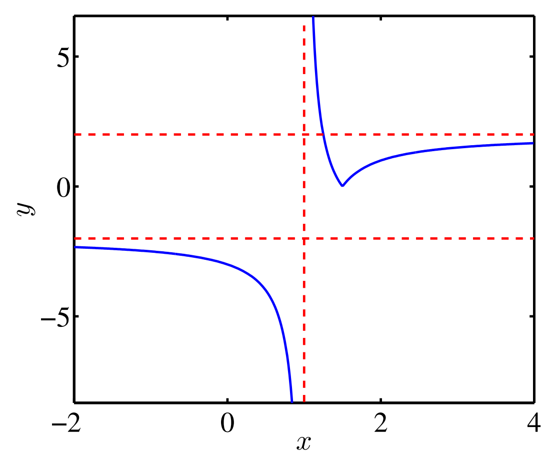


Figure 2: Graph of $g(x)$. Red dashed lines show the horizontal and vertical asymptotes.