The z-Transform and Its Application to the Analysis of LTI Systems
The Direct $z$-Transform

The $z$-transform of a discrete-time signal $x(n)$ is defined as the power series

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

where $z$ is a complex variable.

For convenience, the $z$-transform of a signal $x(n)$ is denoted by

$$X(z) = Z\{x(n)\}$$

whereas the relationship between $x(n)$ and $X(z)$ is indicated by

$$x(n) \longleftrightarrow X(z)$$

Since the $z$-transform is an infinite power series, it exists only for those values of $z$ for which this series converges. The region of convergence (ROC) of $X(z)$ is the set of all values of $z$ for which $X(z)$ attains a finite value. Thus any time we cite a $z$-transform we should also indicate its ROC.
Example

Determine the $z$-transforms of the following finite-duration signals.

(a) $x_1(n) = [1, 2, 5, 7, 0, 1]$

(b) $x_2(n) = [1, 2, 5, 7, 0, 1]$

(c) $x_3(n) = [0, 0, 1, 2, 5, 7, 0, 1]$

(d) $x_4(n) = [2, 4, 5, 7, 0, 1]$

(e) $x_5(n) = \delta(n)$

(f) $x_6(n) = \delta(n - k), k > 0$

(g) $x_7(n) = \delta(n + k), k > 0$

Solution. From definition (3.1.1), we have

(a) $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-3}, \text{ROC: entire } z\text{-plane except } z = 0$

(b) $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}, \text{ROC: entire } z\text{-plane except } z = 0 \text{ and } z = \infty$

(c) $X_3(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}, \text{ROC: entire } z\text{-plane except } z = 0$

(d) $X_4(z) = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}, \text{ROC: entire } z\text{-plane except } z = 0 \text{ and } z = \infty$

(e) $X_5(z) = 1$ [i.e., $\delta(n) \leftrightarrow 1$], \text{ROC: entire } z\text{-plane}

(f) $X_6(z) = z^{-1}$ [i.e., $\delta(n - k) \leftrightarrow z^{-1}$], $k > 0$, \text{ROC: entire } z\text{-plane except } z = 0$

(g) $X_7(z) = z^k$ [i.e., $\delta(n + k) \leftrightarrow z^k$], $k > 0$, \text{ROC: entire } z\text{-plane except } z = \infty
EXAMPLE 3.1.2

Determine the z-transform of the signal

\[ x(n) = \left(\frac{1}{2}\right)^n u(n) \]

Solution. The signal \( x(n) \) consists of an infinite number of nonzero values

\[ x(n) = \{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots\} \]

The z-transform of \( x(n) \) is the infinite power series

\[ X(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{2^2}z^{-2} + \frac{1}{2^3}z^{-3} + \ldots \]

\[ = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^nz^{-n} = \sum_{n=0}^{\infty} \left(\frac{z^{-1}}{2}\right)^n \]

This is an infinite geometric series. We recall that

\[ 1 + A + A^2 + A^3 + \ldots = \frac{1}{1-A} \quad \text{if} \quad |A| < 1 \]

Consequently, for \( \left|\frac{1}{2}z^{-1}\right| < 1 \), or equivalently, for \( |z| > \frac{1}{2} \), \( X(z) \) converges to

\[ X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \quad \text{ROC:} \quad |z| > \frac{1}{2} \]

We see that in this case, the z-transform provides a compact alternative representation of the signal \( x(n) \).
Region of convergence for $X(z)$ and its corresponding causal and anticausal components.
Example

Determine the $z$-transform of the signal

$$x(n) = \alpha^n u(n) = \begin{cases} 
\alpha^n, & n \geq 0 \\
0, & n < 0 
\end{cases}$$

Solution. From the definition (3.1.1) we have

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

If $|\alpha z^{-1}| < 1$ or equivalently, $|z| > |\alpha|$, this power series converges to $1/(1 - \alpha z^{-1})$. Thus we have the $z$-transform pair
The exponential signal \( x(n) = \alpha^n u(n) \) (a), and the ROC of its z-transform (b).

\[
x(n) = \alpha^n u(n) \leftrightarrow X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{ROC: } |z| > |\alpha|
\]

The ROC is the exterior of a circle having radius \(|\alpha|\). Figure 3.1.2 shows a graph of the signal \( x(n) \) and its corresponding ROC. Note that, in general, \( \alpha \) need not be real.

If we set \( \alpha = 1 \) in (3.1.7), we obtain the z-transform of the unit step signal

\[
x(n) = u(n) \leftrightarrow X(z) = \frac{1}{1 - z^{-1}}, \quad \text{ROC: } |z| > 1
\]
Determine the $z$-transform of the signal

$$x(n) = -\alpha^n u(-n - 1) = \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

**Solution.** From the definition (3.1.1) we have

$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha^n)z^{-n} = -\sum_{l=1}^{\infty} (\alpha^{-1}z)^l$$

where $l = -n$. Using the formula

$$A + A^2 + A^3 + \cdots = A(1 + A + A^2 + \cdots) = \frac{A}{1 - A}$$

when $|A| < 1$ gives

$$X(z) = -\frac{\alpha^{-1}z}{1 - \alpha^{-1}z} = \frac{1}{1 - \alpha z^{-1}}$$

provided that $|\alpha^{-1}z| < 1$ or, equivalently, $|z| < |\alpha|$. Thus

$$x(n) = -\alpha^n u(-n - 1) \iff X(z) = -\frac{1}{1 - \alpha z^{-1}}, \quad \text{ROC: } |z| < |\alpha|$$

The ROC is now the interior of a circle having radius $|\alpha|$. This is shown in Fig. 3.1.3.
Anticausal signal $x(n) = -\alpha^n u(-n - 1)$ (a), and the ROC of its $z$-transform (b).
Example

Determine the $z$-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n - 1)$$

**Solution.** From definition (3.1.1) we have

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{l=1}^{\infty} (b^{-1} z)^l$$

The first power series converges if $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$. The second power series converges if $|b^{-1} z| < 1$ or $|z| < |b|$. In determining the convergence of $X(z)$, we consider two different cases.
ROC for z-transform in Example 3.15.

Case 1 \( |b| < |a| \): In this case the two ROC above do not overlap. Consequently, we cannot find values of \( z \) for which both power series converge simultaneously. Clearly, in this case, \( X(z) \) does not exist.

Case 2 \( |b| > |a| \): In this case there is a ring in the \( z \)-plane where both power series converge simultaneously, as shown in Fig. 3.1.4(b). Then we obtain

\[
X(z) = \frac{1}{1 - az^{-1}} \cdot \frac{1}{1 - bz^{-1}}
\]

\[
= \frac{a + b - z - abz^{-1}}{z - az^{-1} - bz^{-1}}
\]

The ROC of \( X(z) \) is \( |a| < |z| < |b| \).
Properties of the z-Transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Time Domain</th>
<th>Z-Domain</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notation</td>
<td>(x(n))</td>
<td>(X(z))</td>
<td>(R_{22} =</td>
</tr>
<tr>
<td>(x_1(n))</td>
<td></td>
<td>(X_1(z))</td>
<td>(R_{12})</td>
</tr>
<tr>
<td>(x_2(n))</td>
<td></td>
<td>(X_2(z))</td>
<td>(R_{21})</td>
</tr>
<tr>
<td>Linearity</td>
<td>(a_1x_1(n) + a_2x_2(n))</td>
<td>(a_1X_1(z) + a_2X_2(z))</td>
<td>At least the intersection of (R_{22}) and (R_{12})</td>
</tr>
<tr>
<td>Time shifting</td>
<td>(x(n-k))</td>
<td>(z^{-k}X(z))</td>
<td>That of (X(z)), except (z = 0) if (k &gt; 0) and (z = \infty) if (k &lt; 0)</td>
</tr>
<tr>
<td>Scaling in the z-domain</td>
<td>(a^n x(n))</td>
<td>(X(a^{-n}z))</td>
<td>(</td>
</tr>
<tr>
<td>Time reversal</td>
<td>(x(-n))</td>
<td>(X(z^{-1}))</td>
<td>(</td>
</tr>
<tr>
<td>Conjugation</td>
<td>(x^*(n))</td>
<td>(X^<em>(z^</em>))</td>
<td>(R_{21})</td>
</tr>
<tr>
<td>Real part</td>
<td>(\text{Re}{x(n)})</td>
<td>(\frac{1}{2}[X(z) + X^<em>(z^</em>)])</td>
<td>Includes ROC</td>
</tr>
<tr>
<td>Imaginary part</td>
<td>(\text{Im}{x(n)})</td>
<td>(\frac{1}{2}[X(z) - X^<em>(z^</em>)])</td>
<td>Includes ROC</td>
</tr>
<tr>
<td>Differentiation in the z-domain</td>
<td>(nx(n))</td>
<td>(-z \frac{dX(z)}{dz})</td>
<td>(r_2 =</td>
</tr>
<tr>
<td>Convolution</td>
<td>(x_1(n) * x_2(n))</td>
<td>(X_1(z)X_2(z))</td>
<td>At least, the intersection of (R_{12}) and (R_{22})</td>
</tr>
<tr>
<td>Correlation</td>
<td>(r_{x_1x_2}(\ell))</td>
<td>(R_{x_1x_2}(z) = X_1(z)X_2(z^{-\ell}))</td>
<td>At least, the intersection of ROC of (X_1(z)) and (X_2(z^{-\ell}))</td>
</tr>
<tr>
<td>Initial value theorem</td>
<td>If (x(n)) causal</td>
<td>(x(0) = \lim_{n \to \infty} X(z))</td>
<td>(r_{12} &lt;</td>
</tr>
<tr>
<td>Multiplication</td>
<td>(x_1(n)x_2(n))</td>
<td>(\frac{1}{2\pi j} \oint_{C} X_1(z)X_2^*(z^{-1}) e^{-zv} dv)</td>
<td>At least, (r_{12} &lt;</td>
</tr>
<tr>
<td>Parseval’s relation</td>
<td>[\sum_{n=-\infty}^{\infty}</td>
<td>x(n)</td>
<td>^2]</td>
</tr>
</tbody>
</table>
### TABLE 3.3 Some Common $z$-Transform Pairs

<table>
<thead>
<tr>
<th>Signal, $x(n)$</th>
<th>$z$-Transform, $X(z)$</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $\delta(n)$</td>
<td>1</td>
<td>All $z$</td>
</tr>
<tr>
<td>2 $u(n)$</td>
<td>$\frac{1}{1-z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>3 $a^n u(n)$</td>
<td>$\frac{1}{1-az^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>4 $na^n u(n)$</td>
<td>$\frac{az^{-1}}{(1-az^{-1})^2}$</td>
<td>$</td>
</tr>
<tr>
<td>5 $-a^n u(-n-1)$</td>
<td>$\frac{1}{1-az^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>6 $-na^n u(-n-1)$</td>
<td>$\frac{az^{-1}}{(1-az^{-1})^2}$</td>
<td>$</td>
</tr>
<tr>
<td>7 $(\cos \omega n) u(n)$</td>
<td>$\frac{1 - z^{-1} \cos \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>8 $(\sin \omega n) u(n)$</td>
<td>$\frac{z^{-1} \sin \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>9 $(a^n \cos \omega n) u(n)$</td>
<td>$\frac{1 - az^{-1} \cos \omega_0}{1 - 2az^{-1} \cos \omega_0 + a^2 z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>10 $(a^n \sin \omega n) u(n)$</td>
<td>$\frac{az^{-1} \sin \omega_0}{1 - 2az^{-1} \cos \omega_0 + a^2 z^{-2}}$</td>
<td>$</td>
</tr>
</tbody>
</table>
Pole-zero location

\[
X(z) = \frac{B(z)}{A(z)} = \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{(z - p_1)(z - p_2) \cdots (z - p_N)}
\]

\[
X(z) = G z^{N-M} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}
\]
Example of first order system

Determine the pole-zero plot for the signal

\[ x(n) = a^n u(n), \quad a > 0 \]

**Solution.** From Table 3.3 we find that

\[ X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \]

ROC: \(|z| > a\)

Thus \(X(z)\) has one zero at \(z_1 = 0\) and one pole at \(p_1 = a\). The pole-zero plot is shown in Fig. 3.3.1. Note that the pole \(p_1 = a\) is not included in the ROC since the \(z\)-transform does not converge at a pole.

Pole-zero plot for the causal exponential signal \(x(n) = a^n u(n)\).
Time Domain Behaviour:

Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.
System Function of LTI Systems

\[ Y(z) = H(z)X(z) \]

\[ H(z) = \frac{Y(z)}{X(z)} \]

\[ H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \]
System Function derived from Difference Equation

\[ y(n) = - \sum_{k=1}^{N} a_k y(n - k) + \sum_{k=0}^{M} b_k x(n - k) \]

\[ Y(z) = - \sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k} \]

\[ Y(z) \left( 1 + \sum_{k=1}^{N} a_k z^{-k} \right) = X(z) \left( \sum_{k=0}^{M} b_k z^{-k} \right) \]

\[ \frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \]
System Function of FIR System

Let $a_k = 0$ for $1 \leq k \leq N$

Then:

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

There is no pole except at zero.
Example

Determine the system function and the unit sample response of the system described by the difference equation

\[ y(n) = \frac{1}{2} y(n - 1) + 2x(n) \]

Solution. By computing the \( z \)-transform of the difference equation, we obtain

\[ Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z) \]

Hence the system function is

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2} z^{-1}} \]

This system has a pole at \( z = \frac{1}{2} \) and a zero at the origin. Using Table 3.3 we obtain the inverse transform

\[ h(n) = 2\left(\frac{1}{2}\right)^n u(n) \]

This is the unit sample response of the system.
Inversion of z-Transform

1. Direct evaluation of by contour integration.

\[ x(n) = \frac{1}{2\pi j} \oint_{C} X(z)z^{n-1}dz \]

2. Expansion into a series of terms, in the variables \(z\), and \(z^{-1}\).

Inverse $z$-Transform by Power Series Expansion

**EXAMPLE**

Determine the inverse $z$-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

which converges when

- (a) ROC: $|z| > 1$
- (b) ROC: $|z| < 0.5$
Example

Determine the inverse $z$-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

when

(a) ROC: $|z| > 1$

(b) ROC: $|z| < 0.5$
Solution (a)

(a) Since the ROC is the exterior of a circle, we expect $x(n)$ to be a causal signal. Thus we seek a power series expansion in negative powers of $z$. By dividing the numerator of $X(z)$ by its denominator, we obtain the power series

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{-4} + \cdots$$

By comparing this relation with (3.1.1), we conclude that

$$x(n) = \{1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots\}$$
Solution (b)

(b) In this case the ROC is the interior of a circle. Consequently, the signal $x(n)$ is anticausal. To obtain a power series expansion in positive powers of $z$, we perform the long division in the following way:

\[
\begin{array}{c|cccccccc}
& 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots \\
\hline
\frac{1}{2}z^{−2} − \frac{3}{2}z^{−1} + 1 & 1 & 3z + 2z^2 \\
& 3z + 2z^2 \hline
& 3z - 9z^2 + 6z^3 \\
& 7z^2 - 6z^3 \hline
& 7z^2 - 21z^3 + 14z^4 \\
& 15z^3 - 14z^4 \hline
& 15z^3 - 45z^4 + 30z^5 \\
& 31z^4 - 30z^5 \hline
\end{array}
\]

Thus

\[
X(z) = \frac{1}{1 - \frac{3}{2}z^{−1} + \frac{1}{2}z^{−2}} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots
\]

In this case $x(n) = 0$ for $n \geq 0$. By comparing this result to (3.1.1), we conclude that

\[
x(n) = \{ \cdots, 62, 30, 14, 6, 2, 0, 0 \}
\]
Example

Determine the inverse $z$-transform of

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|$$

**Solution.** Using the power series expansion for $\log(1 + x)$, with $|x| < 1$, we have

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

Thus

$$x(n) = \begin{cases} (-1)^{n+1} \frac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

Expansion of irrational functions into power series can be obtained from tables.
Inversion using partial-fraction expansion

Let \( X(z) \) be a proper rational function, that is,

\[
X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}
\]

where

\[ a_N \neq 0 \quad \text{and} \quad M < N \]

To simplify our discussion we eliminate negative powers of \( z \) by multiplying both the numerator and denominator of (3.4.12) by \( z^N \). This results in

\[
X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}
\]

which contains only positive powers of \( z \). Since \( N > M \), the function

\[
\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}
\]

is also always proper.
Inversion using partial-fraction expansion

Our task in performing a partial-fraction expansion is to express this as a sum of simple fractions. We distinguish two cases.

**Distinct poles.** Suppose that the poles $p_1, p_2, \ldots, p_N$ are all different (distinct). Then we seek an expansion of the form

$$
\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}
$$

The problem is to determine the coefficients $A_1, A_2, \ldots, A_N$. There are two ways to solve this problem, as illustrated in the following example.
Example

Determine the partial-fraction expansion of the proper function

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Solution. First we eliminate the negative powers, by multiplying both numerator and denominator by $z^2$. Thus

$$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

The poles of $X(z)$ are $p_1 = 1$ and $p_2 = 0.5$. Consequently, the expansion is

$$\frac{X(z)}{z} = \frac{z}{(z - 1)(z - 0.5)} = \frac{A_1}{z - 1} + \frac{A_2}{z - 0.5}$$

A very simple method to determine $A_1$ and $A_2$ is to multiply the equation by the denominator term $(z - 1)(z - 0.5)$. Thus we obtain

$$z = (z - 0.5)A_1 + (z - 1)A_2$$
Now if we set \( z = p_1 = 1 \) in (3.4.18), we eliminate the term involving \( A_2 \). Hence

\[
1 = (1 - 0.5)A_1
\]

Thus we obtain the result \( A_1 = 2 \). Next we return to (3.4.18) and set \( z = p_2 = 0.5 \), thus eliminating the term involving \( A_1 \), so we have

\[
0.5 = (0.5 - 1)A_2
\]

and hence \( A_2 = -1 \). Therefore, the result of the partial-fraction expansion is

\[
\frac{X(z)}{z} = \frac{2}{z - 1} - \frac{1}{z - 0.5}
\]
General Partial-Fraction Expansion Procedure (Single Poles)

The example given above suggests that we can determine the coefficients $A_1, A_2, \ldots, A_N$, by multiplying both sides by each of the terms $(z - p_k), k = 1, 2, \ldots, N$, and evaluating the resulting expressions at the corresponding pole positions, $p_1, p_2, \ldots, p_N$. Thus we have, in general,

\[
\frac{(z - p_k)X(z)}{z} = \frac{(z - p_k)A_1}{z - p_1} + \cdots + A_k + \cdots + \frac{(z - p_k)A_N}{z - p_N}
\]

Consequently, with $z = p_k$, (3.4.20) yields the $k$th coefficient as

\[
A_k = \left. \frac{(z - p_k)X(z)}{z} \right|_{z=p_k}, \quad k = 1, 2, \ldots, N
\]
Example

Determine the partial-fraction expansion of

\[ X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}} \]

**Solution.** To eliminate negative powers of \( z \) we multiply both numerator and denominator by \( z^2 \). Thus

\[ \frac{X(z)}{z} = \frac{z + 1}{z^2 - z + 0.5} \]

The poles of \( X(z) \) are complex conjugates

\[ p_1 = \frac{1}{2} + \frac{1}{2} \left( -1 \right) \]

\[ p_2 = \frac{1}{2} - \frac{1}{2} \left( -1 \right) \]

and

Thus

\[ \frac{X(z)}{z} = \frac{z + 1}{(z - p_1)(z - p_2)} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} \]

we obtain

\[ A_1 = \left. \frac{(z - p_1)X(z)}{z} \right|_{z = p_1} = \left. \frac{z + 1}{z - p_2} \right|_{z = p_1} = \frac{1}{2} + \frac{1}{2} \left( -1 \right) = \frac{1}{2} - \frac{1}{2} \]

\[ A_2 = \left. \frac{(z - p_2)X(z)}{z} \right|_{z = p_2} = \left. \frac{z + 1}{z - p_1} \right|_{z = p_2} = \frac{1}{2} - \frac{1}{2} \left( -1 \right) = \frac{1}{2} + \frac{1}{2} \]
Multiple Order Poles (Example)

Determine the partial fraction expansion of

\[ X(z) = \frac{1}{(z+1)(z-1)^2} \]  

**Solution.** First, we express (3.4.23) in terms of positive powers of \( z \), in the form

\[ X(z) = \frac{A_1}{z + 1} + \frac{A_2}{z - 1} + \frac{A_3}{(z - 1)^2} \]  

(3.4.24)

\( X(z) \) has a simple pole at \( p_1 = -1 \) and a double pole \( p_2 = p_3 = 1 \). In such a case the appropriate partial fraction expansion is

\[ X(z) = \frac{z^2}{(z + 1)(z - 1)^2} = \frac{A_1}{z + 1} + \frac{A_2}{z - 1} + \frac{A_3}{(z - 1)^2} \]  

(3.4.25)

The problem is to determine the coefficients \( A_1 \), \( A_2 \), and \( A_3 \).

We proceed as in the case of distinct poles. To determine \( A_1 \), we multiply both sides of (3.4.24) by \( (z + 1) \) and evaluate the result at \( z = -1 \). Thus (3.4.25) becomes

\[ \frac{(z + 1)X(z)}{z} = A_1 + \frac{A_2}{z - 1} + \frac{A_3}{(z - 1)^2} \]  

which, when evaluated at \( z = -1 \), yields

\[ A_1 = \frac{(z + 1)X(z)}{z} \bigg|_{z = -1} = 1 \]

Next, if we multiply both sides of (3.4.24) by \( (z - 1)^2 \), we obtain

\[ \frac{(z - 1)^2X(z)}{z} = A_2 + (z - 1)A_3 \]  

(3.4.26)

Now, if we evaluate (3.4.25) at \( z = 1 \), we obtain \( A_3 \). Thus

\[ A_2 = \frac{(z - 1)^2X(z)}{z} \bigg|_{z = 1} = 1 \]

The remaining coefficient \( A_3 \) can be obtained by differentiating both sides of (3.4.25) with respect to \( z \) and evaluating the result at \( z = 1 \). Note that it is not necessary formally to carry out the differentiation of the right-hand side of (3.4.25), since all terms except \( A_2 \) vanish when we set \( z = 1 \). Thus

\[ A_3 = \frac{d}{dz} \left( \frac{(z - 1)^2X(z)}{z} \right) \bigg|_{z = 1} = \frac{3}{4} \]  

(3.4.27)
The generalization of the procedure in the example above to the case of an \( m \)th-order pole \( (z - p_k)^m \) is straightforward. The partial-fraction expansion must contain the terms

\[
\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \ldots + \frac{A_{mk}}{(z - p_k)^m}
\]

The coefficients \( \{A_{ik}\} \) can be evaluated through differentiation.
Causality and Stability

- A system is causal if,
  - \( h(n) = 0 \) for \( n < 0 \)
- So, an LTI system is causal if and only if the ROC of \( H(z) \) is exterior of a circle with radius \( r < \infty \).

- An LTI System is BIBO stable if the unit circle lies in the region of convergence of \( H(z) \).
A linear time-invariant system is characterized by the system function

\[ H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}} \]

\[ = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}} \]

Specify the ROC of \( H(z) \) and determine \( h(n) \) for the following conditions:

(a) The system is stable.
(b) The system is causal.
(c) The system is anticausal.
Causality and Stability (Example)

Solution. The system has poles at $z = \frac{1}{2}$ and $z = 3$.

(a) Since the system is stable, its ROC must include the unit circle and hence it is $\frac{1}{2} < |z| < 3$. Consequently, $h(n)$ is noncausal and is given as

$$h(n) = \left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n - 1)$$

(b) Since the system is causal, its ROC is $|z| > 3$. In this case

$$h(n) = \left(\frac{1}{2}\right)^n u(n) + 2(3)^n u(n)$$

This system is unstable.

(c) If the system is anticausal, its ROC is $|z| < 0.5$. Hence

$$h(n) = -\left[\left(\frac{1}{2}\right)^n + 2(3)^n\right] u(-n - 1)$$

In this case the system is unstable.
One-sided z-Transform

The one-sided or unilateral z-transform of a signal $x(n)$ is defined by

$$X^+(z) \equiv \sum_{n=0}^{\infty} x(n)z^{-n}$$

We also use the notations $Z^+\{x(n)\}$ and

$$x(n) \leftrightarrow_{Z^+} X^+(z)$$
Determine the $z$-transforms of the following finite-duration signals.

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$
(b) $x_2(n) = \{1, 2, 5, 7, 0, 1\}$
(c) $x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$
(d) $x_4(n) = \{2, 4, 5, 7, 0, 1\}$
(e) $x_5(n) \rightarrow \delta(n)$
(f) $x_6(n) = \delta(n - k), k > 0$
(g) $x_7(n) = \delta(n + k), k > 0$
One-sided z-Transform (Examples)

Solution.

\[ x_1(n) = \{1, 2, 5, 7, 0, 1\} \xleftarrow{\mathrm{z^+}} X_1^+(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5} \]

\[ x_2(n) = \{1, 2, 5, 7, 0, 1\} \xleftarrow{\mathrm{z^+}} X_2^+(z) = 5 + 7z^{-1} + z^{-3} \]

\[ x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\} \xleftarrow{\mathrm{z^+}} X_3^+(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7} \]

\[ x_4(n) = \{2, 4, 5, 7, 0, 1\} \xleftarrow{\mathrm{z^+}} X_4^+(z) = 5 + 7z^{-1} + z^{-3} \]

\[ x_5(n) = \delta(n) \xleftarrow{\mathrm{z^+}} X_5^+(z) = 1 \]

\[ x_6(n) = \delta(n - k), \quad k > 0 \xleftarrow{\mathrm{z^+}} X_6^+(z) = z^{-k} \]

\[ x_7(n) = \delta(n + k), \quad k > 0 \xleftarrow{\mathrm{z^+}} X_7^+(z) = 0 \]
**One-sided z-Transform (Properties)**

**Shifting Property**

**Case 1: Time delay**

If

$$x(n) \xrightarrow{z} X^+(z)$$

then

$$x(n-k) \xrightarrow{z} z^{-k}[X^+(z) + \sum_{n=1}^{k} x(-n)z^n], \quad k > 0 \quad (3.6.2)$$

In case $x(n)$ is causal, then

$$x(n-k) \xrightarrow{z} z^{-k}X^+(z) \quad (3.6.3)$$

**Proof**

From the definition (3.6.1) we have

$$Z^+[x(n-k)] = z^{-k} \left[ \sum_{l=-k}^{-1} x(l)z^{-l} + \sum_{l=0}^{\infty} x(l)z^{-l} \right]$$

$$= z^{-k} \left[ \sum_{l=-1}^{-k} x(l)z^{-l} + X^+(z) \right]$$

By changing the index from $l$ to $n = -l$, the result in (3.6.2) is easily obtained.
Example

Determine the one-sided $z$-transform of the signals

(a) $x(n) = a^n u(n)$
(b) $x_1(n) = x(n - 2)$ where $x(n) = a^n$

Solution.
(a) From (3.6.1) we easily obtain

$$X^+(z) = \frac{1}{1 - az^{-1}}$$

(b) We will apply the shifting property for $k = 2$. Indeed, we have

$$Z^+\{x(n - 2)\} = z^{-2}[X^+(z) + x(-1)z + x(-2)z^2]$$

$$= z^{-2}X^+(z) + x(-1)z^{-1} + x(-2)$$

Since $x(-1) = a^{-1}, x(-2) = a^{-2}$, we obtain

$$X_1^+(z) = \frac{z^{-2}}{1 - az^{-1}} + a^{-1}z^{-1} + a^{-2}$$
Properties (Time Advance)

Case 2: Time advance

If

\[ x(n) \overset{z^+}{\longrightarrow} X^+(z) \]

then

\[ x(n + k) \overset{z^+}{\longrightarrow} z^k \left[ X^+(z) - \sum_{n=0}^{k-1} x(n)z^{-n} \right], \quad k > 0 \tag{3.6.5} \]

Proof  From (3.6.1) we have

\[ Z^+\{x(n + k)\} = \sum_{n=0}^{\infty} x(n + k)z^{-n} = z^k \sum_{l=0}^{\infty} x(l)z^{-l} \]

where we have changed the index of summation from \( n \) to \( l = n + k \). Now, from (3.6.1) we obtain

\[ X^+(z) = \sum_{l=0}^{\infty} x(l)z^{-l} = \sum_{l=0}^{k-1} x(l)z^{-l} + \sum_{l=k}^{\infty} x(l)z^{-l} \]

By combining the last two relations, we easily obtain (3.6.5).
Example (Time Advance)

With \(x(n)\), as given in Example 3.6.2, determine the one-sided \(z\)-transform of the signal

\[x_2(n) = x(n + 2)\]

**Solution.** We will apply the shifting theorem for \(k = 2\). From (3.6.5), with \(k = 2\), we obtain

\[Z^+(x(n + 2)) = z^2 X^+(z) - x(0)z^2 - x(1)z\]

But \(x(0) = 1\), \(x(1) = a\), and \(X^+(z) = 1/(1 - az^{-1})\). Thus

\[Z^+(x(n + 2)) = \frac{z^2}{1 - az^{-1}} - z^2 - az\]
Properties (Time Advance)

With \(x(n)\), as given in Example 3.6.2, determine the one-sided \(z\)-transform of the signal

\[ x_2(n) = x(n + 2) \]

**Solution.** We will apply the shifting theorem for \(k = 2\). From (3.6.5), with \(k = 2\), we obtain

\[ Z^+\{x(n + 2)\} = z^2 X^+(z) - x(0)z^2 - x(1)z \]

But \(x(0) = 1\), \(x(1) = a\), and \(X^+(z) = 1/(1 - az^{-1})\). Thus

\[ Z^+\{x(n + 2)\} = \frac{z^2}{1 - az^{-1}} - z^2 - az \]
Final Value Theorem

**Final Value Theorem.** If

\[ x(n) \xrightarrow{z^+} X^+(z) \]

then

\[ \lim_{n \to \infty} x(n) = \lim_{z \to 1} (z - 1)X^+(z) \quad (3.6.6) \]

The limit in (3.6.6) exists if the ROC of \((z - 1)X^+(z)\) includes the unit circle.

The proof of this theorem is left as an exercise for the reader.

This theorem is useful when we are interested in the asymptotic behavior of a signal \(x(n)\) and we know its \(z\)-transform, but not the signal itself. In such cases, especially if it is complicated to invert \(X^+(z)\), we can use the final value theorem to determine the limit of \(x(n)\) as \(n\) goes to infinity.
Final Value Theorem (Example)

The impulse response of a relaxed linear time-invariant system is \( h(n) = \alpha^n u(n), \ |\alpha| < 1 \). Determine the value of the step response of the system as \( n \to \infty \).

**Solution.** The step response of the system is

\[
y(n) = h(n) * x(n)
\]

where

\[
x(n) = u(n)
\]

Obviously, if we excite a causal system with a causal input the output will be causal. Since \( h(n), x(n), y(n) \) are causal signals, the one-sided and two-sided \( z \)-transforms are identical. From the convolution property (3.2.17) we know that the \( z \)-transforms of \( h(n) \) and \( x(n) \) must be multiplied to yield the \( z \)-transform of the output. Thus

\[
Y(z) = \frac{1}{1 - \alpha^z} \frac{1}{1 - z^{-1}} = \frac{z^2}{(z - 1)(z - \alpha)}, \quad \text{ROC: } |z| > |\alpha|
\]

Now

\[
(z - 1)Y(z) = \frac{z^2}{z - \alpha}, \quad \text{ROC: } |z| < |\alpha|
\]

Since \( |\alpha| < 1 \), the ROC of \((z - 1)Y(z)\) includes the unit circle. Consequently, we can apply (3.6.6) and obtain

\[
\lim_{n \to \infty} y(n) = \lim_{z \to 1} \frac{z^2}{z - \alpha} = \frac{1}{1 - \alpha}
\]