Lecture 8

Multirate Digital Signal Processing

Introduction

- We are already familiar with the sampling theorem. We know that in order to be able to recover an analog signal from its samples, we need to have $R \geq 2W$ samples per second. That is the sample rate is proportional to the bandwidth of the signal.
- Since the bandwidth of the signal is different in different parts of a system, we need to have different rates at different points.
- Take as an example a signal consisting of N voice channels each with a bandwidth of W Hz. The total bandwidth of the signal is NW. So, we need 2NW samples.
- Assume that we would like to separate these voice signals and have N individual voice channels using N filters working on the compound signal. The out put of each channel has a bandwidth of W Hz. requiring a rate of 2W samples/second.

Introduction

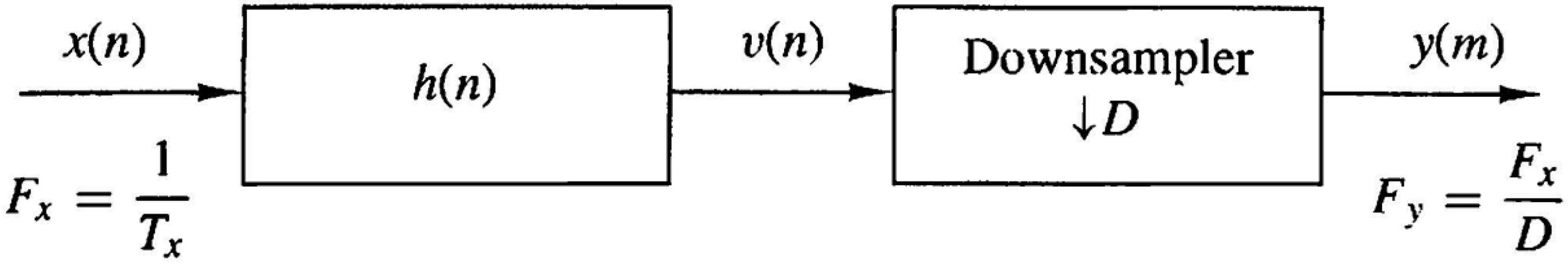
- On the other hand if we have two or more signals with bandwidths $W_1, W_2, \ldots, W_n$ and we want to multiplex them into one signal to go over the medium (cable, air, etc.), we have a system with N input each requiring a rate of $2W_1, 2W_2, \ldots.., 2W_N$. But the output has a bandwidth of $W = \sum_{i=1}^{N} W_i$. So, the rate at the output of the

  multiplexer is $R = 2\sum_{i=1}^{N} W_i$.
- Changing rate at different points of a circuit may be:
- Decimation: Deleting some samples.
- Interpolation: Adding samples between existing samples,
- Changing rate by a rational factor, say, interpolation by a factor a and decimating by a factor D,
- Changing the rate by and arbitrary ratio.

Outline of this lecture

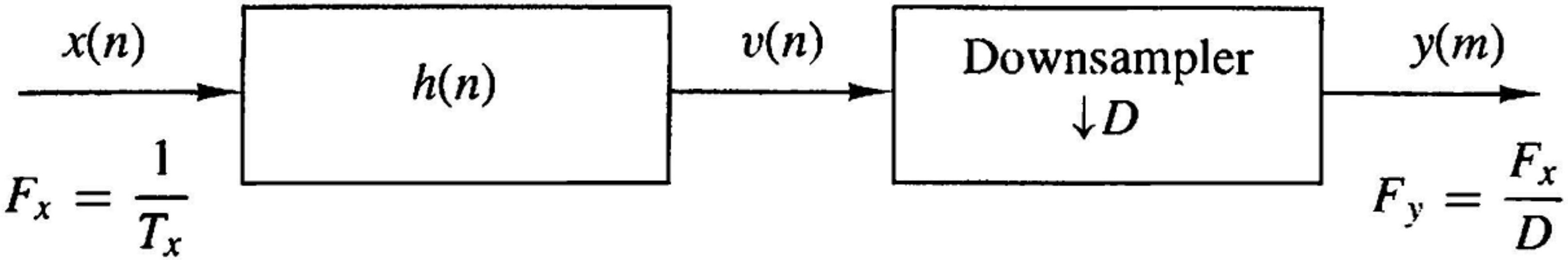
- In this lecture we talk about:
- 1) Decimation by a factor D,
- 2) Interpolation by a factor I,
- 3) Rate conversion by a Rational Factor I/D, and
- 4) Implementation of rate conversion using Polyphase structure.

Decimation by a Factor D

- Decimation consists in taking one sample out of every D samples. By doing that we reduces the number odf samples by a factor D. So if our rate was R=2W now it would be R'=R/D=2W/D. This is less than the number of samples required for the original signal with bandwidth W Hz. So, we need to first filter the signal to reduce its bandwidth by a factor of D. This figure shows the process of decimation:
- 

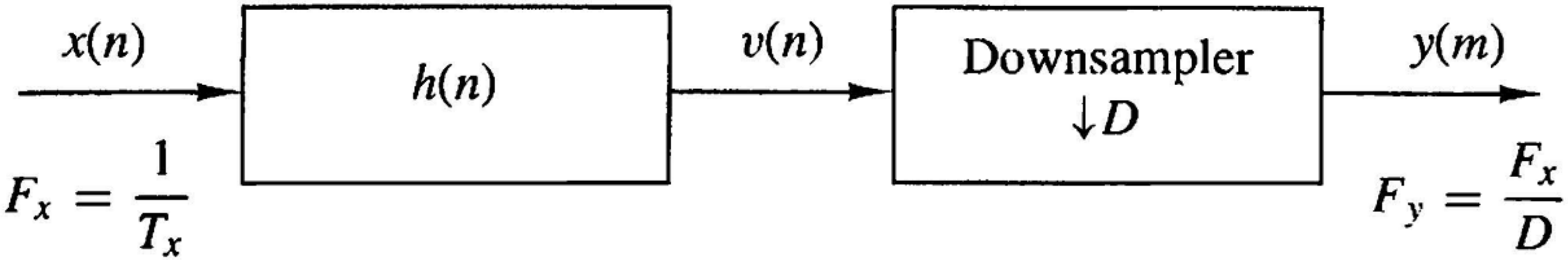
Decimation by a factor D.

Decimation by a Factor D



- The digital signal $x(n)$ has a spectrum $X(\omega)$ that is nonzero in the interval $0 \le \omega \le \pi$, or equivalently, $|F| \le \frac{F_x}{2}$. If we just reduce the rate by simply selecting every Dth sample of $x(n)$, we get an aliased version of $x(n)$ folded around $\frac{F_x}{2D}$.
- To avoid aliasing we need to reduce the bandwidth of $x(n)$ to $F_{max} = \frac{F_x}{2D}$ using the lowpass filter $h(n)$.

Decimation by a Factor D



- The lowpass filter has n impulse response h(n) and a frequency response:
  - $H_D(\omega) = \begin{cases} 1 & \omega \leq \pi/D \\ 0 & otherwise \end{cases}$
- So, the filter eliminates the spectrum of $X(\omega)$ in the range of $\frac{\pi}{D} < \omega < \pi$.

Decimation by a Factor D

- The output of the filter $h(n)$ is $v(n)$ given as:
  - $v(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$
- Which is then downsampled to produce:
  - $y(m) = v(mD) = \sum_{k=0}^{\infty} h(k)x(mD-k)$
- Let $\tilde{v}(n)$ be the downsampled version of $v(n)$:
  - $\tilde{v}(n) = \begin{cases} v(n) & n = 0, \pm D, \pm 2D, \ldots \\ 0 & otherwise \end{cases}$
- $\tilde{v}(n)$ can be viewed as the product of $v(n)$ with a train of impulses $p(n)$ with period D, i.e.,
  - $\tilde{v}(n) = v(n)p(n)$
- and $y(m) = \tilde{v}(mD) = v(mD)p(mD) = v(mD)$.

Decimation by a Factor D

- The discrete Fourier Series of $p(n)$ is:
  - $p(n) = \frac{1}{D}\sum_{k=0}^{D-1} e^{j2\pi kn/D}$
- The z-transform of the output $y(m)$ is:
  - $Y(z) = \sum_{m=-\infty}^{\infty} y(m)z^{-m} = \sum_{m=-\infty}^{\infty} \tilde{v}(mD)z^{-m}$
- Using the fact that $\tilde{v}(m) = 0$ except at multiples of D, we can write $Y(z)\ as$:
  - $Y(z) = \sum_{m=-\infty}^{\infty} \tilde{v}(m)\, z^{-m/D}$

Decimation by a Factor D

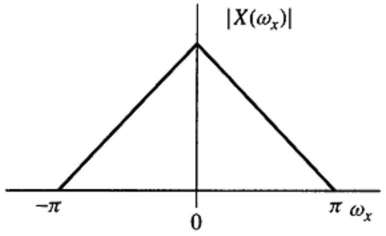
- Substituting $\tilde{v}(m) = v(m)p(m)$ in $Y(z)$ we get:
- $Y(z) = \sum_{m=-\infty}^{\infty} v(m) \left[\frac{1}{D} \sum_{k=0}^{D-1} e^{j2\pi mk/D}\right] z^{-m/D}$
- $= \frac{1}{D} \sum_{k=0}^{D-1} \sum_{m=-\infty}^{\infty} v(m)\left(e^{-j2\pi k/D}.z^{1/D}\right)^{-m}$
- $= \frac{1}{D} \sum_{k=0}^{D-1} V(e^{-j2\pi k/D}.z^{1/D}).$
- Note that $V(z) = H_D(z)X(z)$. So,
  - $Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} H_D\left(e^{-j2\pi k/D}.z^{1/D}\right) X\left(e^{-j2\pi k/D}.z^{1/D}\right).$
- Computing $Y(z)$ on the unit circle, i.e., $z = e^{j\omega_y}$, we get:
  - $Y(\omega_y) = \frac{1}{D} \sum_{k=0}^{D-1} H_D\left(\frac{\omega_y - 2\pi k}{D}\right) X\left(\frac{\omega_y - 2\pi k}{D}\right).$

Decimation by a Factor D

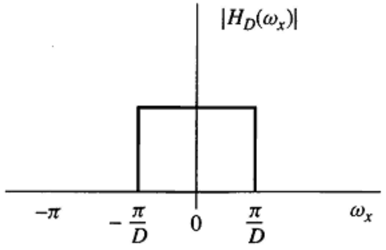
- Properly designing $H_D(\omega)$, we can avoid aliasing and get:
  - $Y(\omega_y) = \frac{1}{D} H_D\left(\frac{\omega_y}{D}\right) X\left(\frac{\omega_y}{D}\right) = \frac{1}{D} X\left(\frac{\omega_y}{D}\right).$
- Note that the sampling rate of $y(m)$ is $F_y = \frac{1}{T_y}$ while the sampling rate of x(n) is $F_x = \frac{1}{T_x}$ and since $T_y = DT_x$,we have $F_y = \frac{F_x}{D}$.
- In digital domain $\omega_y = \frac{2\pi F}{F_y}$ and $\omega_x = \frac{2\pi F}{F_x}$. So,
  - $\omega_y = D\omega_x.$

Decimation by a Factor D

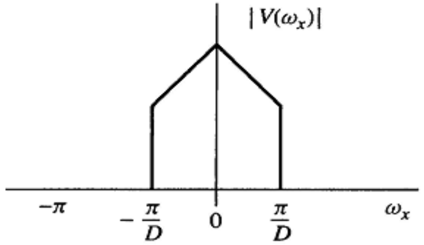
- As an example, consider $x(n)$ with spectrum $X(\omega_x)$:



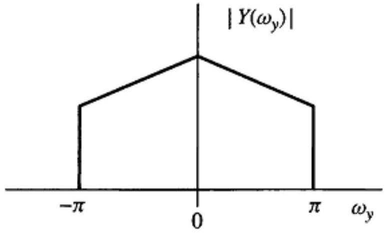
- Filtering with:



- we get:



- Or as a function of ω_y



Interpolation by a Factor I

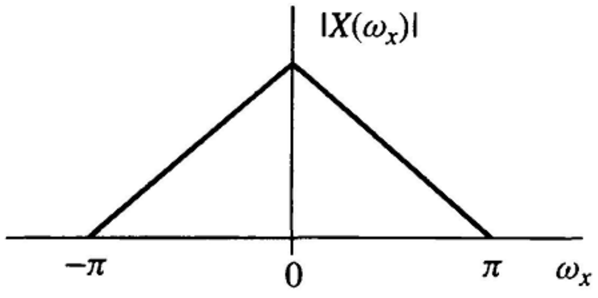
- Interpolation is a process where we increase the number os samples by a factor I, i.e., we have to insert $I-1$ samples between every two consecutive samples.
- Assume that we would like to interpolate $x(n)$ by a factor I. Let the output be $y(m)$ with rate $F_y = IF_x$ where F_x is the rate of $x(n)$.
- First form a sequence v(m) with rate the same rate as$y(m)$, i.e., $F_y = IF_x$ by placing $I-1$ zeros between samples of $x(n)$:
  - $v(m) = \begin{cases} x(\frac{m}{I}) & m = 0, \pm I, \pm 2I, \ldots \\ 0 & otherwise \end{cases}$

Interpolation by a Factor I

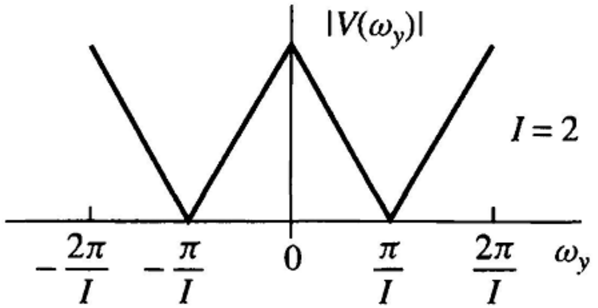
- This sequence has z-transform:
  - $V(z) = \sum_{m=-\infty}^{\infty} v(m) z^{-m} = \sum_{m=-\infty}^{\infty} x(m) z^{-mI} = X(z^I)$.
- Evaluation this on the unit circle, we obtain the spectrum of $v(n)$ as:
  - $V(\omega_y) = X(\omega_y I)$,
- where ω_y is the frequency variable related to the new sampling rate $F_y = IF_x$. So,
  - $\omega_y = \frac{2\pi F}{F_y} = \frac{2\pi F}{IF_x} = \frac{\omega_x}{I}$

Interpolation by a Factor

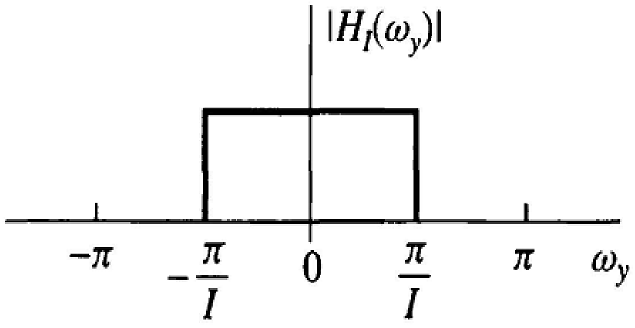
- Assume that the spectrum of $x(n)$ ·



- Then the spectrum of $v(m)$ is:

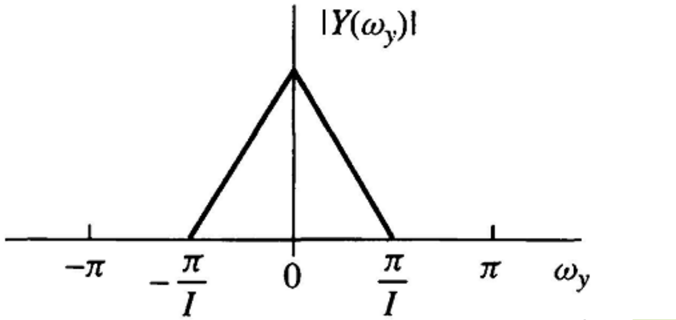


- Since only the spectrum in the range $0 \leq \omega_y \leq \pi/I$ is unique, we filter the spectrum of $v(m)$ using:
  - $$H_I(\omega_y) = \begin{cases} C, & 0 \leq \omega_y \leq \pi/I \\ 0, & otherwise \end{cases}$$



Interpolation by a Factor I

- By filtering the spectrum of $v(m)$ weobtain the spectrum of $y(m)$ as:
  - $Y(\omega_y) = \begin{cases} CX(\omega_y I), & 0 \le \omega_y \le \pi/I \\ 0, & otherwise \end{cases}$



- To find the scaling factor C, we let $y(m) = x(\frac{m}{I})$ m= $0, \pm I, \pm 2I, \ldots$. In particular, for $m = 0$, we have:
- $y(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega_y)d\omega_y = \frac{C}{2\pi}\int_{-\pi/I}^{\pi/I} X(\omega_y I)d\omega_y$
- Since $\omega_y = \omega_x/I$, we have:
  - $y(0) = \frac{C}{I}\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega_x)d\omega_x = \frac{C}{I}x(0).$
- Therefore, in order to have $y(0) = x(0)$, we should have $C = I$.

Interpolation by a Factor I

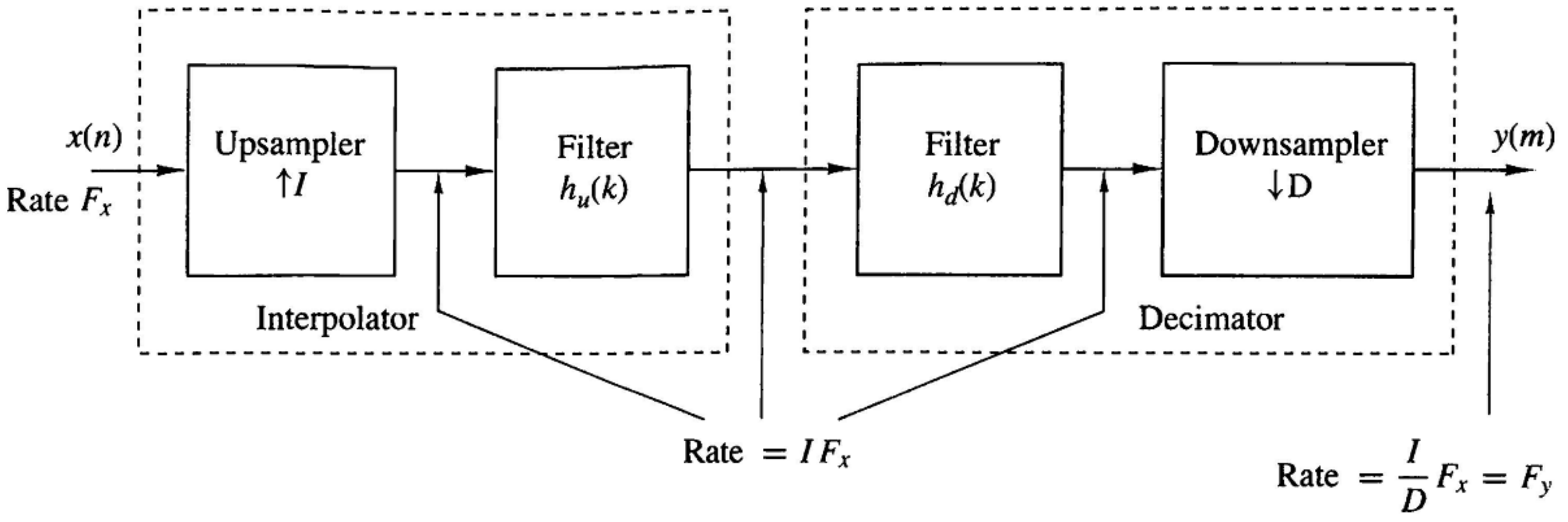
- The output $y(m)$ is the result of convolving the sequence $v(m)$ with the filter's unit sample response $h(n)$:
  - $y(m) = \sum_{k=-\infty}^{\infty} h(m-k)v(k)$.
- Since $v(k) = 0$ except at multiples of I, where,
  - $v(kI) = x(k)$,
- we have:
  - $y(m) = \sum_{k=-\infty}^{\infty} h(m-kI)x(k)$.

Sampling rate conversion by a factor I/D

- We have learned how to do decimation by a factor D and interpolation by a factor I. So, we are in a position to do rate conversion by a factor I/D.
- To do this, we first interpolate by a factor I, i.e., we place $I - 1$ samples between any two samples and then do decimation by a factor D by taking one sample out of every D samples.
- As an example assume that we have a signal with a bandwidth of 1 MHz. According to Nyquist theorem we need at least 2 Msamples/sec. Assume that we have taken exactly 2 Msamples/sec. but later have been told that filtering with the minimum sampling rate is impossible and we need to add 25% more samples, i.e., we need to have a sampling rate of 2.5 Msamples/sec. To do this we need to change the sampling rate by a factor of $\frac{2.5}{2} = \frac{5}{4}$, i.e., $I = 5$ and $D = 4$. So first we increase the number of samples to 10 million per second and then take one out of 4 to get 2.5 million samples.

Sampling rate conversion by a factor I/D

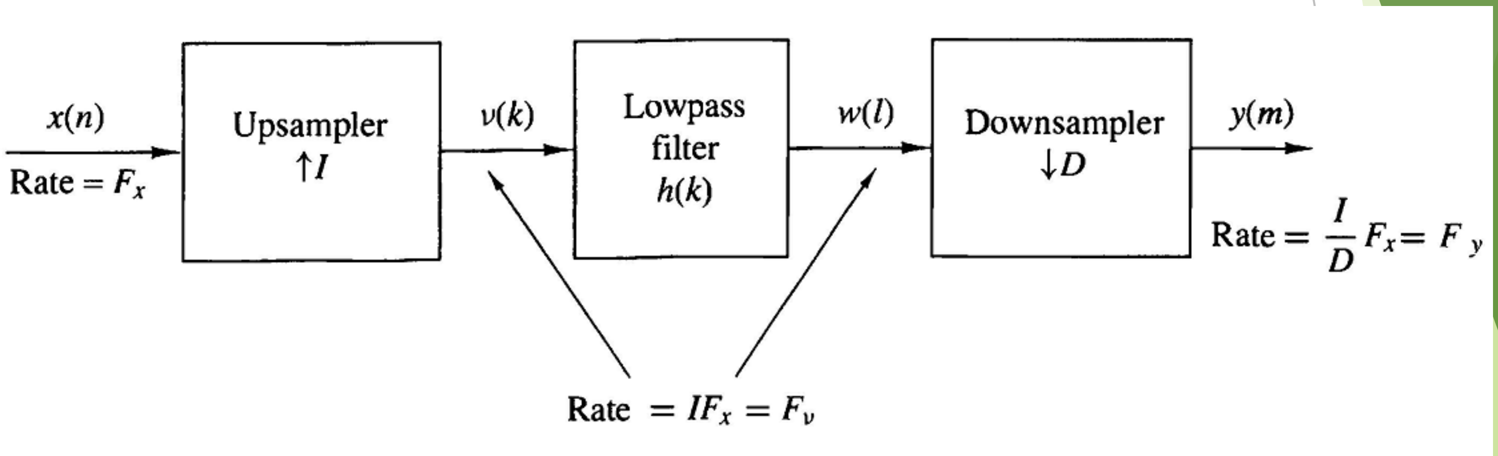
- Rate conversion by a factor I/D can be implemented by interpolating by a factor I followed by decimation by a factor D as shown:



- Note that the two filters $\{h_u(n)\}$ and $\{h_d(n)\}$ both operate at the same rate, IF_x. So, they can be combined into a single lowpass filter.

Sampling rate conversion by a factor I/D

- Combining the two filters $\{h_u(n)\}$ and $\{h_d(n)\}$ into a single lowpass filter, we get:



- The frequency response of the lowpass filter should ideally be:

  - $$H(\omega_v) = \begin{cases} I & 0 \le |\omega_v| \le \min(\frac{\pi}{D}, \frac{\pi}{I}) \\ 0 & otherwise \end{cases}$$

- where $\omega_v = \frac{2\pi F}{F_v} = \frac{2\pi F}{IF_x} = \frac{\omega_x}{I}$.

Sampling rate conversion by a factor I/D

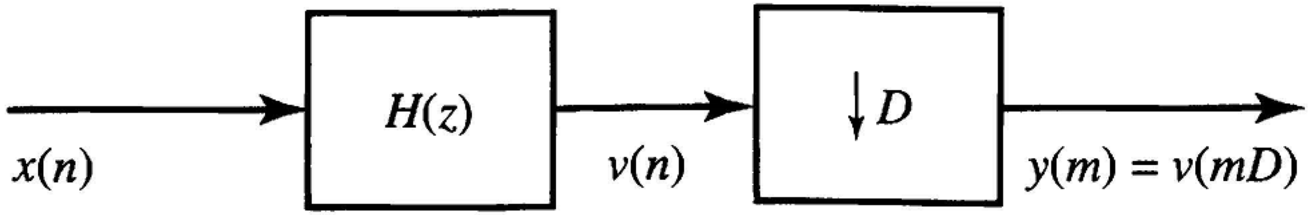
- In time-domain, the output of the sampler is:
  - $$\mathrm{v}(l) = \begin{cases} x(l/I) & l = 0, \pm I, \pm 2I, \ldots \\ 0 & otherwise \end{cases}$$
- and the output of the filter is:
  - $w(l) = \sum_{k=-\infty}^{\infty} h(l-k)v(k) = \sum_{k=-\infty}^{\infty} h(l-kI)x(k)$.
- So, the output $y(m)$ is given as:
  - $y(m) = w(mD) = \sum_{k=-\infty}^{\infty} h(mD-kI)x(k)$.

Sampling rate conversion by a factor I/D

- In frequency-domain, we combine the results of interpolation and decimation.
- The spectrum of the filter output is:
  - $$V(\omega_v) = H(\omega_v)X(\omega_v I) = \begin{cases} IX(\omega_v I), & 0 \le |\omega_v| \le \min(\frac{\pi}{D}, \frac{\pi}{I}) \\ 0 & otherwise \end{cases}$$
- The spectrum of the output sequence $y(m)$ obtained by decimating $v(n)$ by a factor D is:
  - $$Y(\omega_y) = \frac{1}{D}\sum_{k=0}^{D-1} V\left(\frac{\omega_y - 2\pi k}{D}\right),$$
- where $\omega_y = D\omega_v$.
- Since the filter prevents aliasing, we have
  - $$Y(\omega_y) = \begin{cases} \frac{1}{D} X\left(\frac{\omega_y}{D}\right), & 0 \le |\omega_y| \le \min(\pi, \frac{\pi D}{I}) \\ 0 & otherwise \end{cases}.$$

Efficient Implementation of Rate Conversion

- Let's start with decimation. The circuit to do decimation by D consists of a lowpass filter and a down sampler:



- Note that the filter operates at the high rate of the input while, the rate at which we compute the output samples is 1/D of the input rate.
- In order to reduce the processing rate, we can use a structure called the polyphaser.

Polyphase Structure

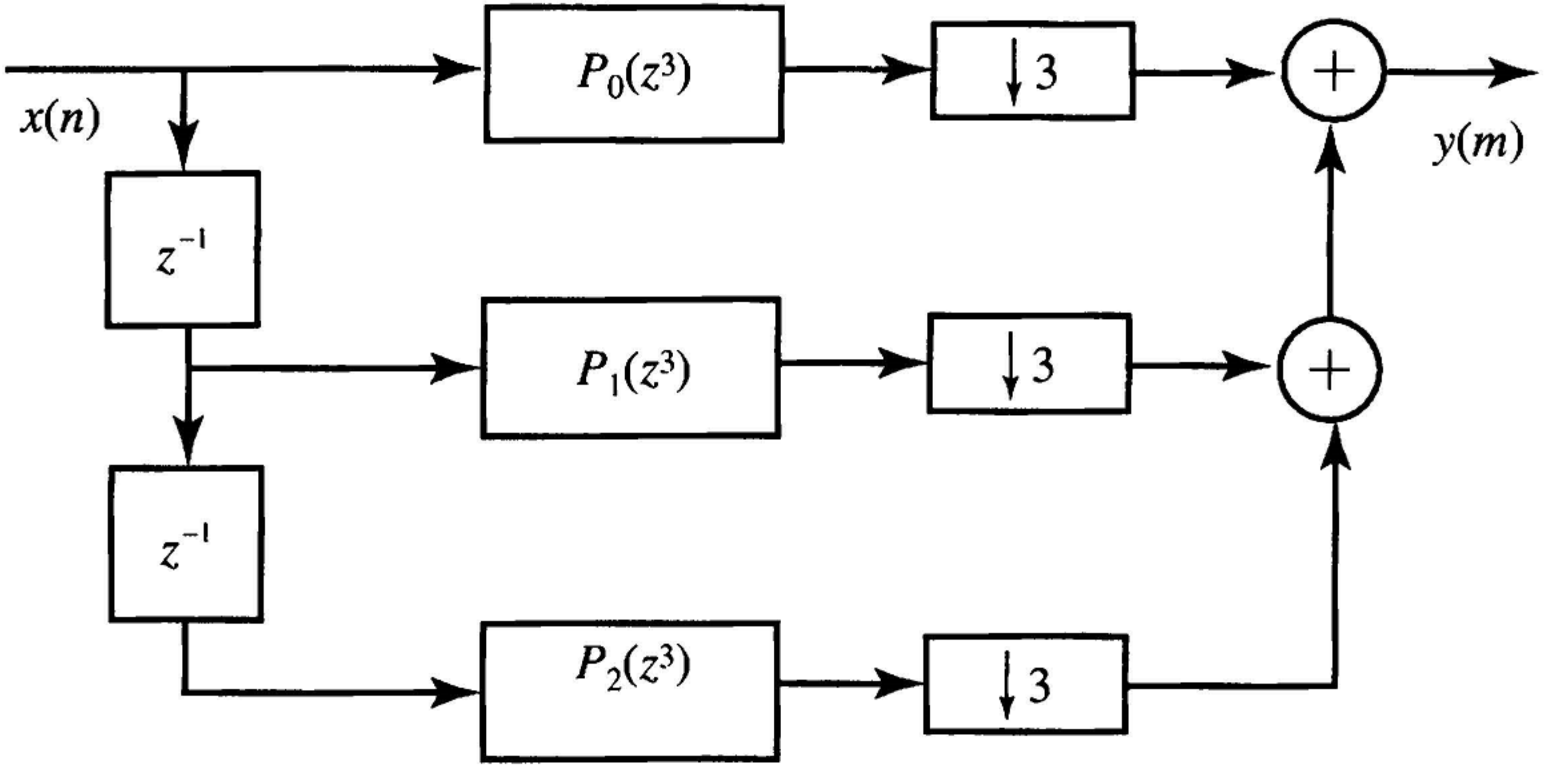
- In order to understand the concepts, we use a polyphaser filter with three branches (M=3). Generalization to any value of M is straightforward.
- Let the filter $H(z)$ be given as:
  - $H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + \cdots.$
- Let's rearrange $H(z)$ as:
  - $H(z) = h(0) + h(3)z^{-3} + h(6)z^{-6} + \cdots$
  - $+h(1)z^{-1} + h(4)z^{-4} + h(7)z^{-7} + \cdots$
  - $+h(2)z^{-2} + h(5)z^{-5} + h(8)z^{-8} + \ldots.$
- Or, equivalently:
  - $H(z) = h(0) + h(3)z^{-3} + h(6)z^{-6} + \cdots$
  - $+z^{-1}[h(1) + h(4)z^{-3} + h(7)z^{-6} + \cdots]$
  - $+z^{-2}[h(2) + h(5)z^{-3} + h(8)z^{-6} + \ldots.]$

Polyphase Structure

- Denoting:
  - $P_0(z^3) = h(0) + h(3)z^{-3} + h(6)z^{-6} + \cdots$
    - $P_1(z^3) = h(1) + h(4)z^{-3} + h(7)z^{-6}$
    - $P_2(z^3) = h(2) + h(5)z^{-3} + h(8)z^{-6}$
- We can write $H(z)$ as:
  - $H(z) = P_0(z^3) + z^{-1}P_1(z^3) + z^{-2}P_2(z^3).$

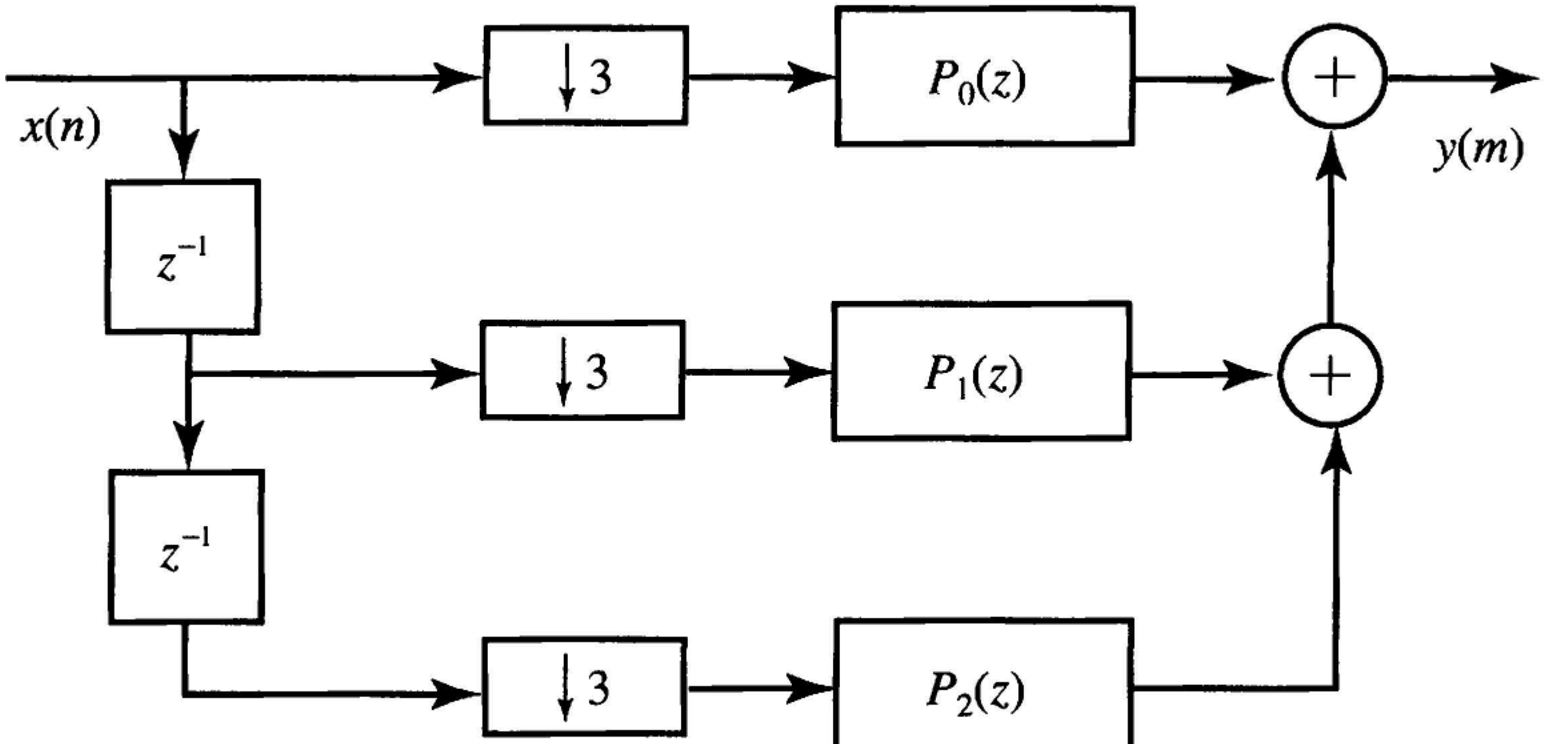
Polyphase Structure: Decimation

- This can be implemented as:



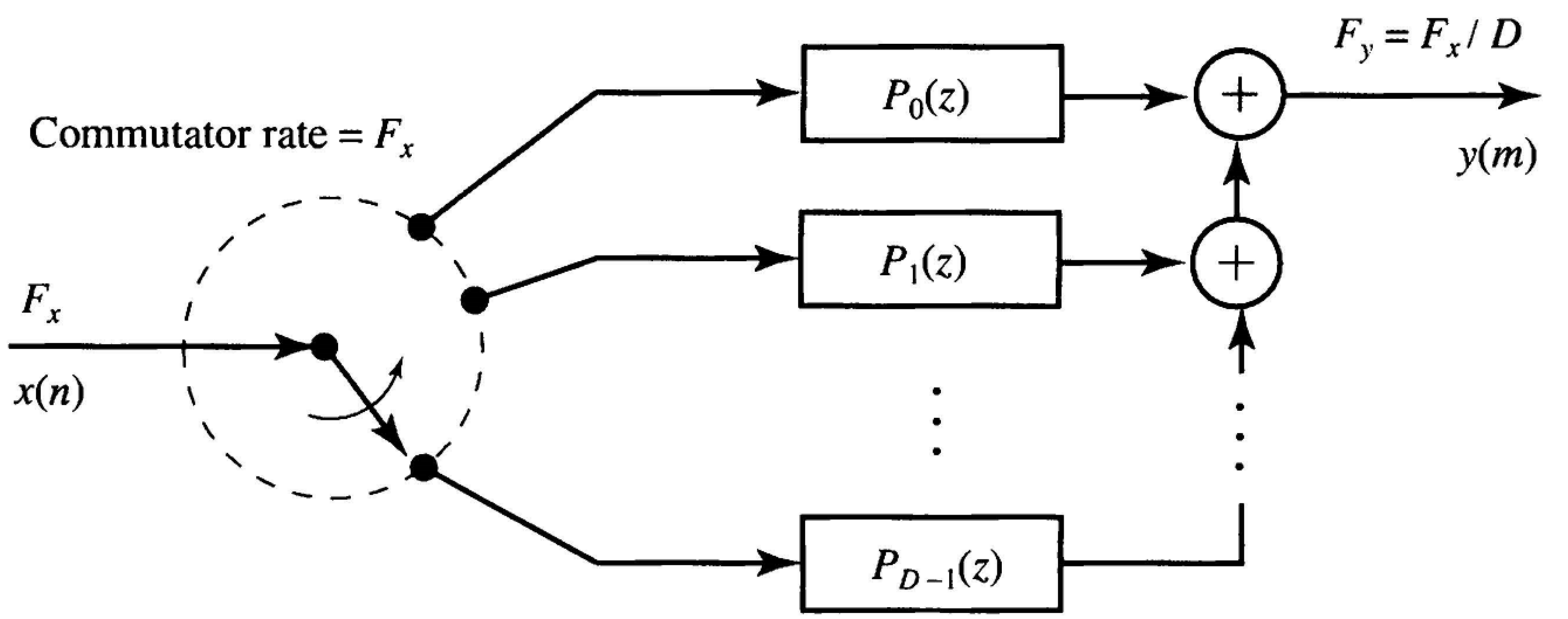
Polyphase Structure: Decimation

- We can down sample first. Let's substitute $z \longrightarrow z^{1/3}$ in:
  - $H(z) = P_0(z^3) + z^{-1}P_1(z^3) + z^{-2}P_2(z^3).$
- to get:
  - $H(z^{1/3}) = P_0(z) + z^{-1/3}P_1(z) + z^{-2/3}P_2(z).$
- Then we have:



Polyphase Structure: Decimation

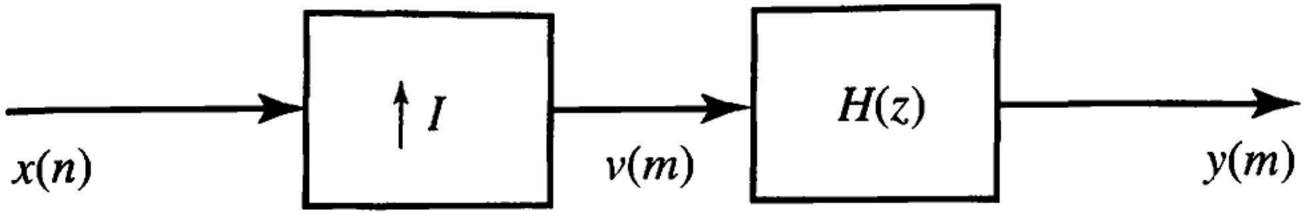
- Another way to look at the decimation using polyphaser filter is to consider a commutator followed by the polyphaser filter:



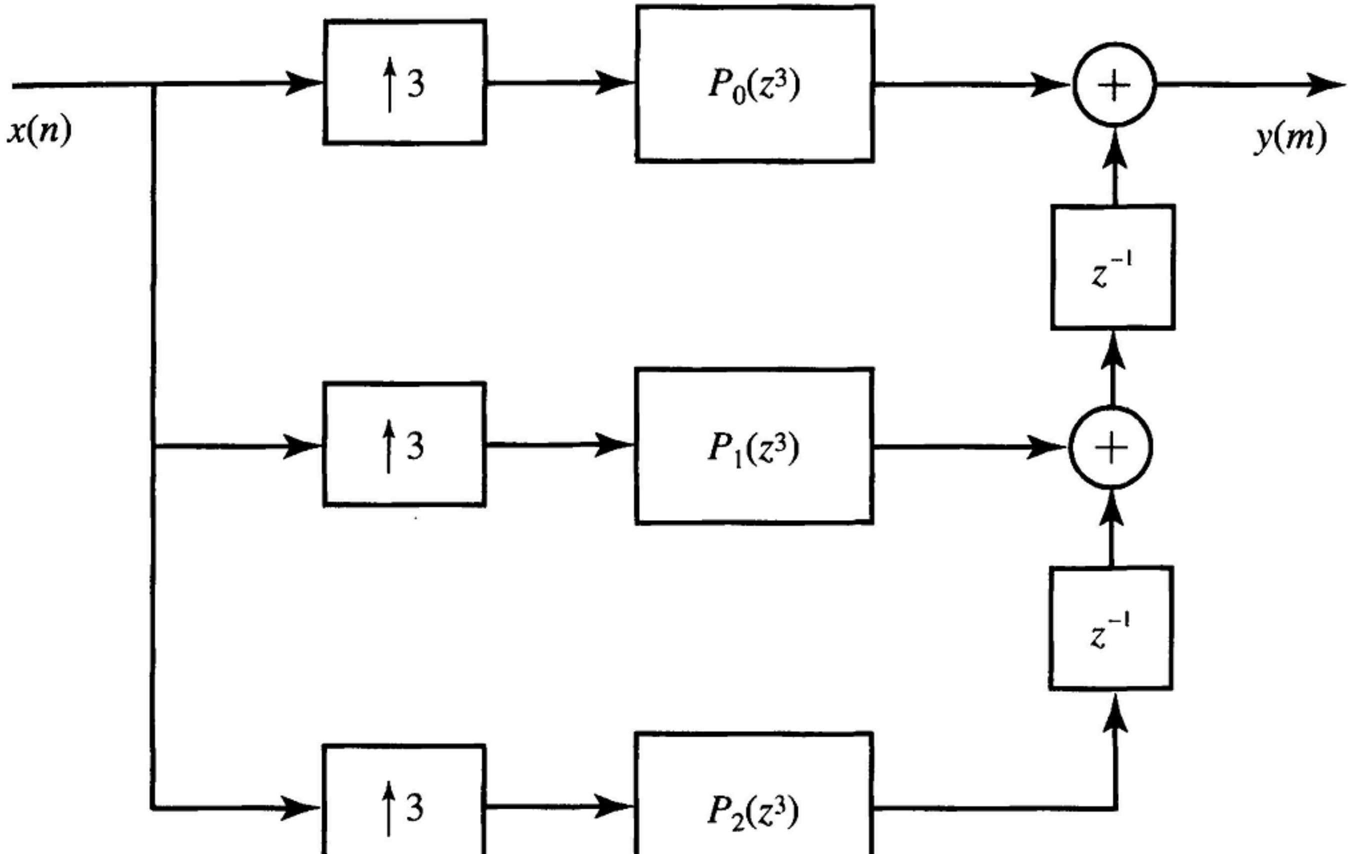
- A commutator is like a switch placing the input samples at the input of filter branches.

Polyphase Structure: Interpolation

- An interpolator has an upsampler and a lowpass filter:

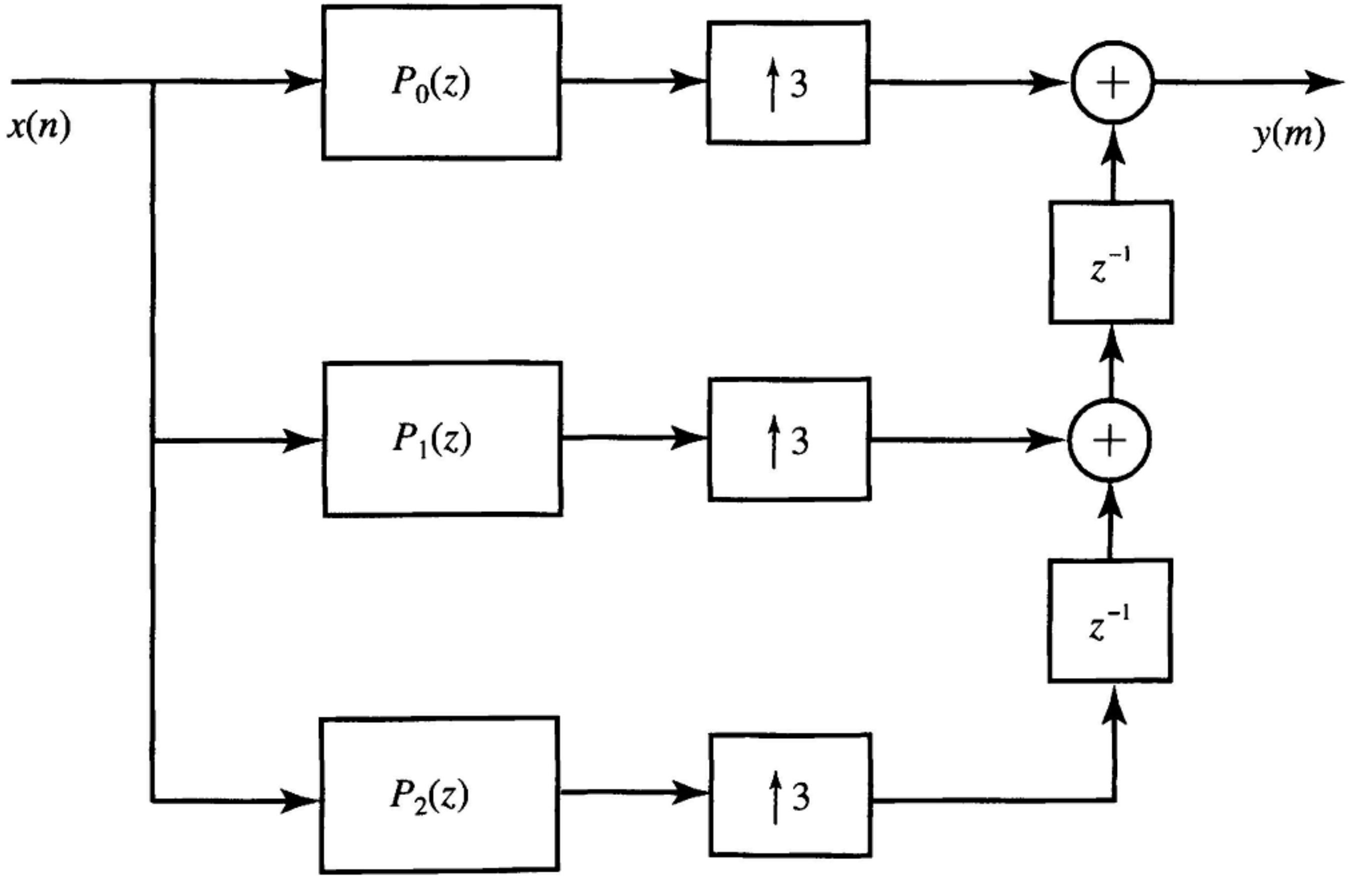


- The problem is that
- filtering is done at high
- rate using polyphaser
- structure, the interpolator
- can be implemented as:



Polyphase Structure: Interpolation

- Similar to the case of decimation, the filtering and upsampling functions can be swapped to get:



Polyphase Structure: Interpolation

- The interpolator can be implemented using polyphaser filters and a commutator. For each input sample the commutator reads I samples at the output of all filters.