# MECH 344/X

# Machine Element Design

**Time: M _ _ _ _14:45 - 17:30**

# Lecture 2

# Contents of today's lecture

## 4 Static Body Stresses

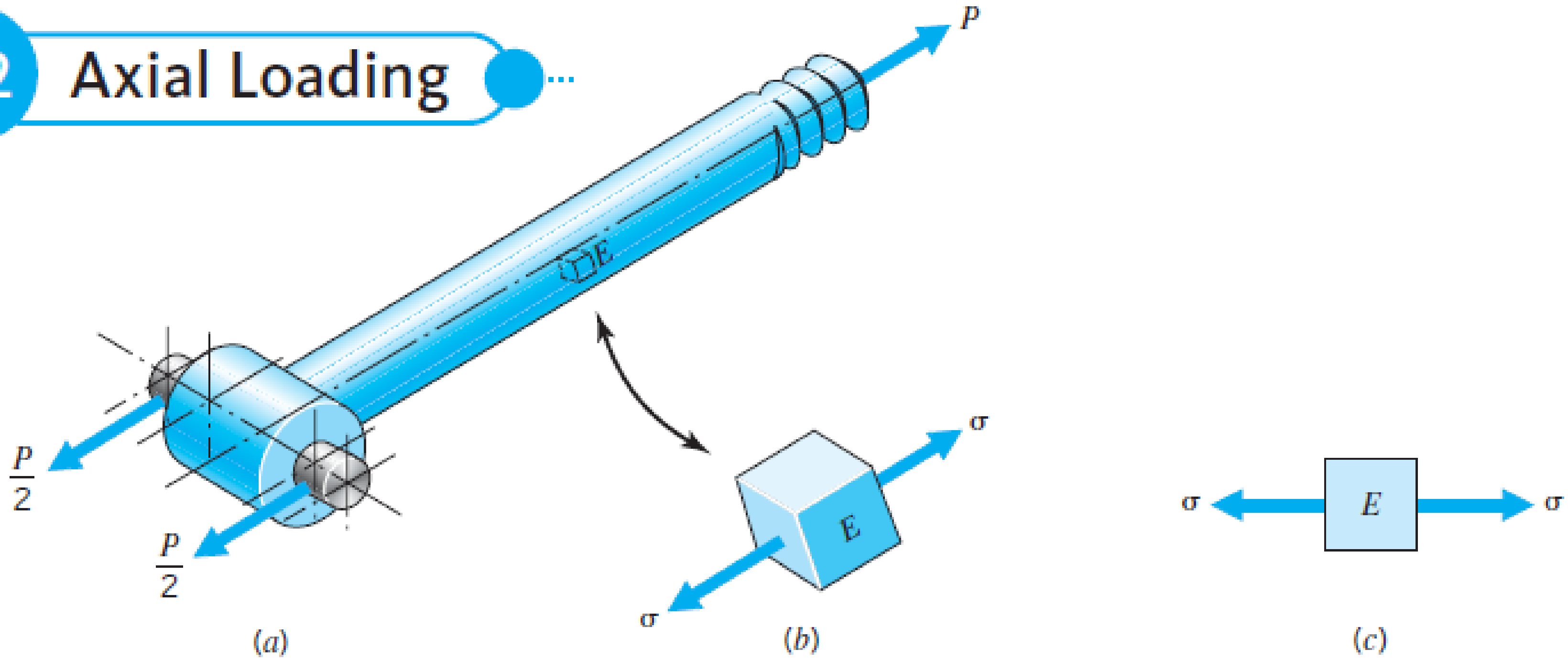
- Introduction to Static Stresses
- Axial, Shear and Torsional Loading
- Bending in Straight and Curved Beams
- Transverse Shear in Beam
- Mohr Circle
- Combined Stresses
- Stress Concentration
- Residual Stresses

## 4.1 Introduction

- After we identify the external loads on a member, we need to see what the resulting stresses on the member due to those loads are.
- Body stresses, existing within the member as a whole
- Surface or contact stresses in localized regions where external loads are applied.
- Stresses resulting from static loading, as opposed to stresses caused by impact or fatigue loading.
- Convention capital letter S for material strength ($S_u$, $S_y$ for ultimate, yield strength) and using Greek letters $\sigma$ and $\tau$ for normal and shear stress.

This chapter is going to be a review of subjects you learned in mechanics of materials

# 4.2 Axial Loading

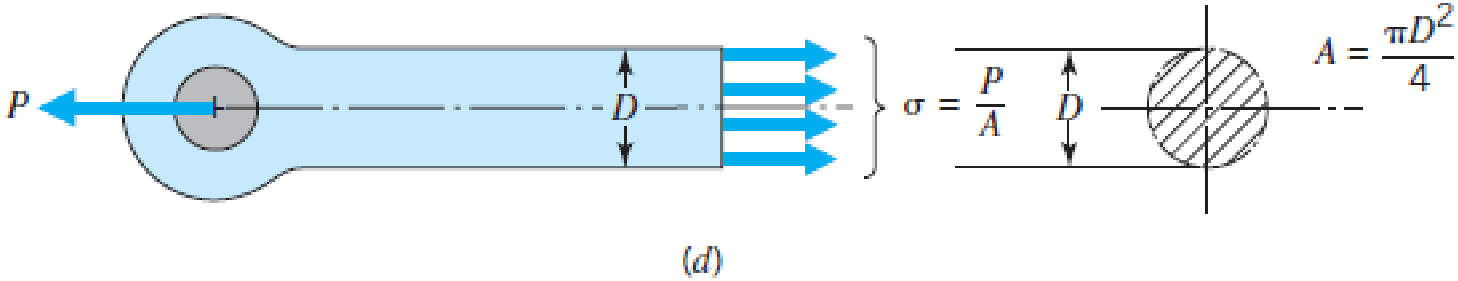



(a)
Isometric view of tensile link loaded through a pin at one end and a nut at the other.

(b)
Enlarged view of element *E*

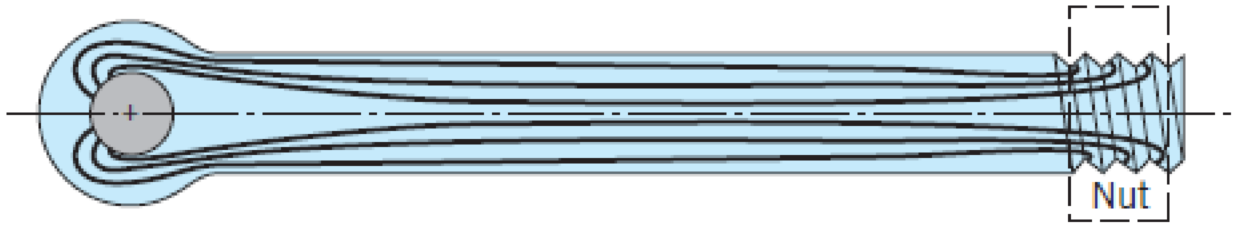
(c)
Direct view of element *E*

- Figure illustrates a case of simple *tension*. P for tension and –P for compression, in both cases axial.
- Small block *E* represents an arbitrarily located small element of material
- It is important to remember that the stresses are acting on faces *perpendicular to the paper*.
- This is made clear by the isometric view in Figure 4.1*b*.




Equilibrium of left half showing uniform stress distribution at cutting plane

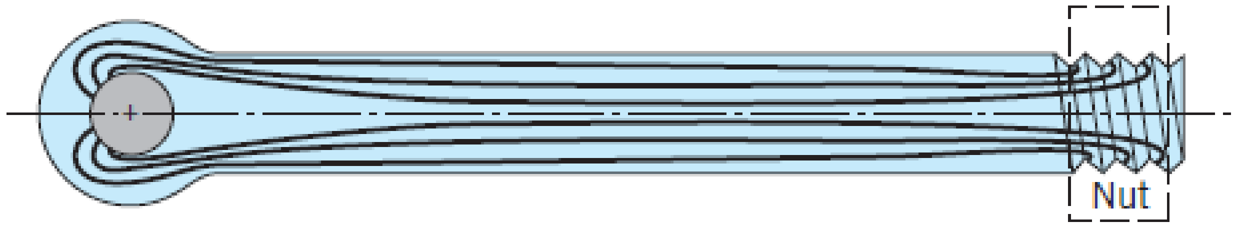
- Figure illustrates equilibrium of the left portion of the link
- From this we have perhaps the simplest formula in all of engineering: $\sigma = P/A$
- It is important to remember that although this formula is always correct as an expression for the average stress in any cross section, disastrous errors can be made by naively assuming that it also gives the correct value of maximum stress in the section.
- Unless several important requirements are fulfilled, the maximum stress will be several times greater than P/A



(e)

View showing "lines of force" through the link

- Figure illustrates lines of force on the link
- Maximum stress = P/A only if load distribution is uniform
  1. The section being considered is well removed from the loaded ends. Uniform distribution is reached at points about three d from the end
  2. The load is applied exactly along the centroidal axis. If, there is eccentricity, bending moment will come into play
  3. The bar is a perfect straight cylinder, with no holes, notches, threads, internal imperfections giving rise to stress concentration

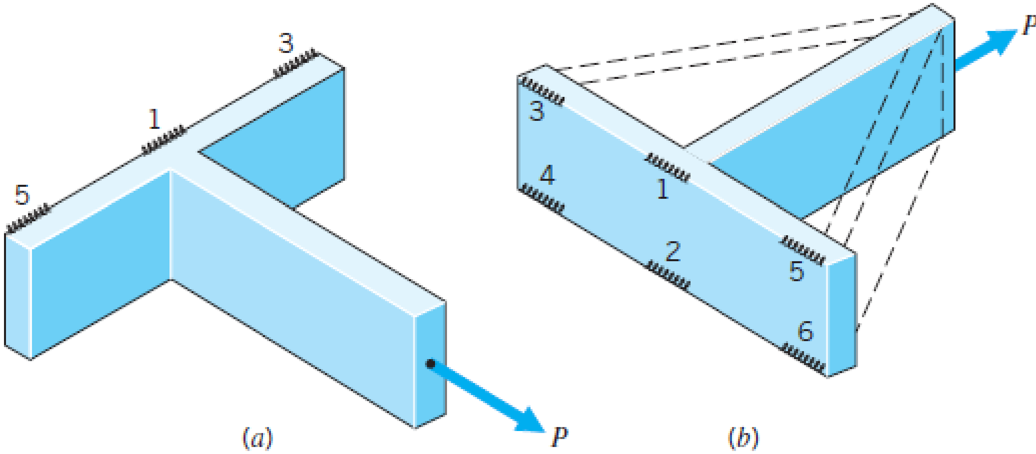



(e)

View showing "lines of force" through the link

- Figure illustrates lines of force on the link
- Maximum stress = P/A only if load distribution is uniform

4. The bar is free of stress when the external loads are removed. - No residual stresses due to manufacture and past mechanical/thermal loading
5. The bar comes to stable equilibrium when loaded. If, the bar is in compression, or if it long, buckling occurs, and elastically unstable.
6. The bar is homogeneous. Not a composite material (where 2 different Es make a material so that one with larger E gets stressed more)

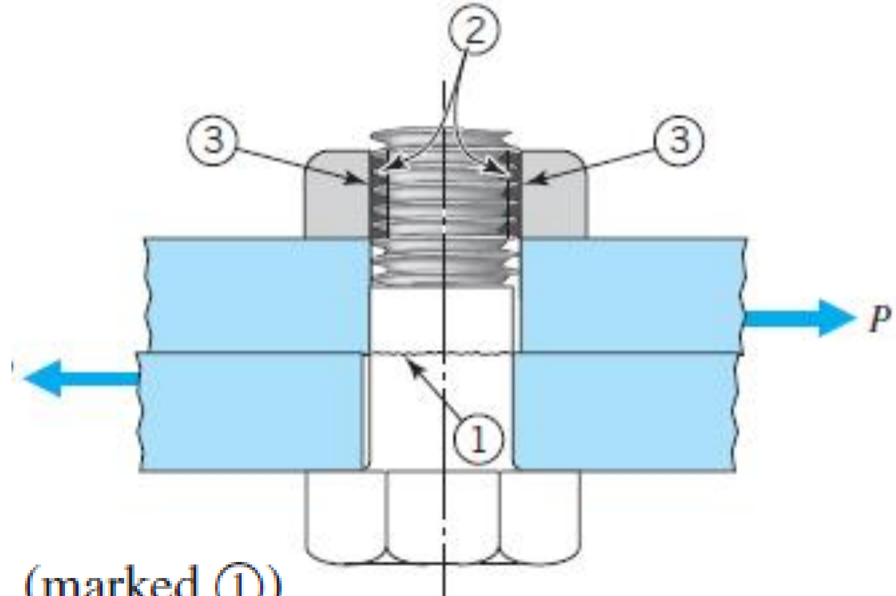



- Figure shows where unexpected failure can result from assuming

$$\sigma = P/A$$

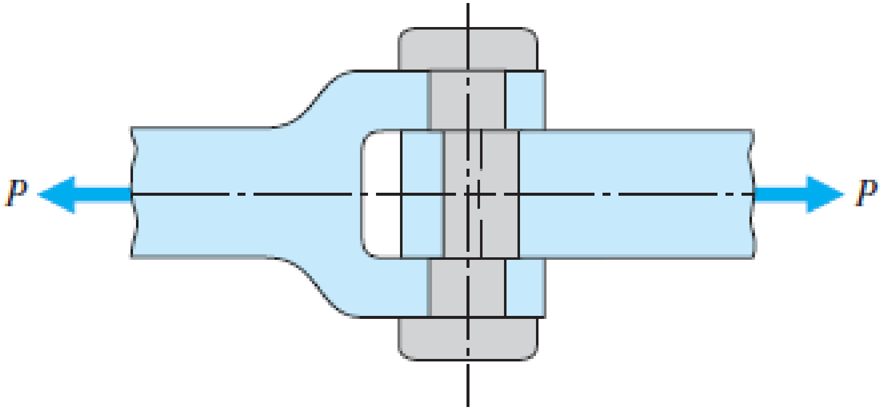
- P is 600 N and 6 identical welds are used. The average load 100 N per weld
- 6 welds represent redundant force paths of different stiffnesses.
- The paths to welds 1 and 2 are much stiffer; they may carry nearly all the load.
- A more uniform distribution could be obtained by adding two side plates
- One might despair of ever using P/A as an acceptable value of maximum stress for relating to the strength properties of the material.
- The student should acquire increasing insight for making “engineering judgments” relative to these factors as his or her study progresses and experience grows.

# 4.3 Direct Shear Loading




- Direct shear loading involves equal and opposite forces, collinear that the material between them experiences shear stress, with negligible bending.
- In figure neglecting interface friction direct shear with $\tau = P/A$ happens @ (marked ①)
- If the nut in Figure is tightened to produce an initial bolt tension of P, the direct shear stresses at the root of the bolt threads (area 2 ), and at the root of the nut threads (area 3), have average values P/A.
- The thread root areas involved are cylinders of a height equal to the nut thickness. (for V threads)
- If the shear stress is excessive, shearing or "stripping" of the threads occurs in the bolt or nut, whichever is weaker.
- Direct shear is used in metal cutting, rivets, pins, keys, splines etc..

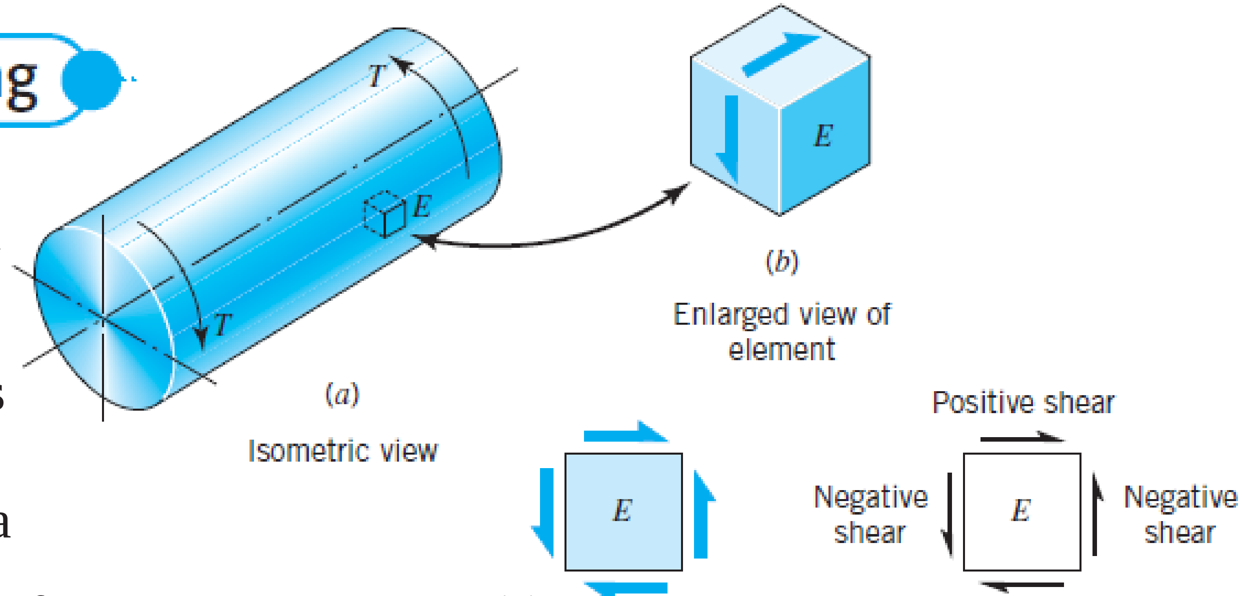
- Figure shows a hinge pin loaded in double shear, where the load P is carried in shear through two areas; hence, the area A used in Eq. is twice the CSA of pin $\tau = P/A$




- Direct shear loading does not produce pure shear (as does torsional loading), and the actual stress distribution is complex.
- It involves fits between the mating members and relative stiffnesses. The maximum shear stress will always be somewhat in excess of the P/A value
- In the design of machine and structural members, however, it is commonly used in conjunction with appropriately conservative values of working shear stress.
- Furthermore, to produce total shear fracture of a ductile member, the load must simultaneously overcome the shear strength in every element of material in the shear plane.
- Thus, for total fracture the equation will work if $\tau$ being set equal to the ultimate shear strength, $S_{us}$ .

# 4.4 Torsional Loading

- Figure illustrates torsional loading
- Direction of T determines that the left face of element E is subjected to a ↓ shear stress, and the right face to an ↑ stress, making a CCW couple, balanced by a corresponding CW couple with shear stresses on top/bottom




(a) Isometric view

(b) Enlarged view of element

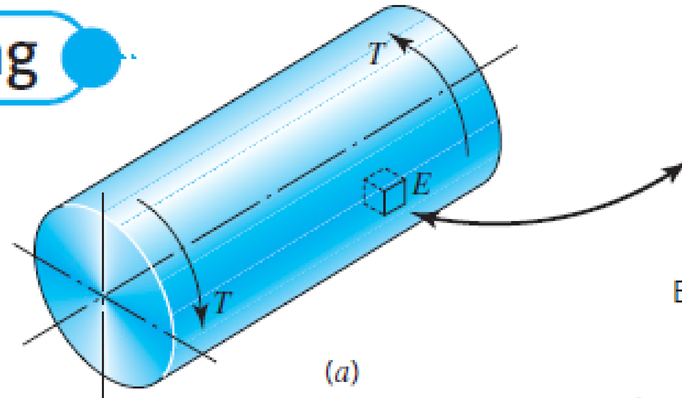
(c) Direct view of element $E$

(d) Shear sign convention

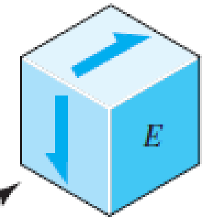
- The state of stress shown on element E is pure shear.
- In axial force + for tension and – for compression; Compression can cause buckling but tension cannot, a chain can withstand tension but not compression, concrete is strong in compression but weak in tension.
- The sign convention for shear loading serves no similar function; + and - shear are basically the same; and the sign convention is purely arbitrary.
- In this book CCW –ve and CW +ve

# 4.4 Torsional Loading

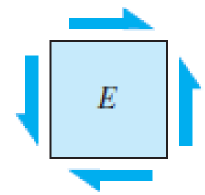
- For round bar in torsion stresses vary from 0 at center to max at surface
- τ = Tr/J where r is the radius and J polar moment $\pi d^4/32$
- Max shear stress will be $\tau_{max} = 16T/\pi d^3$



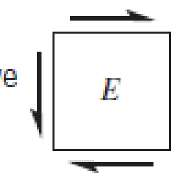

(a)
Isometric view




(b)
Enlarged view of element




(c)
Direct view of element E

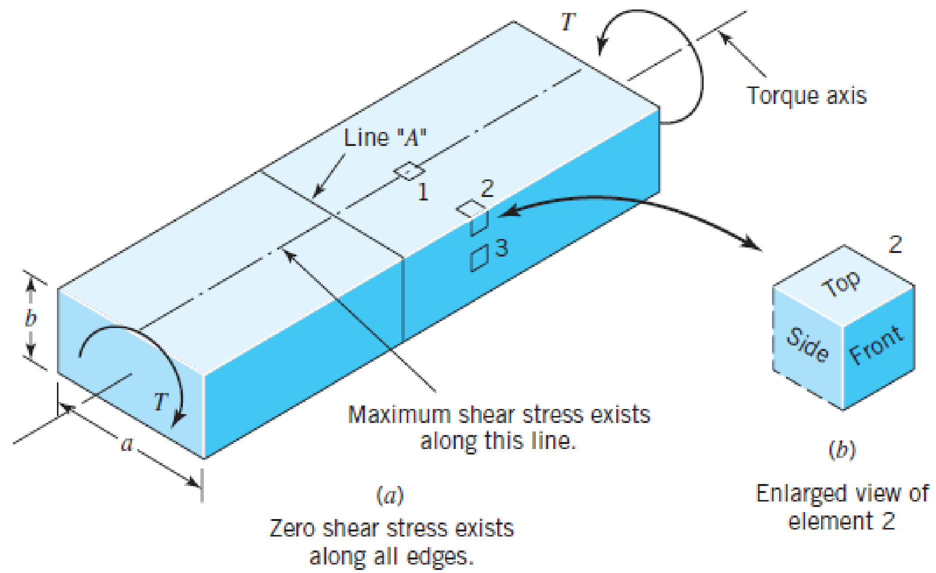



(d)
Shear sign convention

- Important assumptions for this equation are

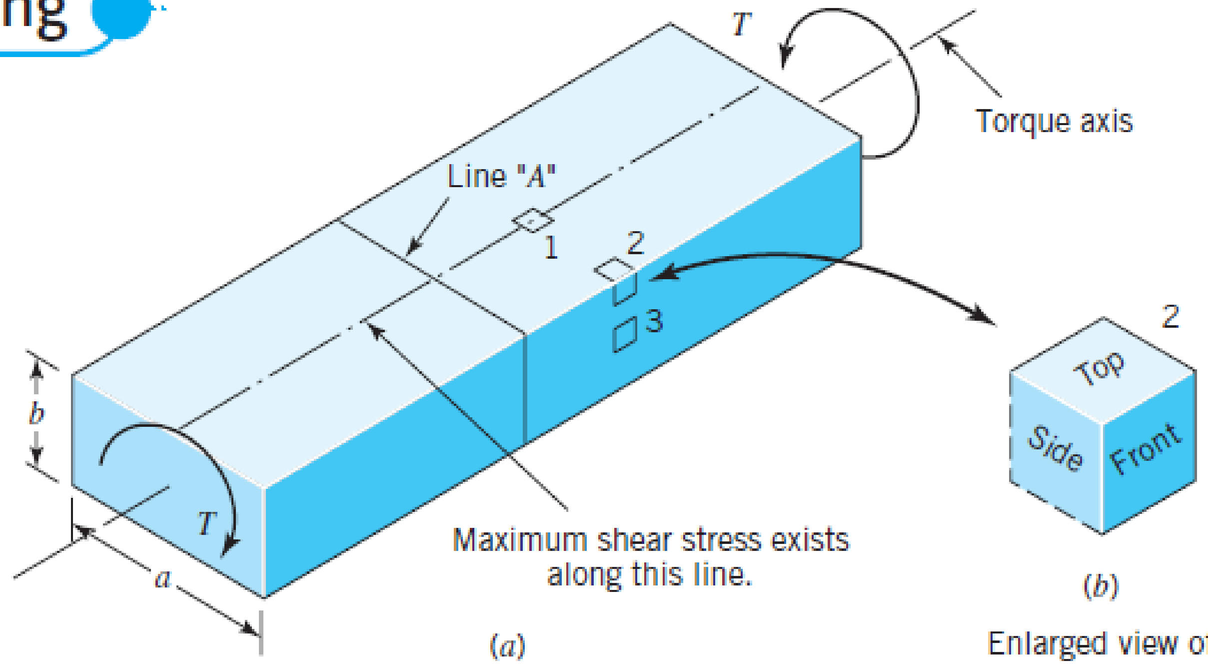
1. The bar must be straight and round (either solid or hollow), and the torque must be applied about the longitudinal axis.
2. The material must be homogeneous and perfectly elastic within the stress range involved.
3. The cross section considered must be sufficiently remote from points of load application and from stress raisers (i.e., holes, notches, keyways, surface gouges, etc.).

- For bars of nonround cross section, the analysis gives erroneous results.
- This can be demonstrated with an ordinary rubber eraser with small square elements 1, 2, and 3 as shown in Figure.
- When the eraser is twisted, Eq. implies highest shear stress would be at element 2 farthest from the neutral axis.




- Lowest surface stress at element 1 closest to the axis.
- Observation of the twisted eraser shows exactly the opposite; element 2 does not distort at all, while element 1 experiences the greatest distortion
- In Eq. the basic assumption that what are transverse planes before twisting remain planes after twisting. If such a plane is represented by drawing line "A" on the eraser, obvious distortion occurs upon twisting; therefore, the assumption is not valid for a rectangular section.
- The equilibrium requirement of corner element 2 makes it clear that this element must have zero shear stress: (1) the "free" top and front surfaces do not contact anything that could apply shear stresses;

# 4.4 Torsional Loading




(a)
Zero shear stress exists along all edges.

(b)
Enlarged view of element 2

- (2) this being so, equilibrium requirements prevent any of the other four surfaces from having shear. Hence, there is zero shear stress along all edges of the eraser.
- Torsional stress equations for nonround sections are summarized in references

$$\tau_{\max} = T(3a + 1.8b)/a^2b^2$$

# 4.5 Pure Bending Loading, Straight Beams

- Pure Bending – it is rare for beam to be loaded in pure bending. It is useful though to understand the situation
- Mostly shear loading and bending moments act together
- Applying point loads P equidistant at the simply supported beam, absence of shear loading makes this pure bending
- Assumptions used for analysis

1. The segment analyzed is distant from applied loads or external constraints on the beam.
2. The beam is loaded in a plane of symmetry.
3. Cross sections of the beam remain plane and perpendicular to the neutral axis during bending.
4. The material of the beam is homogeneous and obeys Hooke's law.
5. Stresses remain below the elastic limit and deflections are small.
6. The segment is subjected to pure bending with no axial or shear loads.
7. The beam is initially straight.

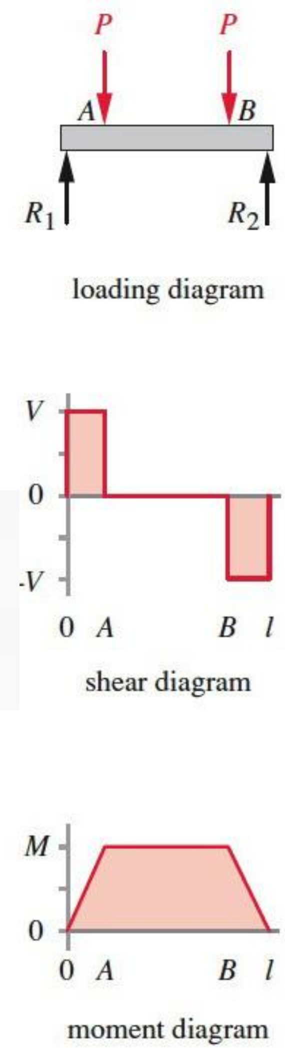
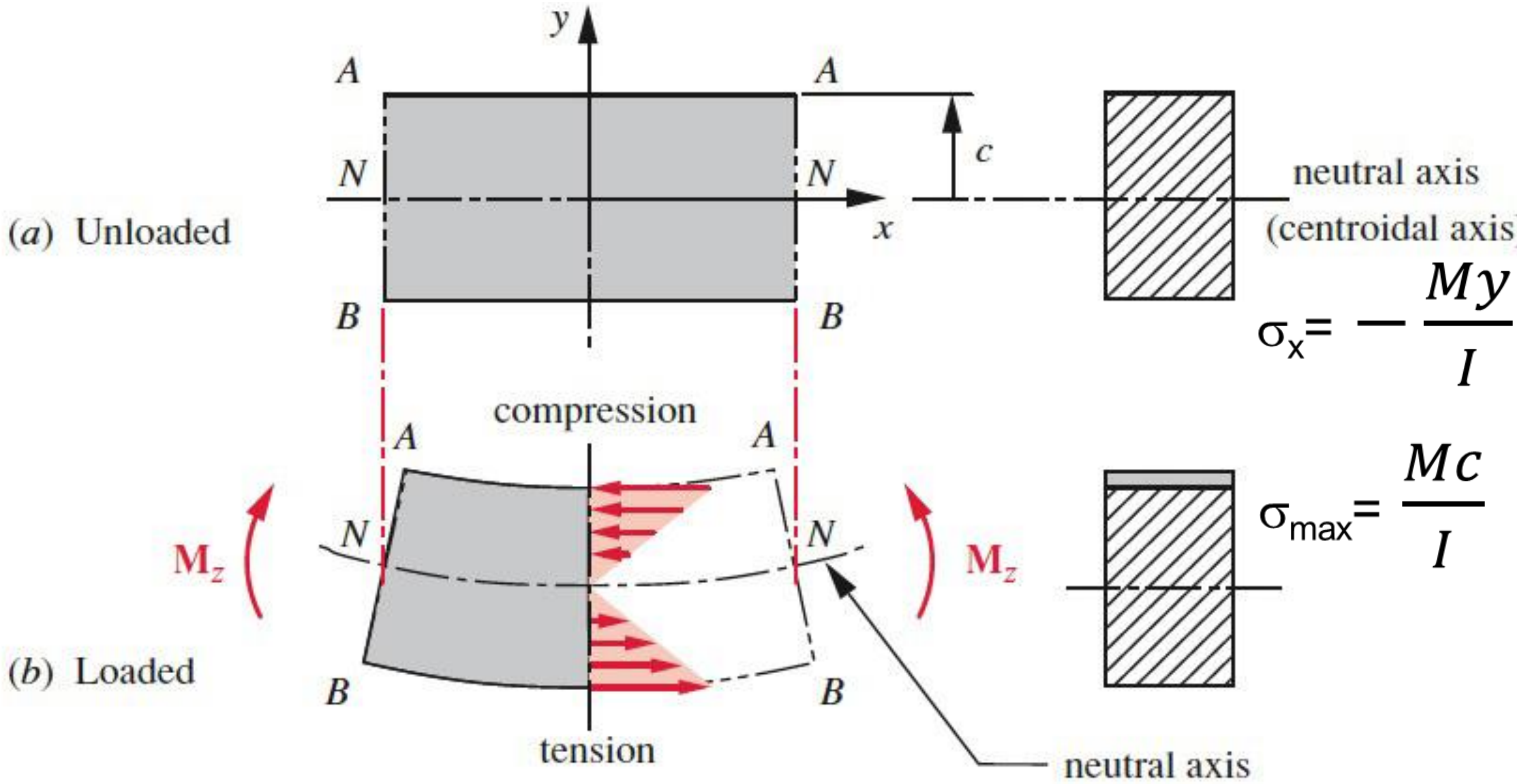



**FIGURE 4-14**
Pure Bending in a Beam




$$\sigma_x = -\frac{My}{I}$$

$$\sigma_{max} = \frac{Mc}{I}$$

FIGURE 4-15

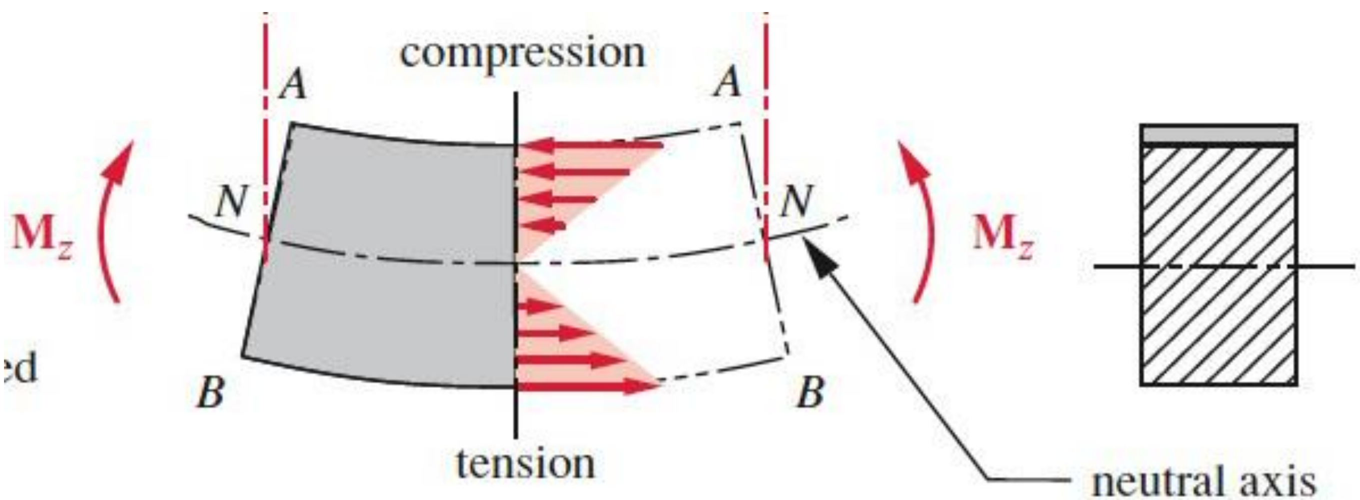
Segment of a Straight Beam in Pure Bending

# 4.5 Pure Bending Loading, Straight Beams

- N-N along the neutral axis, no change in length
- A-A shortents (in compression) and B-B lengthens (in tension
- Bending stress is 0 at N-N and is linearly proportional to distance y away from N-N
- Where M is the bending moment and I is the area moment of Inertia of the beam cross section at the neutral plane, y the distance from N-N
- The max stress at outer plane is
- Where c is distance from neutral plane (it should be same both sections if the beam is symmetrical about the neutral axis

$$\sigma_x = -\frac{My}{I}$$

$$\sigma_{max} = \frac{Mc}{I}$$




- C is taken as +ve initially and proper sign applied based on loading compression –ve and tension +ve

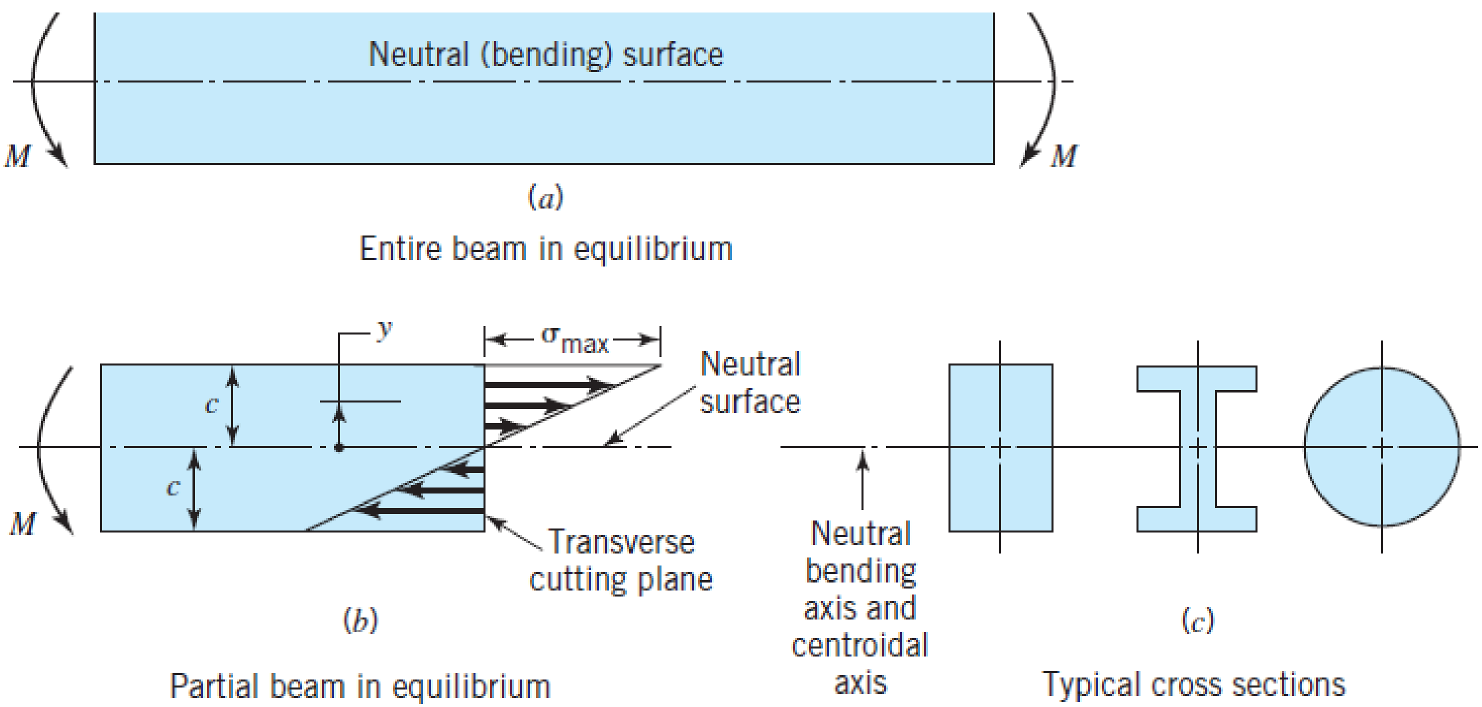
## 4.5 Pure Bending Loading, Straight Beams

- Figure shows bending load applied to beam of CS having 2 axes of symmetry
- cutting-plane stresses $\sigma_{max}$ are obtained from Eq. 4.6 by substituting $c$ for $y$
- *Section modulus Z* ($I/c$) is used, giving $\sigma_{max}$ as

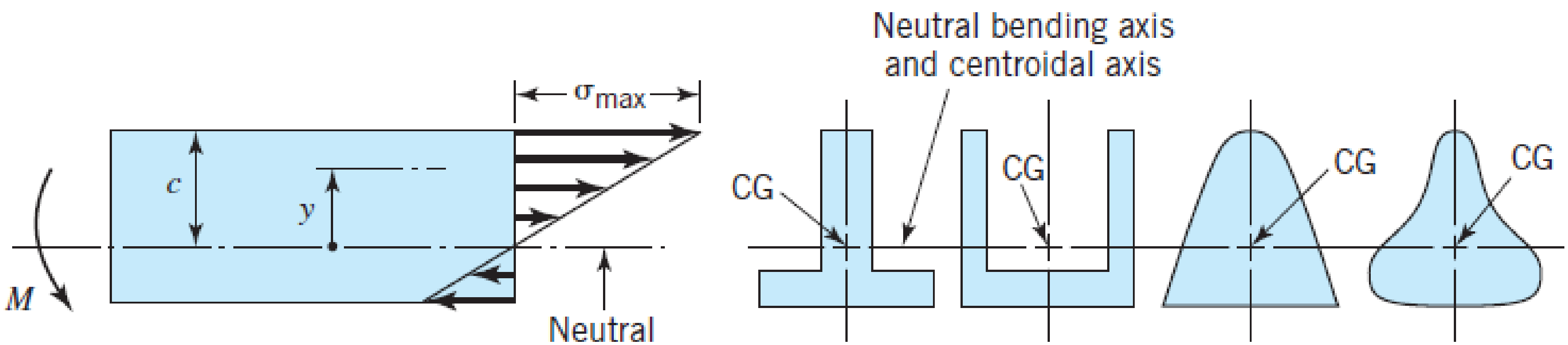
$$\sigma_{max} = M/Z = Mc/I \tag{4.7}$$

For a solid round bar, $I = \pi d^4/64$, $c = d/2$, and $Z = \pi d^3/32$. Hence, for this case

$$\sigma_{max} = 32M/\pi d^3 \tag{4.8}$$



# 4.5 Pure Bending Loading, Straight Beams




(b)

(c)

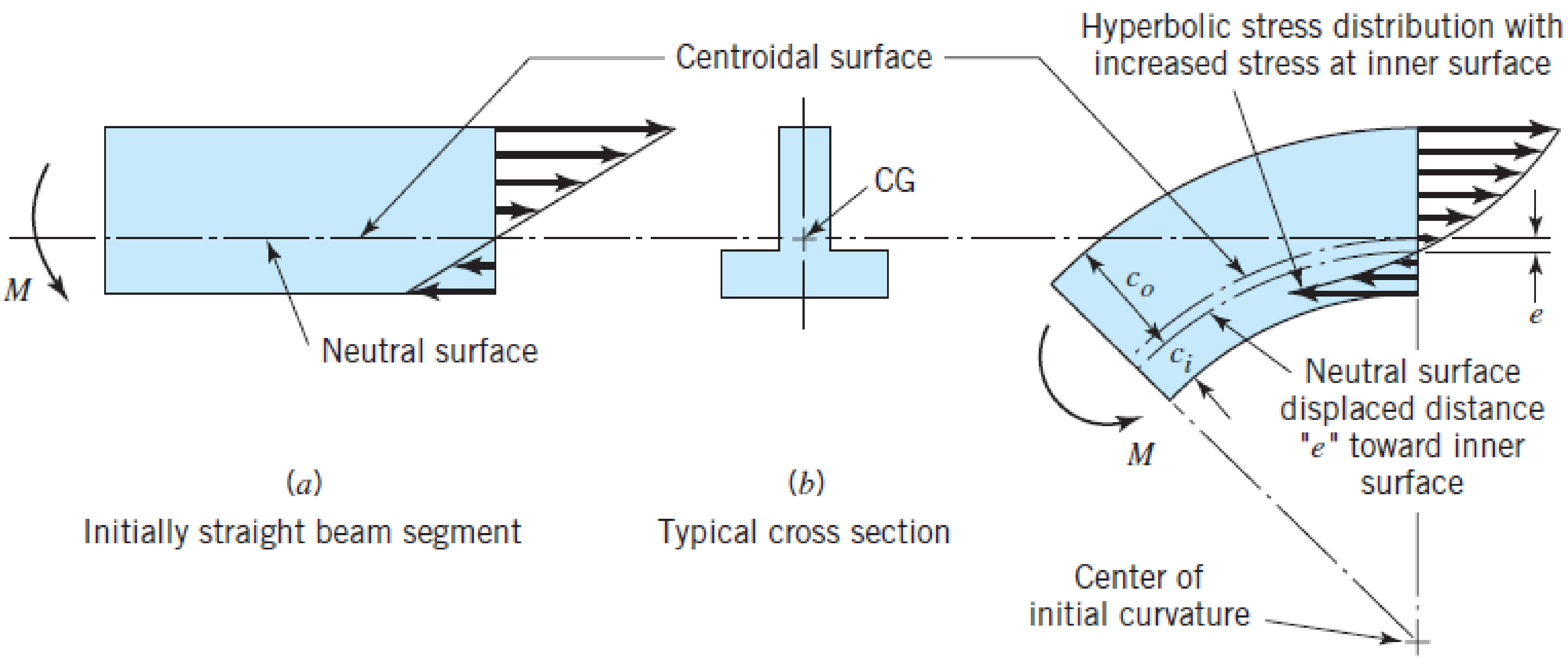
Partial beam in equilibrium

Typical cross sections

- Figure shows bending load applied to beam of CS having 1 axis of symmetry
- The properties for calculating $\sigma_{max}$ for these beams are given in Appendix B

# 4.6 Pure Bending Loading, Curved Beams

- Machines have curved beams like in C clamps, hooks etc, with a ROC) the first 6 assumptions still apply
- If it has a significant curvature the neutral axis will not be coincident with the centroidal axis




(a) Initially straight beam segment

(b) Typical cross section

(c) Initially curved beam segment

# 4.6 Pure Bending Loading, Curved Beams

- If it has a significant curvature the neutral axis will not be coincident with the centroidal axis and the shift e is found from

$$e = r_c - \frac{A}{\int \frac{dA}{r}}$$

- Where A is area, $r_c$ is ROC of centroidal axis and r is the ROC of the differential area dA
- For a rectangular beam this can be $e = r_c - (r_o - r_i)/\mathrm{LN}(r_o / r_i)$

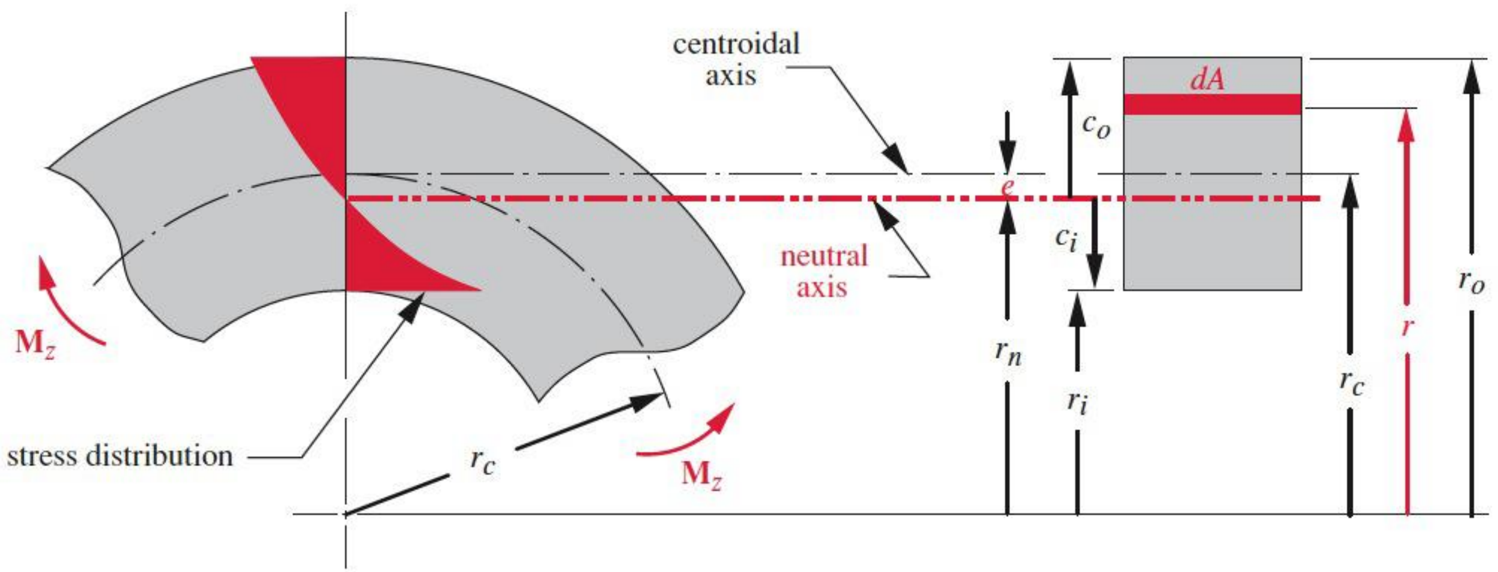



FIGURE 4-16

Segment of a Curved Beam in Pure Bending

# 4.6 Pure Bending Loading, Curved Beams

- Stress distribution is not linear but hyperbolic. Sign convention is +ve moment straightens the beam (tension inside and compression outside)
- For pure bending loads stresses at inner and outer surface is

$$\sigma_i = +\frac{M}{eA}\left(\frac{c_i}{r_i}\right)$$

$$\sigma_o = -\frac{M}{eA}\left(\frac{c_o}{r_o}\right)$$

- And if a force F on the CSA "A" is applied on the curved beam then the stresses will be

$$\sigma_i = +\frac{M}{eA}\left(\frac{c_i}{r_i}\right) + \frac{F}{A}$$

$$\sigma_o = -\frac{M}{eA}\left(\frac{c_o}{r_o}\right) + \frac{F}{A}$$

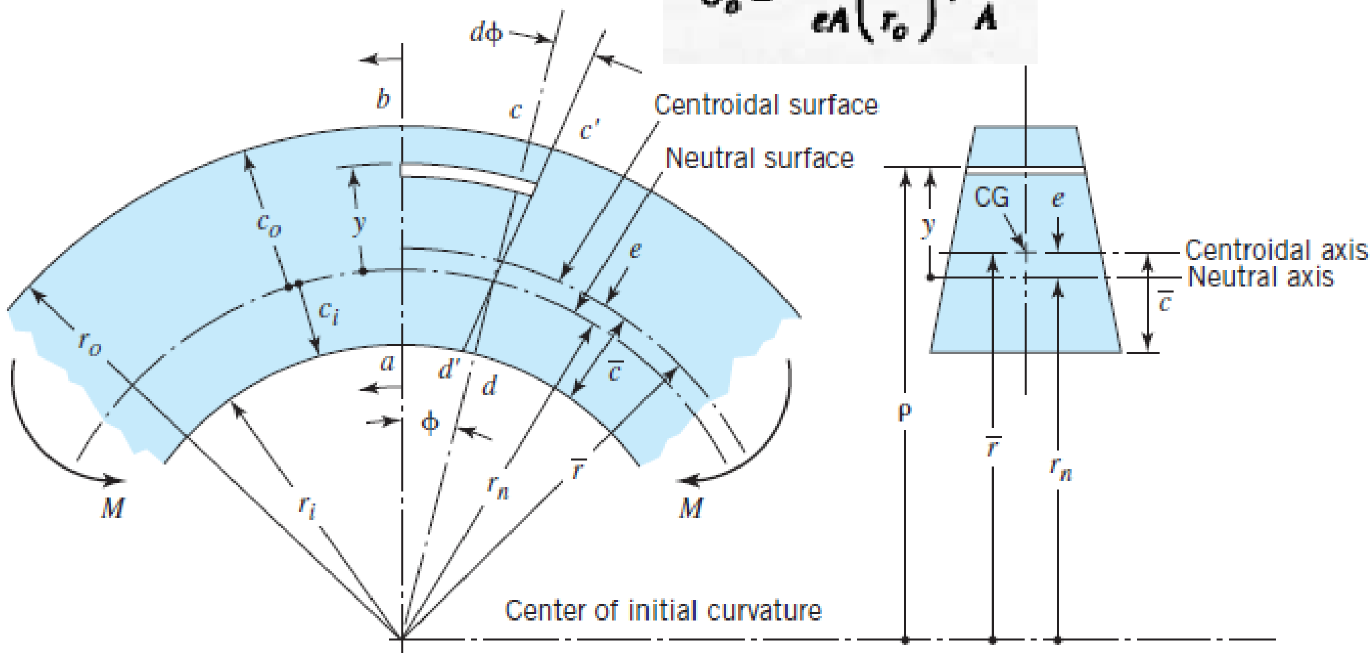



FIGURE 4.10
Curved beam in bending.

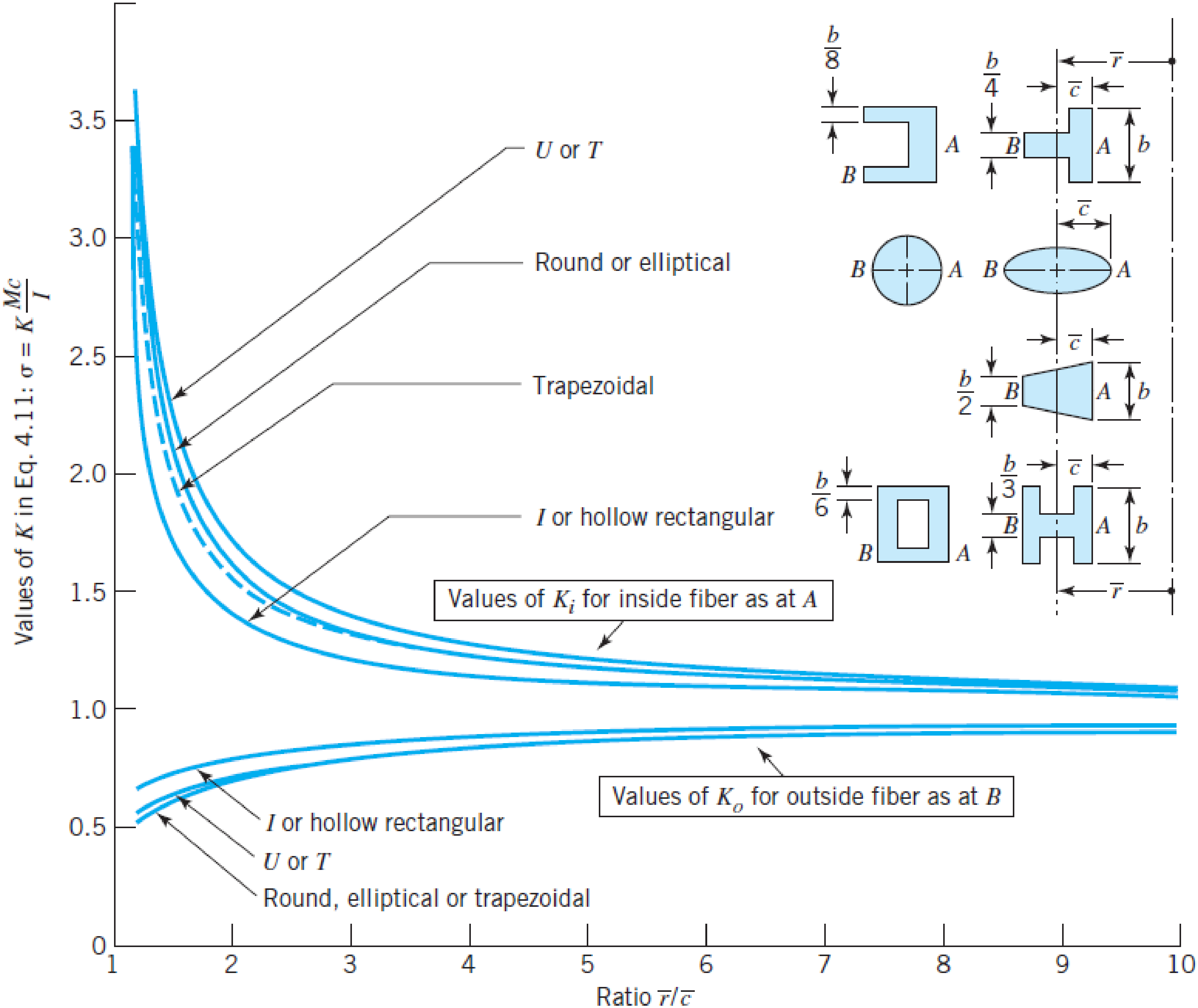
# 4.6 Pure Bending Loading, Curved Beams

$$\sigma_i = +\frac{Mc_i}{eAr_i} \quad \text{and} \quad \sigma_o = -\frac{Mc_o}{eAr_o}$$

Stress values given by Eq differ from the straight-beam "*Mc*/*I*" value by a curvature factor, *K*. Thus, using subscripts *i* and *o* to denote inside and outside fibers, respectively, we have

$$\sigma_i = K_i Mc/I = K_i M/Z \quad \text{and} \quad \sigma_0 = -K_o Mc/I = -K_o M/Z$$

Values of *K* in Figure illustrates a common rule of thumb: "If $\bar{r}$ is at least ten times $\bar{c}$, inner fiber stresses are usually not more than 10 percent above the *Mc*/*I* value." Values of *Ko*, *Ki* , and *e* are tabulated for several cross sections in references.

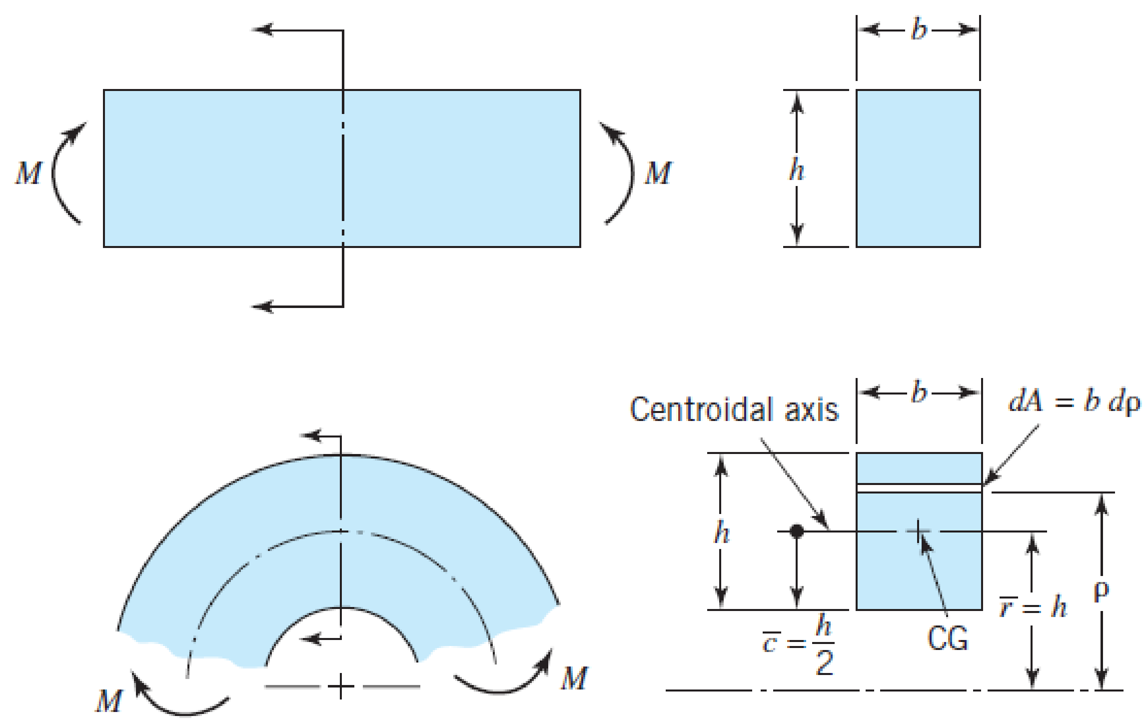

U or T
Round or elliptical
Trapezoidal
I or hollow rectangular
Values of $K_i$ for inside fiber as at A
Values of $K_o$ for outside fiber as at B
I or hollow rectangular
U or T
Round, elliptical or trapezoidal
Values of K in Eq. 4.11: $\sigma = K\frac{Mc}{I}$
Ratio $\bar{r}/\bar{c}$
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
1
2
3
4
5
6
7
8
9
10
$\frac{b}{8}$
$\frac{b}{4}$
$\frac{b}{2}$
$\frac{b}{6}$
$\frac{b}{3}$
$\bar{r}$
$\bar{c}$
A
B
b

## SAMPLE PROBLEM 4.1 Bending Stresses in Straight and Curved Beams

A rectangular beam has an initial curvature equal to the section depth h, as shown in Figure. How do its extreme-fiber-bending stresses compare with those of an identical straight beam?

Known: A straight beam and a curved beam of given cross section and initial curvature are loaded in bending.

Find: Compare the bending stresses between the straight beam and the curved beam.

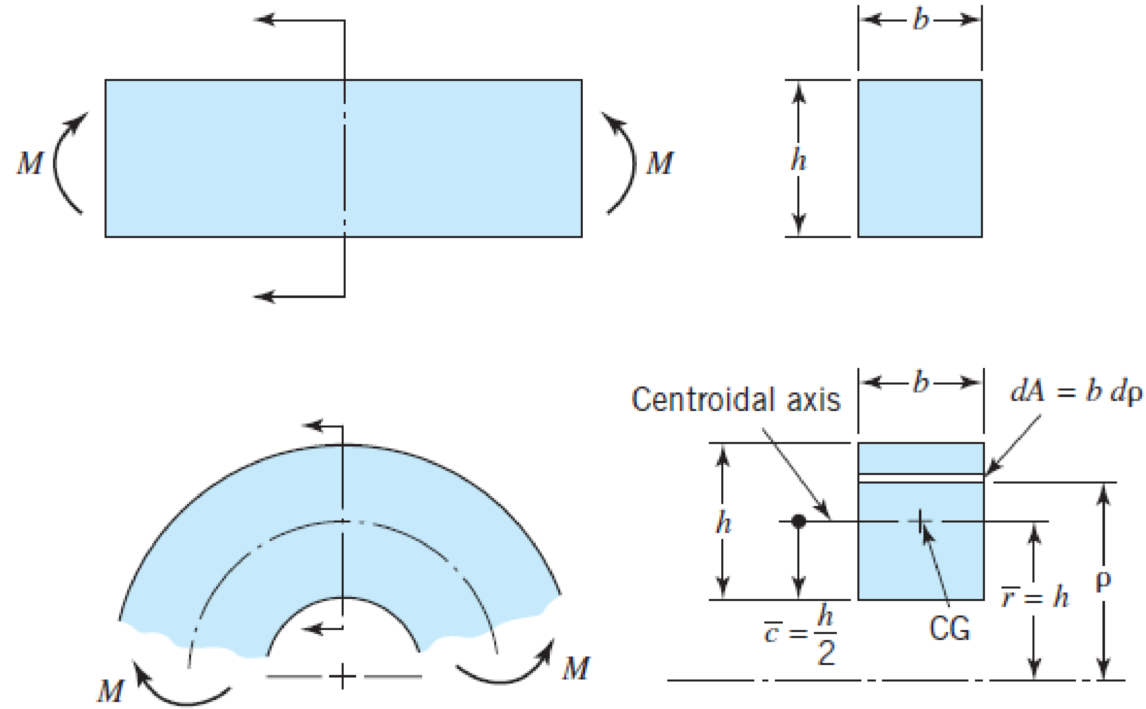



A curved rectangular bar with radius of curvature $\bar{r}$ equal to section depth $h$ (giving $\bar{r}/\bar{c} = 2$) and a straight rectangular bar.

## SAMPLE PROBLEM 4.1 Bending Stresses in Straight and Curved Beams

Assumptions:

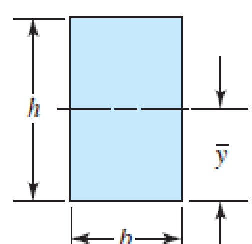
1. The straight bar must initially be straight.
2. The beams are loaded in a plane of symmetry.
3. The material is homogeneous, and all stresses are within the elastic range.
4. The sections where stresses are calculated are not too close to significant stress raisers or to regions where external loads are applied.
5. Initial plane sections remain plane after loading.
6. The bending moment is positive; that is, it tends to straighten an initially curved beam.




**A curved rectangular bar with radius of curvature $\bar{r}$ equal to section depth $h$ (giving $\bar{r}/\bar{c} = 2$) and a straight rectangular bar.**

$A$ = area, in.$^2$
$I$ = moment of inertia, in.$^4$
$J$ = polar moment of inertia, in.$^4$

$Z$ = s
$\rho$ = r
$\bar{y}$ = c




$$A = bh$$

$$I = \frac{bh^3}{12}$$

$$Z = \frac{bh^2}{6}$$

Analysis:

**1.** For the direction of loading shown in Figure 4.12, the conventional straight-beam formula gives

$$\sigma_i = +\frac{Mc}{I} = \frac{6M}{bh^2}, \qquad \sigma_o = -\frac{6M}{bh^2}$$

**2.** From Eq. 4.10,

$$e = \bar{r} - \frac{A}{\int dA/\rho} = h - \frac{bh}{b\int_{r_i}^{r_o} d\rho/\rho} = h - \frac{h}{\ln(r_o/r_i)} = h\left(1 - \frac{1}{\ln 3}\right)$$

$$= 0.089761h$$

- $e = r_c - (r_o - r_i)/LN(r_o / r_i)$
- $r_o = 1.5h$ and $r_i = .5h$ and $r_c = h$

**3.** From Eq. 4.9,

$$\sigma_i = +\frac{M(0.5h - 0.089761h)}{(0.089761h)(bh)(0.5h)} = \frac{9.141M}{bh^2}$$

$$\sigma_o = -\frac{M(0.5h + 0.089761h)}{(0.089761h)(bh)(1.5h)} = -\frac{4.380M}{bh^2}$$

$$\sigma_i = +\frac{M}{eA}\left(\frac{c_i}{r_i}\right)$$

$$\sigma_o = -\frac{M}{eA}\left(\frac{c_o}{r_o}\right)$$

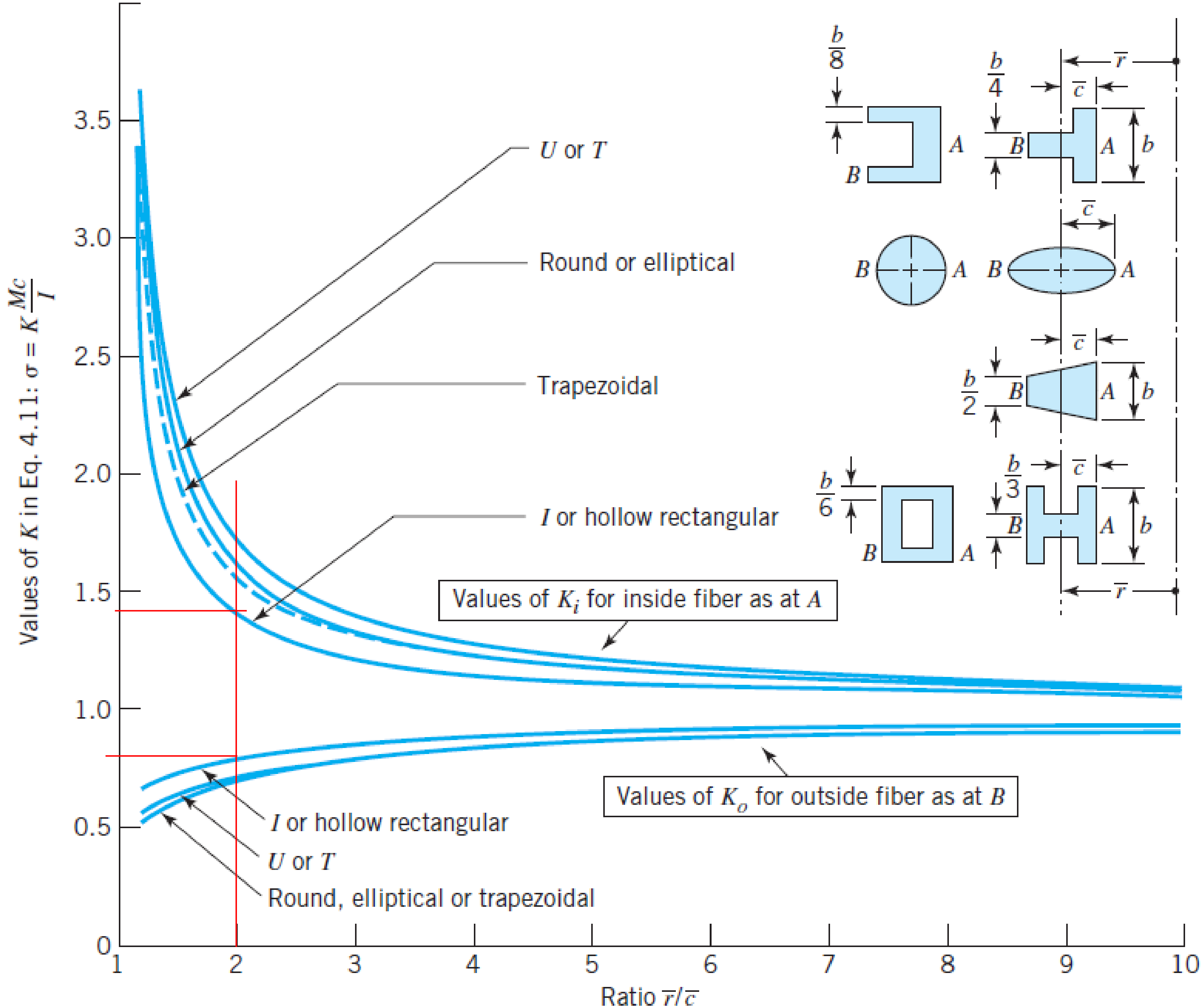
**4.** From Eq. 4.11 with $Z = bh^2/6$,

$$\sigma_i = K_i Mc/I = K_i M/Z$$

$$\sigma_0 = -K_o Mc/I = -K_o M/Z$$

$$K_i = \frac{9.141}{6} = 1.52 \quad \text{and} \quad K_o = \frac{4.380}{6} = 0.73$$

Comment: These values are consistent with those shown for other sections in Figure 4.11 for $\bar{r}/\bar{c} = 2$.

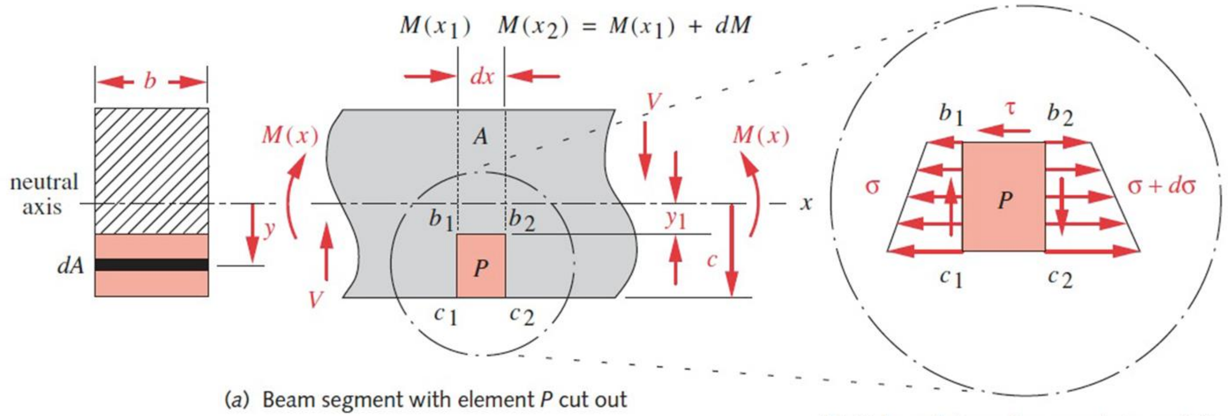


U or T
Round or elliptical
Trapezoidal
I or hollow rectangular
Values of $K_i$ for inside fiber as at $A$
Values of $K_o$ for outside fiber as at $B$
I or hollow rectangular
U or T
Round, elliptical or trapezoidal
Values of $K$ in Eq. 4.11: $\sigma = K\frac{Mc}{I}$
Ratio $\bar{r}/\bar{c}$
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
1
2
3
4
5
6
7
8
9
10
$\frac{b}{8}$
$\frac{b}{4}$
$\frac{b}{2}$
$\frac{b}{6}$
$\frac{b}{3}$
$\bar{r}$
$\bar{c}$
A
B
b

# Shear due to transverse loading



- Common case is both shear and bending moment on the beam.
- Fig shows a point loaded beam shear and moment diagram
- Cutting our segment P of width dx around A, cut from outer side at c upto depth $y_1$ it is seen that the $M(x_1)$ to left $< M(x_2)$ to right and the difference being dM
- Similarly for stresses (can be seen in fig b) since the stresses are proportion to moment




(a) Beam segment with element P cut out

(b) Enlarged view of removed element P

FIGURE 4-19

Segment of a Beam in Bending and Transverse Shear - Shown Removed at Point A in Figure 4-18



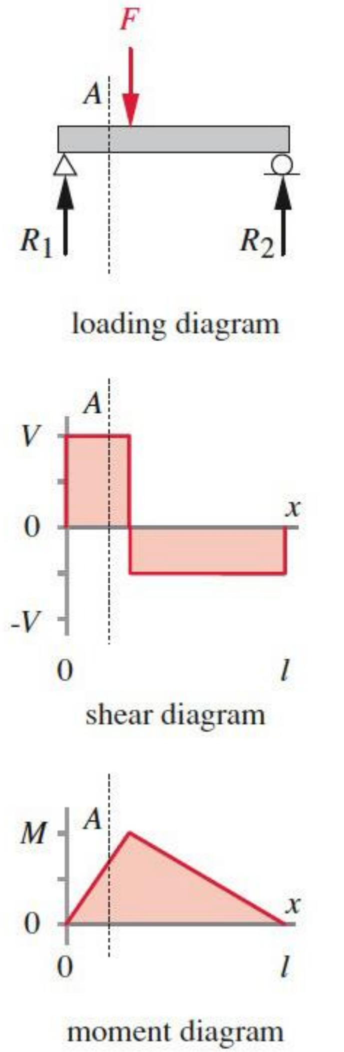



FIGURE 4-18

Shear Force and Bending Moment in a Beam

# Shear due to transverse loading



- Similarly for stresses (can be seen in fig b) since the stresses are proportion to moment. This stress imbalance is countered by shear stress $\tau$ component
- Stress acting on left hand side of p at a distance y from neutral axis is stress times the differential area dA at that point $\sigma dA = \frac{My}{I} dA$
- The total force acting on the left hand side will then be $F_{1x} = \int_{y_1}^{c} \frac{My}{I} dA$
- Similarly for right hand side $F_{2x} = \int_{y_1}^{c} \frac{(M + dM)y}{I} dA$
- Shear force on the top face is $F_{xy} = \tau b dx$
- Where bdx is the area of the top face of the element

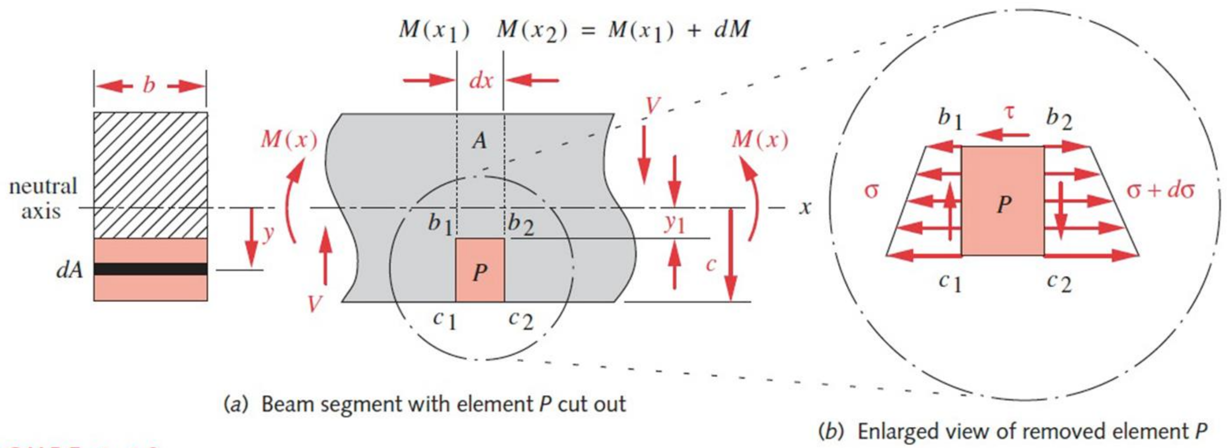



FIGURE 4-19

Segment of a Beam in Bending and Transverse Shear - Shown Removed at Point A in Figure 4-18



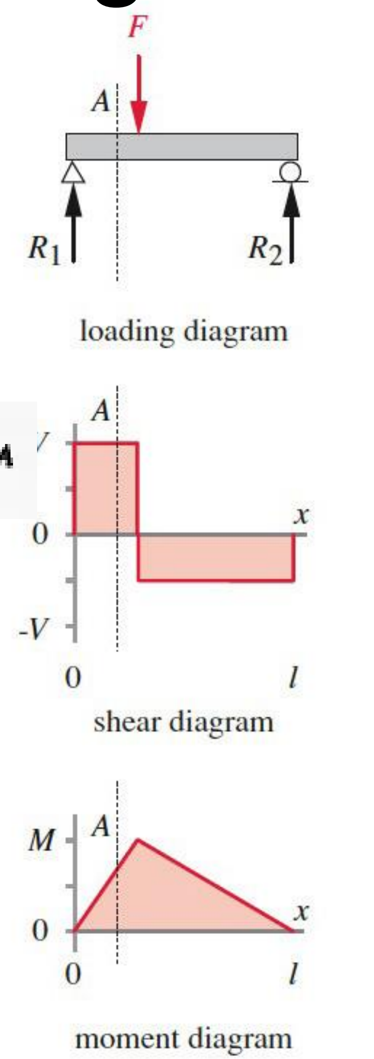



FIGURE 4-18

Shear Force and Bending Moment in a Beam

# Shear due to transverse loading



- For equilibrium, the forces acting on p is 0
- Gives an expression for shear stress as a function of change in momentum wrt x
- Since slope of dM/dx is the magnitude of the shear function V
- Assigning the integral as Q, then
- Shear stresses vary across y
- And becomes 0 when $c=y_1$
- And maximum at neutral axis

$$F_{xy} = F_{2x} - F_{1x}$$

$$\tau b\,dx = \int_{y_1}^{c} \frac{(M+dM)y}{I}\,dA - \int_{y_1}^{c} \frac{My}{I}\,dA$$

$$\tau = \frac{dM}{dx}\frac{1}{Ib}\int_{y_1}^{c} y\,dA$$

$$\tau_{xy} = \frac{V}{Ib}\int_{y_1}^{c} y\,dA$$

$$Q = \int_{y_1}^{c} y\,dA$$

$$\tau_{xy} = \frac{VQ}{Ib}$$

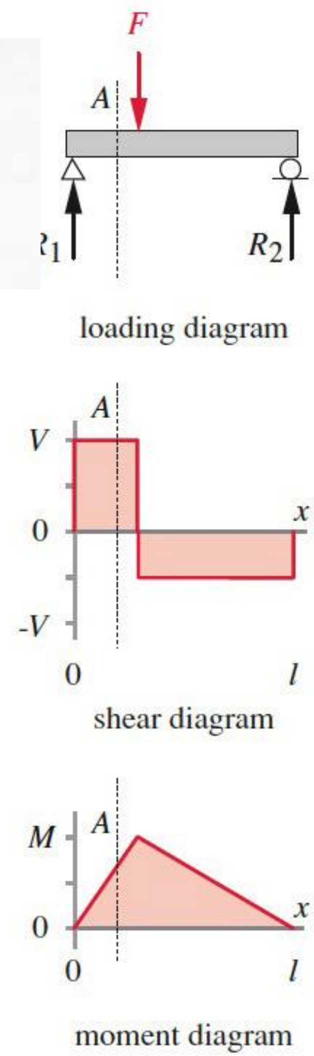
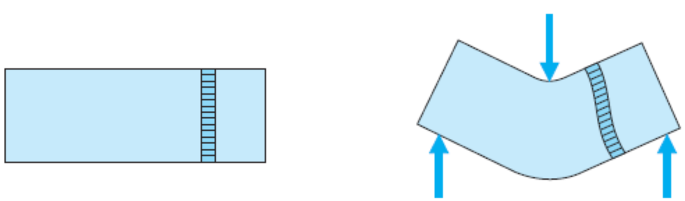



**FIGURE 4-18**
Shear Force and Bending Moment in a Beam

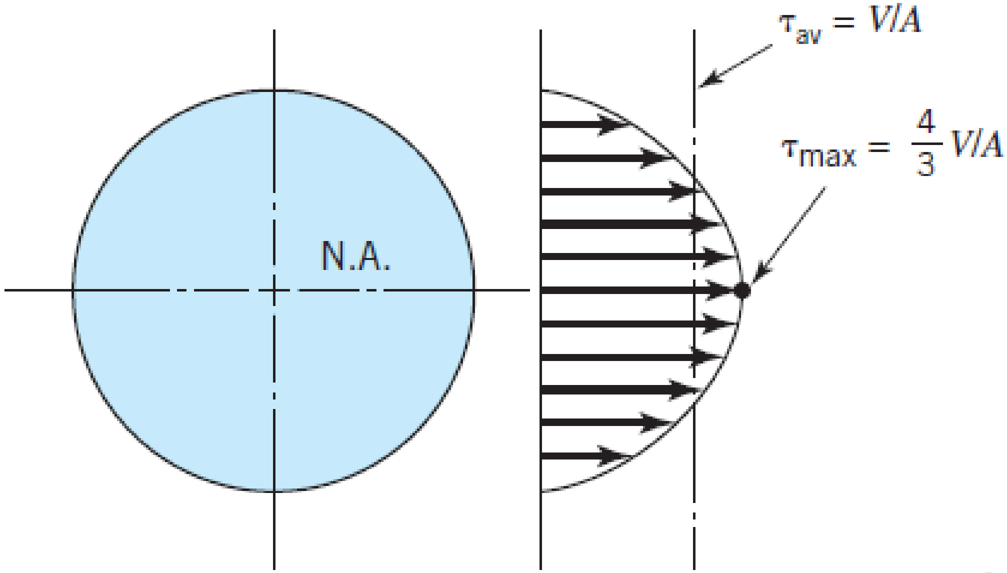


- A common rule of thumb is the shear stress due to transverse loading in a beam will be small enough to ignore if the length to depth ratio of the beam is 10 or more. Short beams below that ratio should be investigated for both transverse shear and bending.



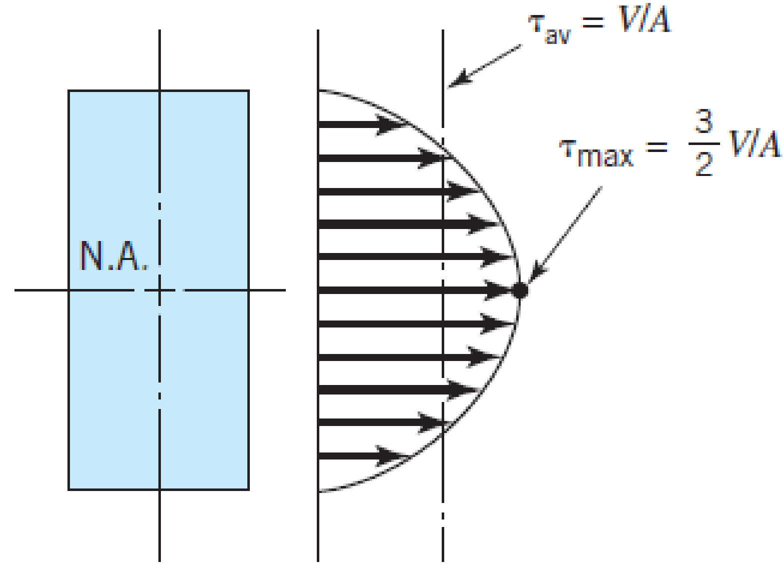
(*a*) Marked and unloaded

(*b*) Loaded as a beam




*solid round* sections

$$\boxed{\tau_{max} = \tfrac{4}{3} V/A}$$




*solid rectangular* sections

$$\boxed{\tau_{max} = \tfrac{3}{2} V/A}$$

For a hollow round section, the stress distribution depends on the ratio of inside to outside diameter, but for *thin-wall tubing*, a good approximation of the maximum shear stress is

$$\tau_{max} = 2V/A \tag{4.15}$$

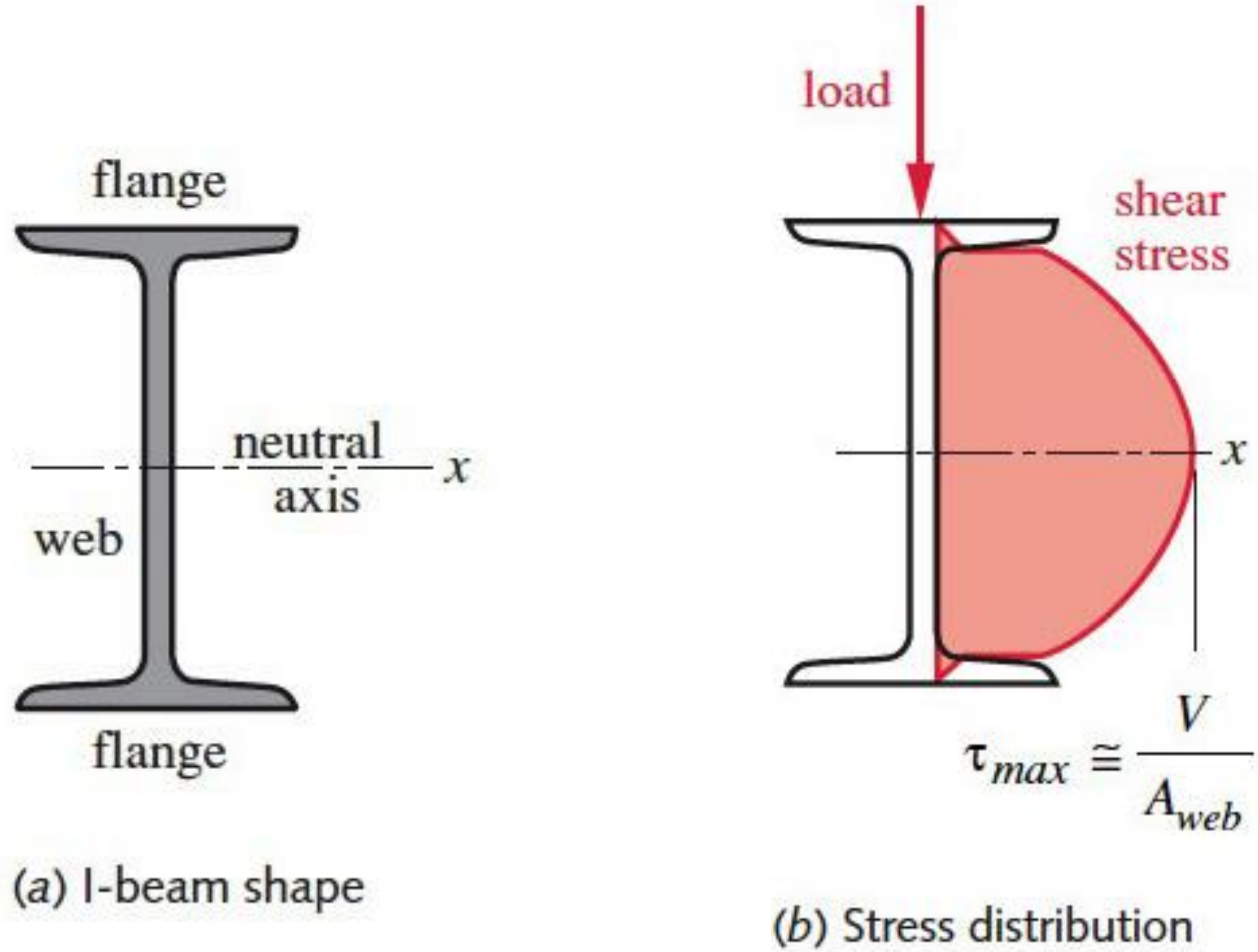



**FIGURE 4-21**

Shear-Stress Distribution and Maximum in an I-Beam

## SAMPLE PROBLEM 4.2 Determine Shear Stress Distribution

Determine the shear stress distribution for the beam and loading shown in Figure 4.18. Compare this with the maximum bending stress.

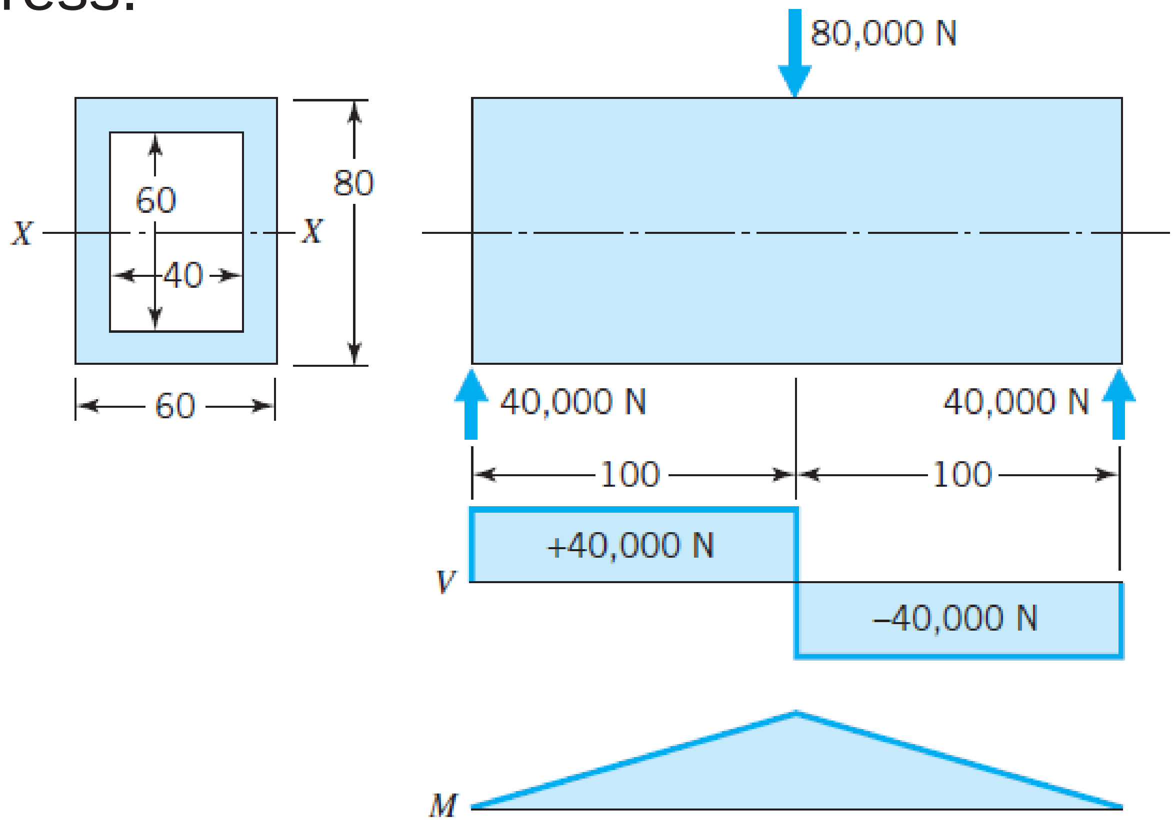
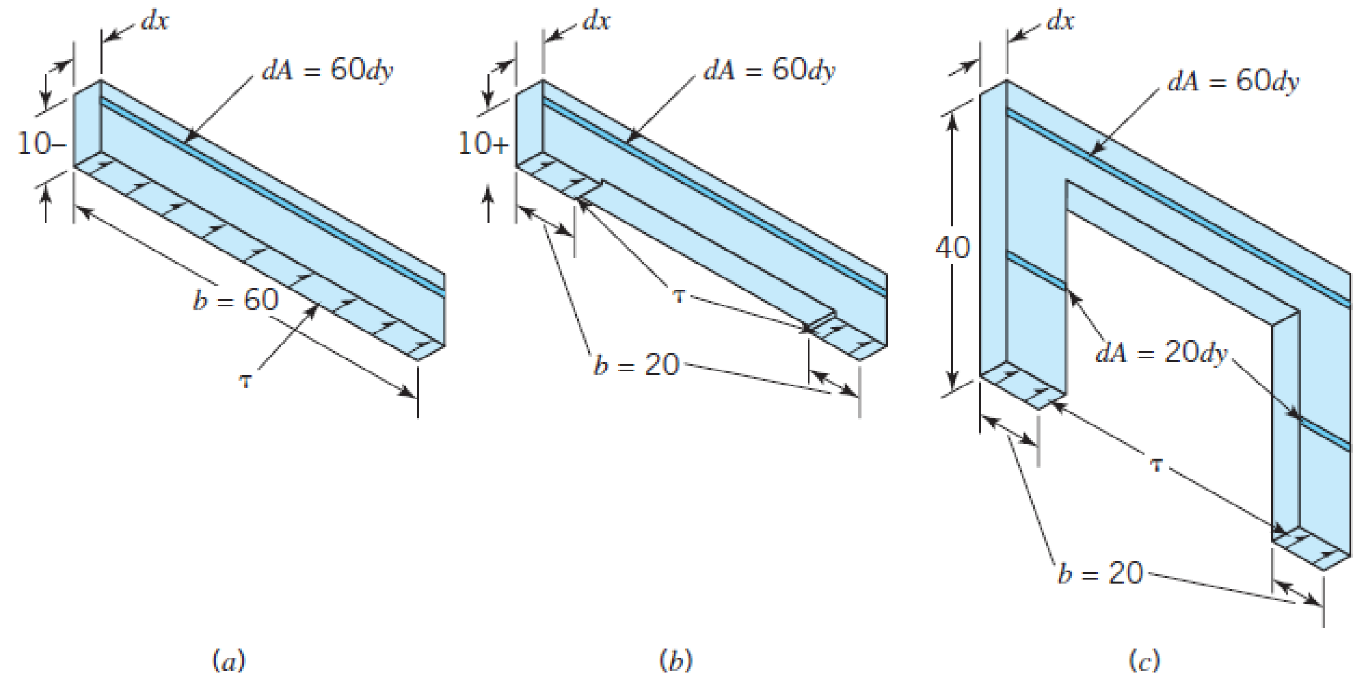



**FIGURE 4.18**

Sample Problem 4.2. Beam shear stress distribution. Note: all dimensions are in millimeters; section properties are $A = 2400\ \text{mm}^2$; $I_x = 1840 \times 10^6\ \text{mm}^4$.




## SOLUTION

**Known:** A rectangular beam with given cross-sectional geometry has a specified central load.

**Find:** Determine the shear stress distribution and the maximum bending stress.

**Assumptions:**

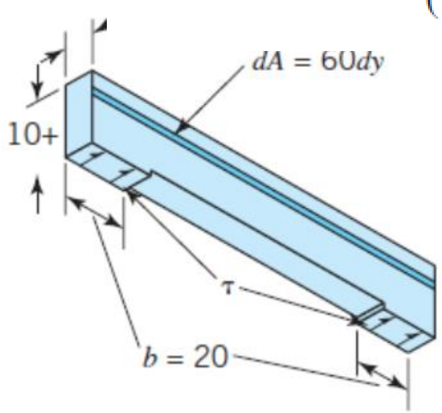
1. The beam is initially straight.
2. The beam is loaded in a plane of symmetry.
3. The shear stress in the beam is uniform across the beam width at each location from the neutral axis.

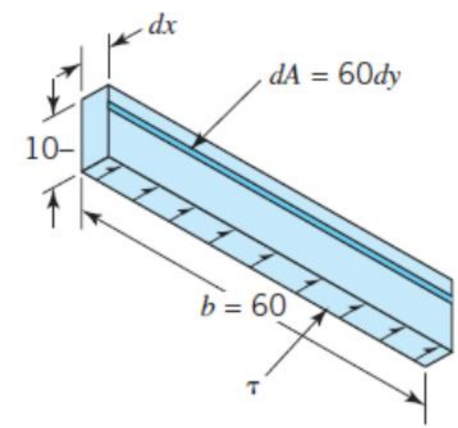
## Analysis:

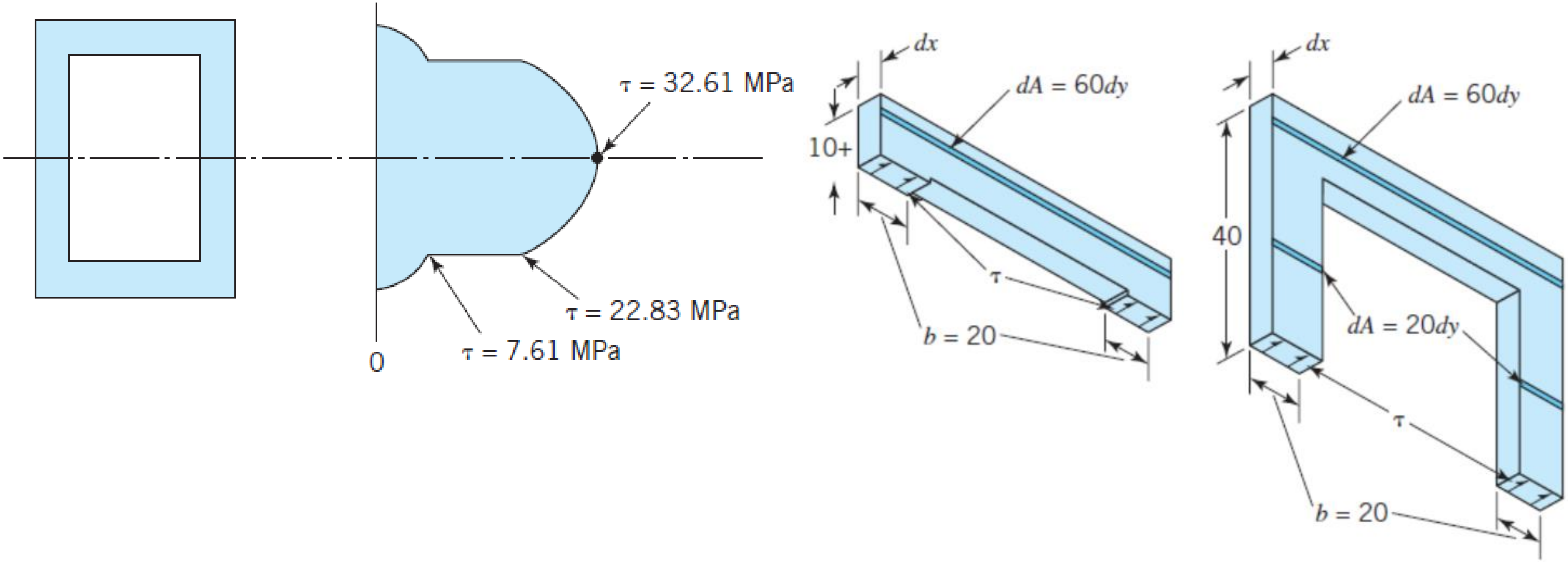
1. With reference to Figure 4.14 and Eq. 4.12, it is known at the outset that $\tau = 0$ at the top and bottom surfaces. This gives a start in plotting the shear stress distribution in Figure 4.20. As the imaginary parallel saw cuts (described in connection with Figure 4.14) proceed down from the top to increasing depth, the areas exposed to the slightly unbalanced bending stresses increase, thereby causing the compensating shear stress at the bottom of the imaginary segment to increase parabolically. This continues to a saw cut depth of 10 mm. Figure 4.19*a* illustrates the imaginary segment just before the saw cuts break through the interior surface of the section. The shear stress at this level (which acts on bottom area 60 · dx) is calculated using Eq. 4.12 as

$$\tau = \frac{V}{Ib}\int_{y=y_0}^{y=c} y\,dA = \frac{40{,}000}{(1.840\times 10^6)(60)}\int_{y=30}^{y=40} y(60dy)$$

$$= \frac{40{,}000}{(1.840\times 10^6)(60)}(60)\left[\frac{y^2}{2}\right]_{y=30}^{y=40} = 7.61\ \text{N/mm}^2\text{, or } 7.61\ \text{MPa}$$






2. With a slightly deeper saw cut, the inner surface is broken through, and the area over which the shear stress acts is suddenly reduced to 20 *dy*, as shown in Figure 4.19*b*. The unbalanced bending forces acting on the segment sides are virtually unchanged. Thus, the only term that changes in Eq. 4.12 is *b*, which is reduced by a factor of 3, thereby giving a shear stress three times as high, or 22.83 MPa.

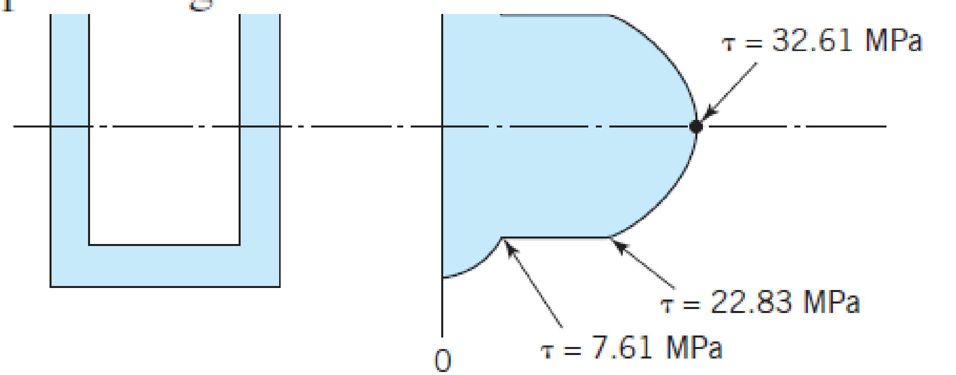
**3.** As the saw cut depth increases until it reaches the neutral axis, the area over which the shear stress acts remains the same, while greater and greater imbalances build up as additional areas $dA$ are exposed. But, as shown in Figure 4.19$c$, these added areas $dA$ are only one-third as large as those in the top portion of the section. Hence, the increased shear stress at the neutral axis is not as great as might at first be expected. When using Eq. 4.12 to find $\tau$ at the neutral axis, note that two integrals are involved, one covering the range of $y$ from 0 to 30 mm and the other from 30 to 40 mm. (The latter integral, of course, has already been evaluated.)

$$\tau = \frac{V}{Ib}\int_{y=y_0}^{y=c} y\,dA = \frac{40{,}000}{(1.840\times 10^6)(20)}\left[\int_{y=0}^{y=30} y(20\,dy) + \int_{y=30}^{y=40} y(60\,dy)\right]$$

$$= \frac{40{,}000}{(1.840\times 10^6)(20)}(20)\left[\frac{y^2}{2}\right]_{y=0}^{y=30} + 22.83$$

$$= 32.61\ \text{N/mm}^2\text{, or } 32.61\ \text{MPa}$$

These calculations enable the shear stress plot in Figure 4.20 to be drawn.

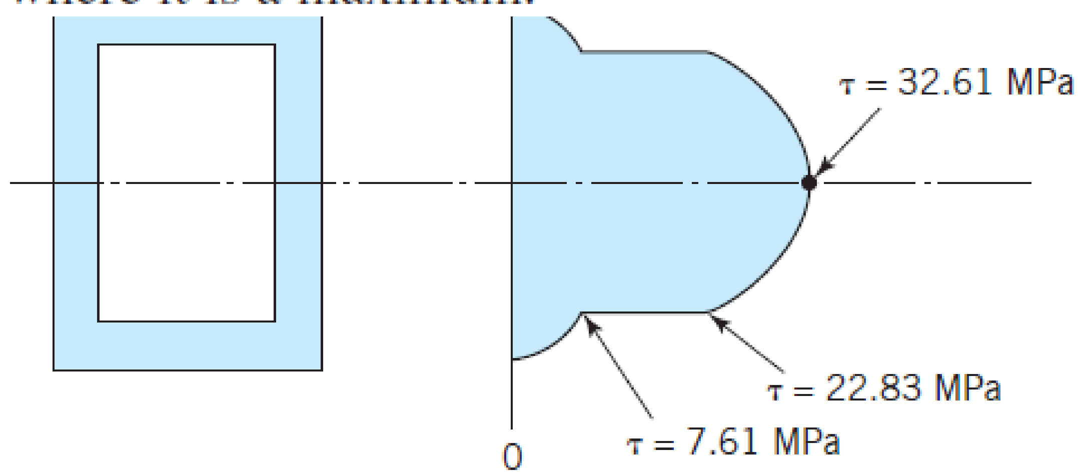


**4.** By way of comparison, the maximum bending stresses occur in the top and bottom surfaces of the beam, halfway along its length, where the bending moment is highest. Here, the bending stress is computed as

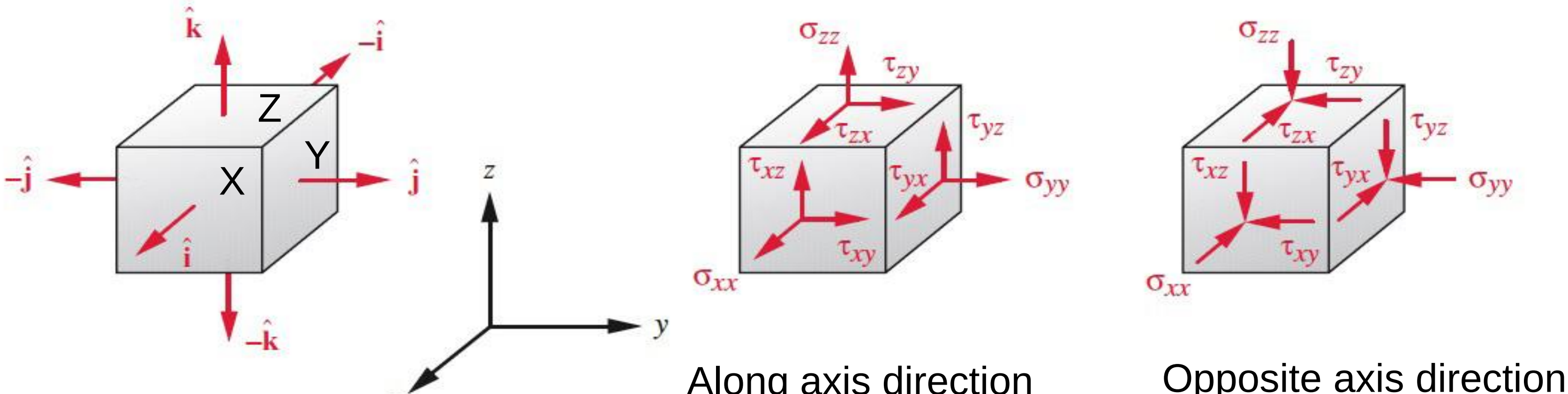
$$\sigma = \frac{Mc}{I} = \frac{(40{,}000 \times 100)(40)}{1.84 \times 10^6} = 86.96 \text{ N/mm}^2$$

$$= 86.96 \text{ MPa}$$

**Comment:** Recalling that the shear stress must be zero at the exposed inner surface of the section, it is apparent that the evenly distributed shear stress assumed in Figure 4.19*a* is incorrect, and that the shear stresses in the outer supported portions of the section at this level will be higher than the calculated value of 7.61 MPa. This is of little importance because, to the degree that shear stresses are of concern, attention will be focused at the level just below, where the calculated value of $\tau$ is three times as high, or at the neutral axis where it is a maximum.



# 4.8 Induced Stresses, Mohr Circle Representation



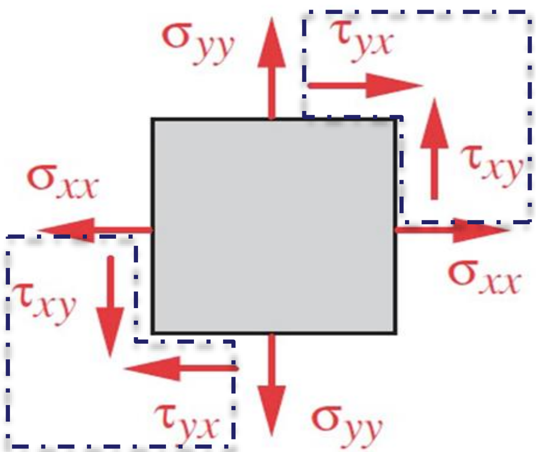

Along axis direction

Opposite axis direction

(a) Surface normals (b) Positive stress components (c) Negative stress components

**FIGURE 4-1**

The Stress Cube, Its Surface Normals, and Its Stress Components



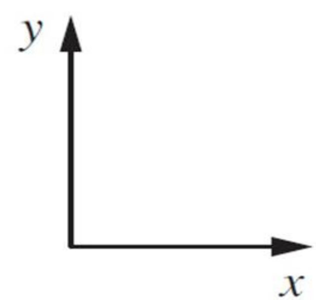
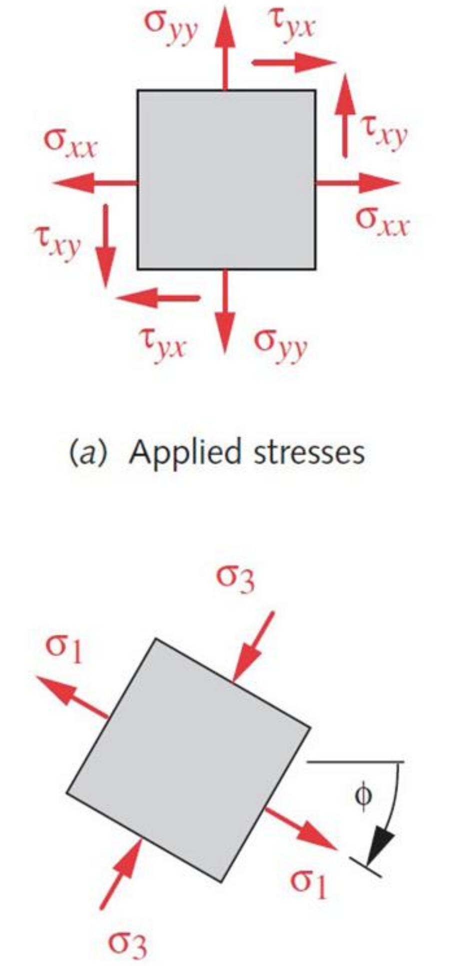




**FIGURE 4-2**

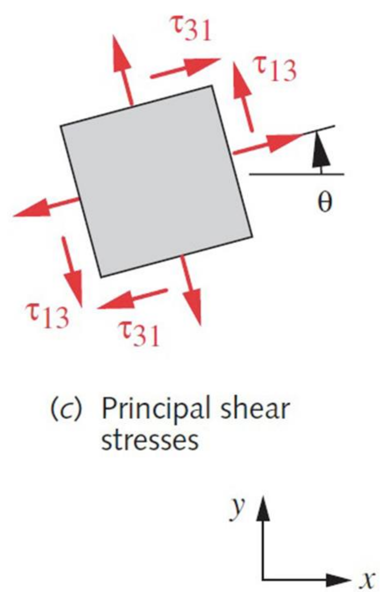
Two-Dimensional Stress Element

- Axis of the stresses is arbitrarily chosen for convenience. Normal and shear stresses at one point will vary with direction along the coordinate system.
- There will be planes where the shear stress is 0, and the normal stresses acting on these planes are principal stresses, and planes as principal planes
- There will be planes where the shear stress components are 0, and the normal stresses acting on these planes are principal stresses, and planes as principal planes
- Direction of surface normals to the planes is principal axes .
- And the normal stresses acting in these directions are principal normal stresses
- The principal shear stresses, act on planes at 45° to the planes of principal normal stresses




(*a*) Applied stresses

(*b*) Principal normal stresses




(*c*) Principal shear stresses

**FIGURE 4-3**

Principal Stresses on a Two-Dimensional Stress Element

- The principal shear stresses from principal normal stresses can be calculated as

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2}$$

$$\tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2}$$

$$\tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2}$$

- Since usually $\sigma_1 > \sigma_2 > \sigma_3$, $\tau_{max}$ is $\tau_{13}$ and the direction of principal shear stresses are 45° to the principal normal planes and are mutually orthogonal

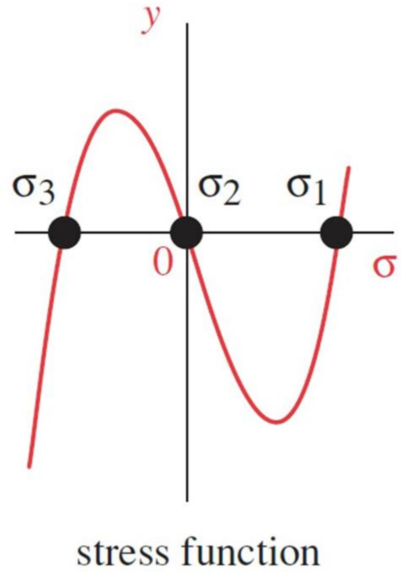
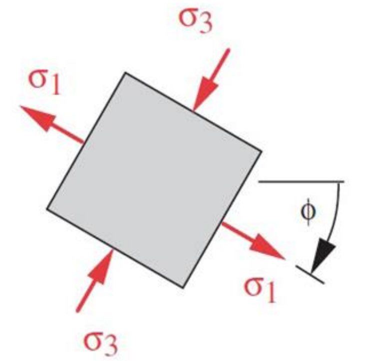



FIGURE 4-4

The Three Roots of the Stress Function for a Plane Stress Case

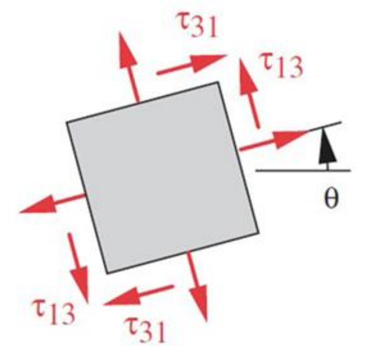




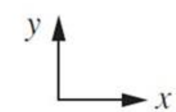

(b) Principal normal stresses

FIGURE 4-3

Principal Stresses on a Two-Dimensional Stress Element



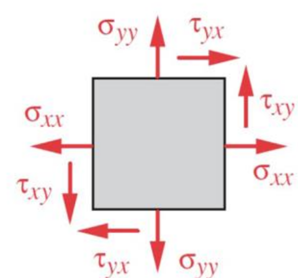

(c) Principal shear stresses

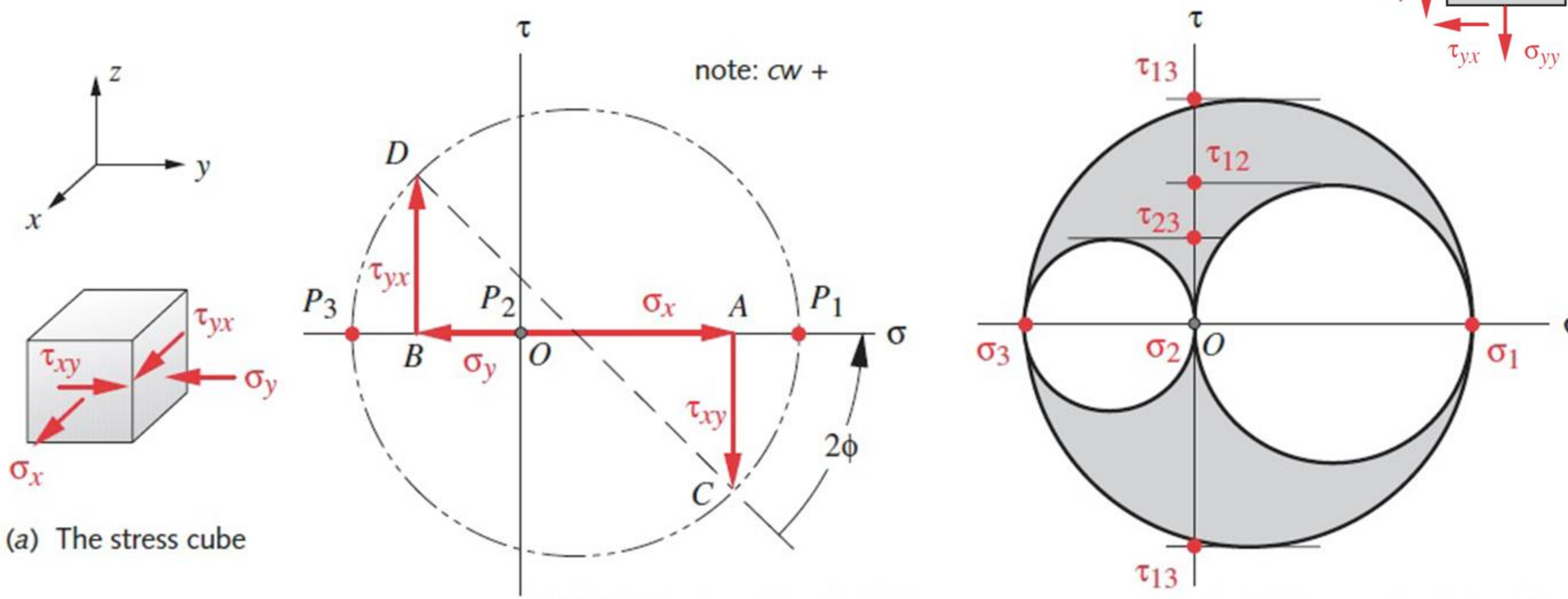


# The Mohr's Circle

- Mohr circle is a graphical method find the principal stresses

**Problem:** **A biaxial stress element as shown in Figure 4-2 has $\sigma_x$ = 40 000 psi, $\sigma_y$ = –20 000 psi, and $\tau_{xy}$ = 30 000 psi *ccw*. Use Mohr's circles to determine the principal stresses. Check the result with a numerical method.**




(*a*) The stress cube

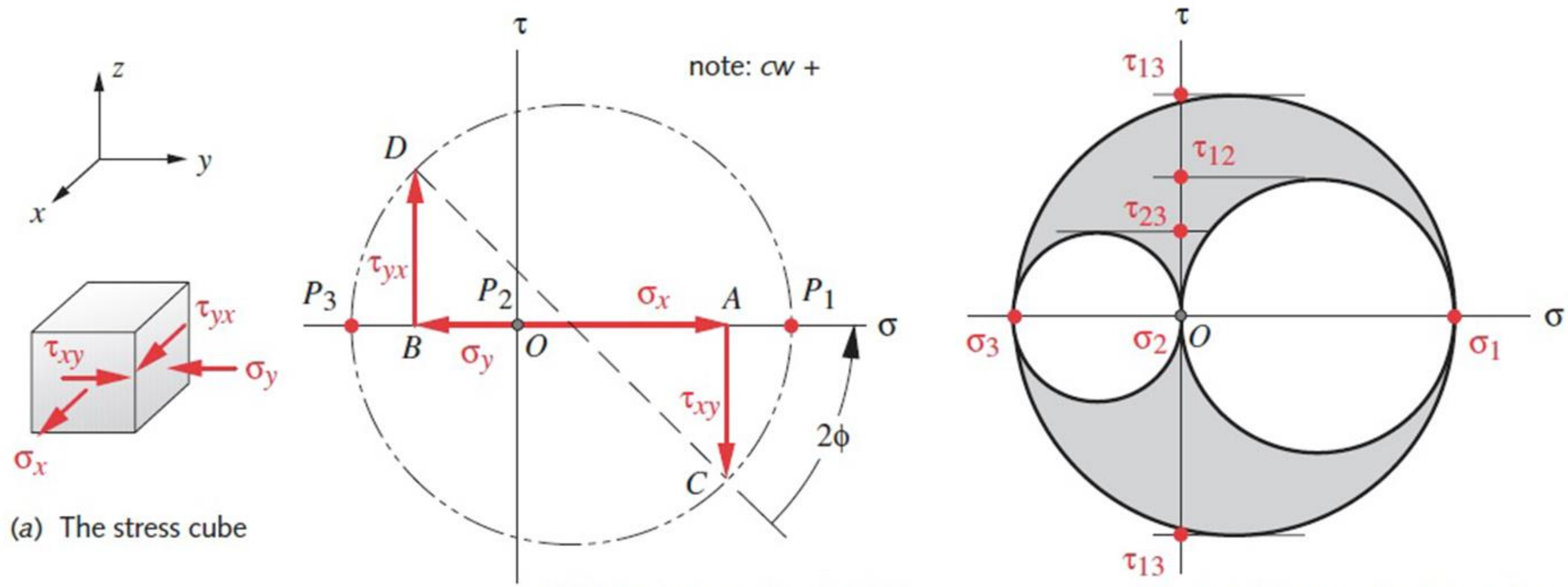
(*b*) Constructing the first circle

(*c*) The three Mohr's circles

**FIGURE 4-5**

The Stress Cube and Mohr's Circles for Example 4-1

# The Mohr's Circle




(*a*) The stress cube

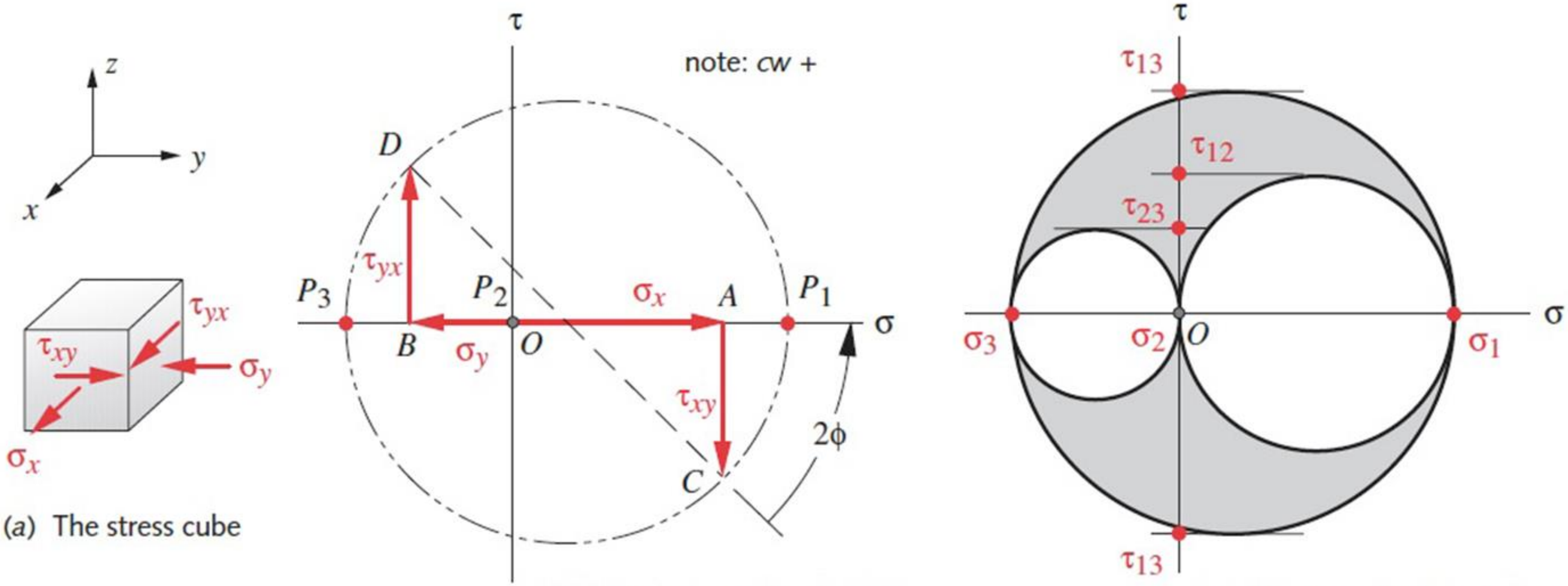
(*b*) Constructing the first circle

(*c*) The three Mohr's circles

7 Two of the three principal normal stresses are then found at the two intersections that this Mohr's circle makes with the normal stress axis at points $P_1$ and $P_3$: $\sigma_1 = 52\,426$ psi at $P_1$ and $\sigma_3 = -32\,426$ psi at $P_3$.

8 Since there were no applied stresses in the $z$ direction in this example, it is a 2-D stress state, and the third principal stress, $\sigma_2$, is zero, located at point $O$, which is also labeled $P_2$.

# The Mohr's Circle




(*a*) The stress cube

(*b*) Constructing the first circle

(*c*) The three Mohr's circles

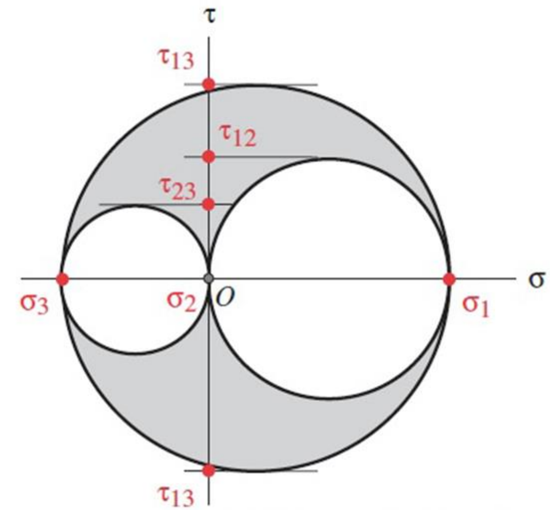
**8** **Since there were no applied stresses in the *z* direction in this example, it is a 2-D stress state, and the third principal stress, $\sigma_2$, is zero, located at point *O*, which is also labeled $P_2$.**

**9** **There are still two other Mohr's circles to be drawn. The three Mohr's circles are defined by the diameters ($\sigma_1 - \sigma_3$), ($\sigma_1 - \sigma_2$), and ($\sigma_2 - \sigma_3$), which are the lines $P_1P_3$, $P_2P_1$, and $P_2P_3$. The three circles are shown in Figure 4-5*c*.**

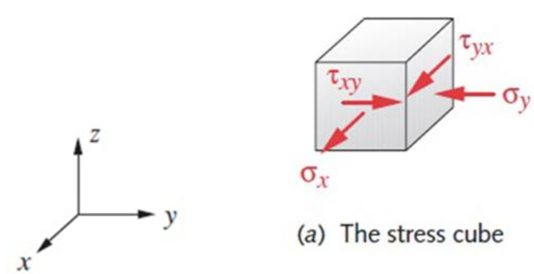
# The Mohr's Circle

10 Extend horizontal tangent lines from the top and bottom extremes of each Mohr's circle to intersect the shear (vertical) axis. This determines the values of the principal shear stresses associated with each pair of principal normal stresses: $\tau_{13} = 42\,426$, $\tau_{12} = 26\,213$, and $\tau_{23} = 16\,213$ psi. Note that despite having only two nonzero principal normal stresses, there are three nonzero principal shear stresses. However, only the largest of these, $\tau_{max} = \tau_{13} = 42\,426$ psi, is of interest for design purposes.

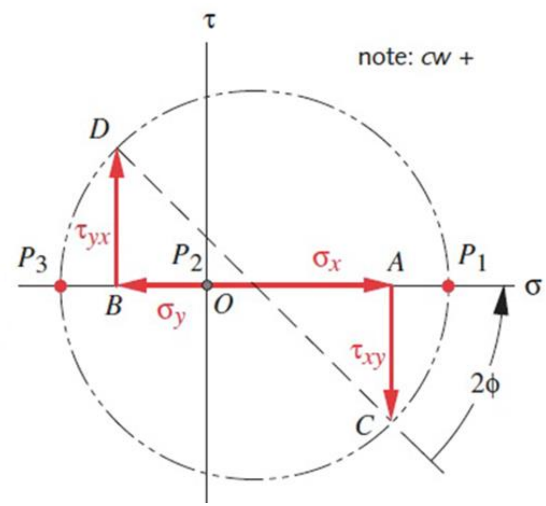
11 We can also determine the angles (with respect to our original *xyz* axes) of the principal normal and principal shear stresses from the Mohr's circle. These angles are only of academic interest if the material is homogeneous and isotropic. If it is not isotropic, its material properties are direction dependent and the directions of the principal stresses are then important. The angle $2\phi = -45°$ in Figure 4-5*a* represents the orientation of the principal normal stress with respect to the *x* axis of our original system. Note that the line *DC* on the Mohr plane is the *x* axis in real space and the angles are measured according to Mohr's left-handed convention (*cw*+). Since angles on the Mohr plane are double those in real space, the angle of the principal stress $\sigma_1$ with respect to the real-space *x* axis, is $\phi = -22.5°$. The stress $\sigma_3$ will be 90° from $\sigma_1$ and the maximum shear stress $\tau_{13}$ will be 45° from the $\sigma_1$ axis in real space.



(c) The three Mohr's circles



(a) The stress cube



## Sample Problem 4.3 Stresses in Stationary Shaft

Figure 4.23 represents a stationary shaft and pulley subjected to a 2000-lb static load. Determine the location of highest stresses in the 1-in.-diameter section, and calculate the stresses at that point.

## Solution

**Known:** A shaft of given geometry is subjected to a known combined loading.

**Find:** Determine the magnitude and location of the highest stresses.

**Schematic and Given Data:**

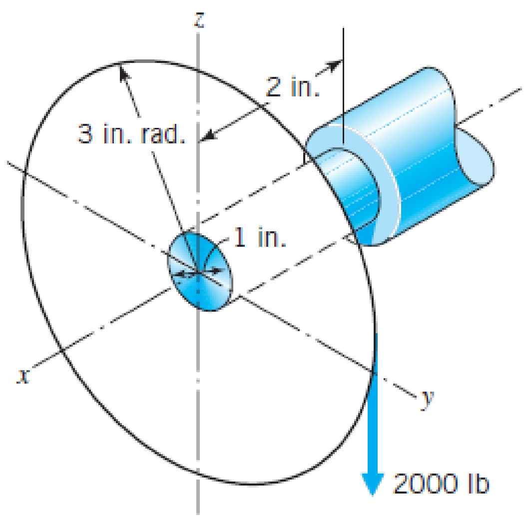



**Figure 4.23**
Shaft subjected to combined loading. For a solid 1-in.-diameter shaft: $A = \pi d^2/4 = 0.785 \text{ in.}^2$; $I = \pi d^4/64 = 0.049 \text{ in.}^4$; and $J = \pi d^4/32 = 0.098 \text{ in.}^4$ (see Appendix B-1).

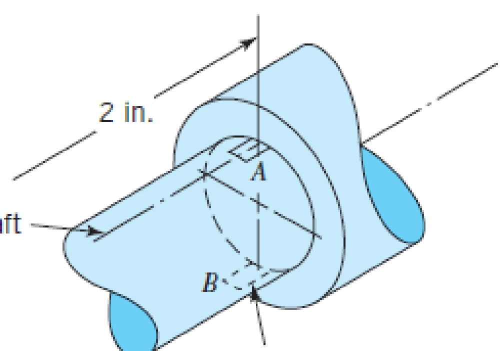



**Figure 4.24**
Location of highest stresses.

## Assumptions:

1. The stress concentration at the 1-in.-diameter shaft step can be ignored.
2. The compressive stress on the shaft surface caused by atmospheric pressure has negligible effects.

## Analysis:

- The shaft is subjected to torsion, bending, and transverse shear. Torsional stresses are max @ shaft surface. Bending stresses are a maximum at points A and B
- Transverse shear stresses are relatively small compared to bending stresses, and equal to zero at points A and B

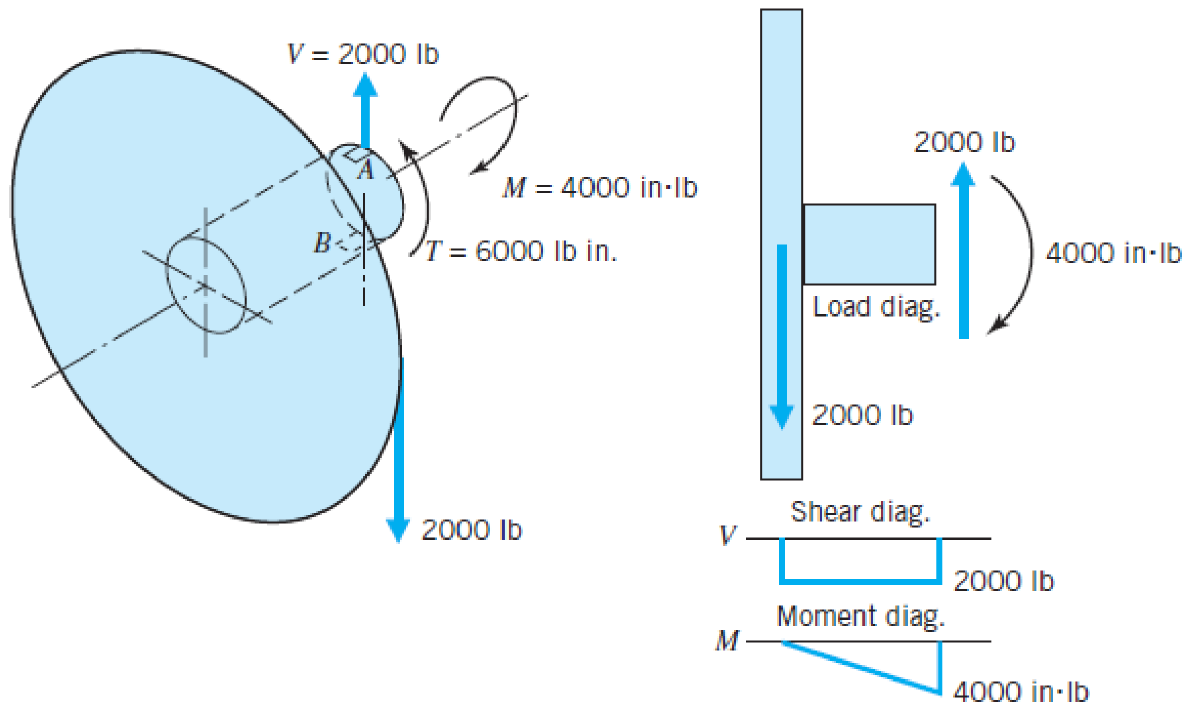



**FIGURE 4.25**
Free-body and load diagrams.

2. In Figure 4.25, imagine the shaft to be cut off at the section containing $A$ and $B$, and consider the member thus obtained as a *free body in equilibrium*. This is a convenient way of being certain that all loads acting on the cutting plane are identified. In this case there are the three loads, $M$, $T$, and $V$, as shown. Note that the free body is indeed in equilibrium, the summation of all forces and moments being zero. Also in Figure 4.25 are the load, shear, and moment diagrams for the isolated free body.

3. Compute the direct stresses associated with loads.

   Bending stresses (tension at $A$; compression at $B$):

$$\sigma_x = \frac{Mc}{I} = \frac{(4000 \text{ in} \cdot \text{lb})\left(\frac{1}{2}\text{in.}\right)}{0.049 \text{ in.}^4} = 40{,}816 \text{ psi} \approx 40.8 \text{ ksi}$$

   Torsional stresses (over the entire surface):

$$\tau_{xy} = \frac{Tr}{J} = \frac{(6000 \text{ lb} \cdot \text{in.})\left(\frac{1}{2}\text{in.}\right)}{0.098 \text{ in.}^4} = 30{,}612 \text{ psi} \approx 30.6 \text{ ksi}$$

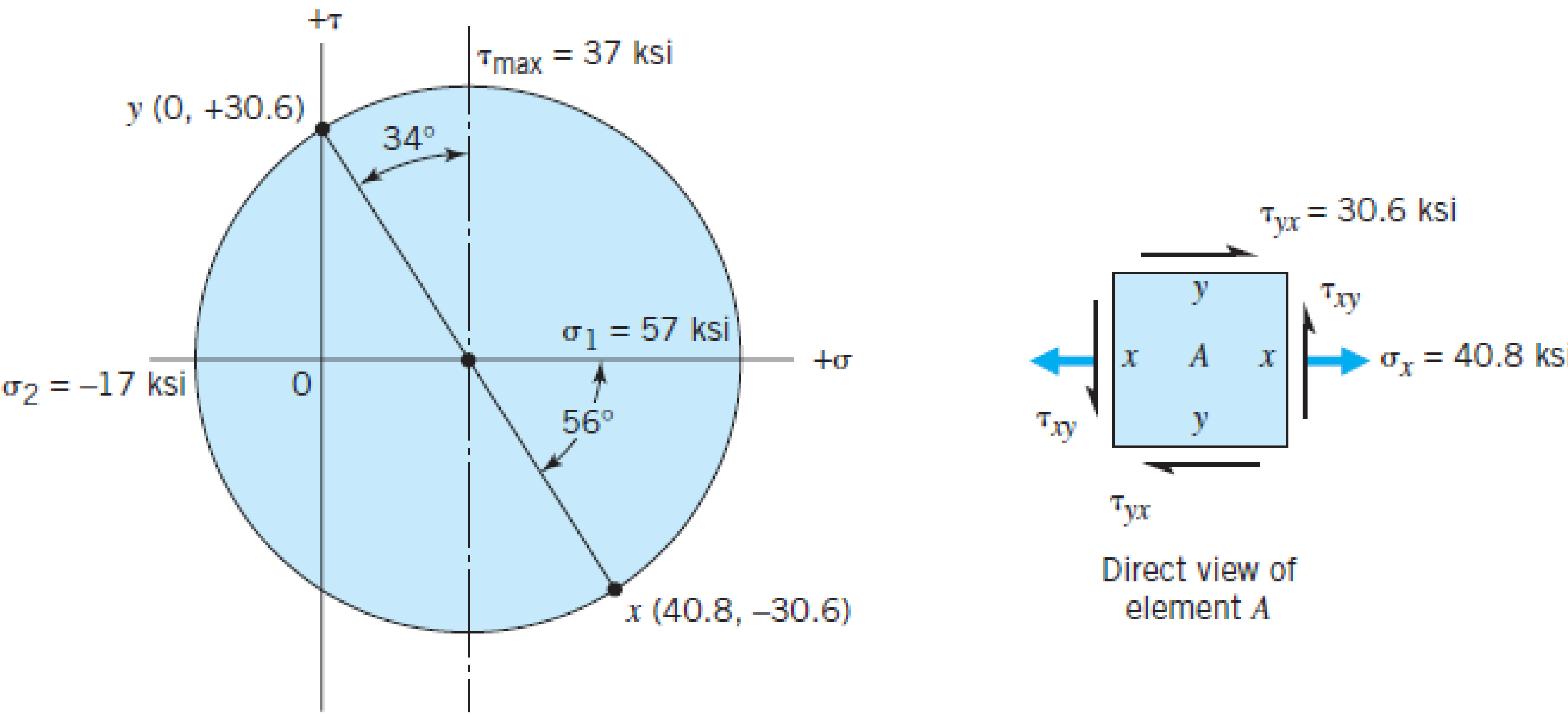



**FIGURE 4.27**

Mohr circle representation at point *A* of Figure 4.25.

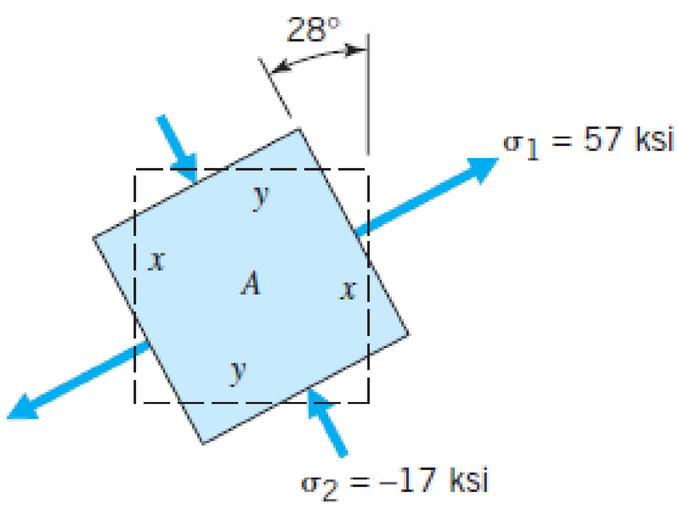



**FIGURE 4.28**
Principal element at $A$ (direct view) shown in relation to $x$ and $y$ faces.

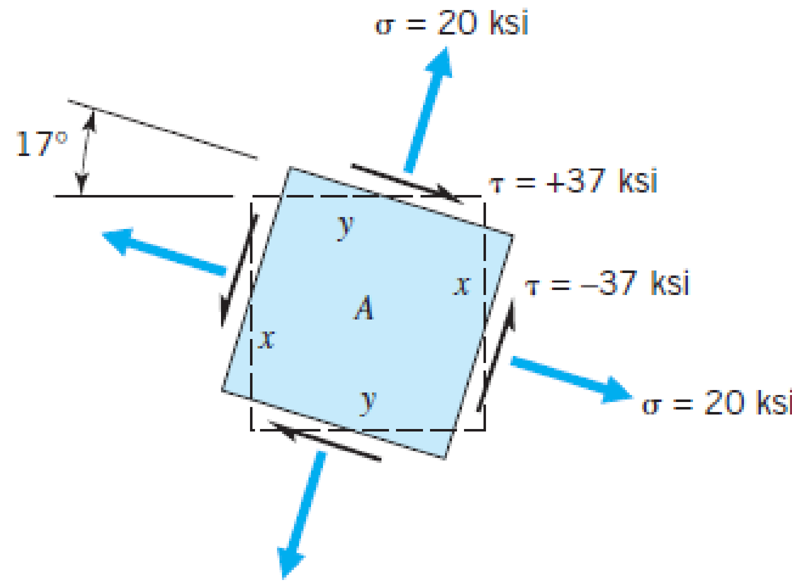



**FIGURE 4.29**
Maximum shear element at $A$ (direct view) shown in relation to $x$ and $y$ faces.

6. Figure 4.28 shows the magnitude and orientation of the highest normal stresses. It may also be of interest to represent similarly the highest shear stresses. This is done in Figure 4.29. Observe again the rules of
   a. rotating in the *same direction* on the element and the circle, and
   b. using angles on the circle that are *twice* those on the element.

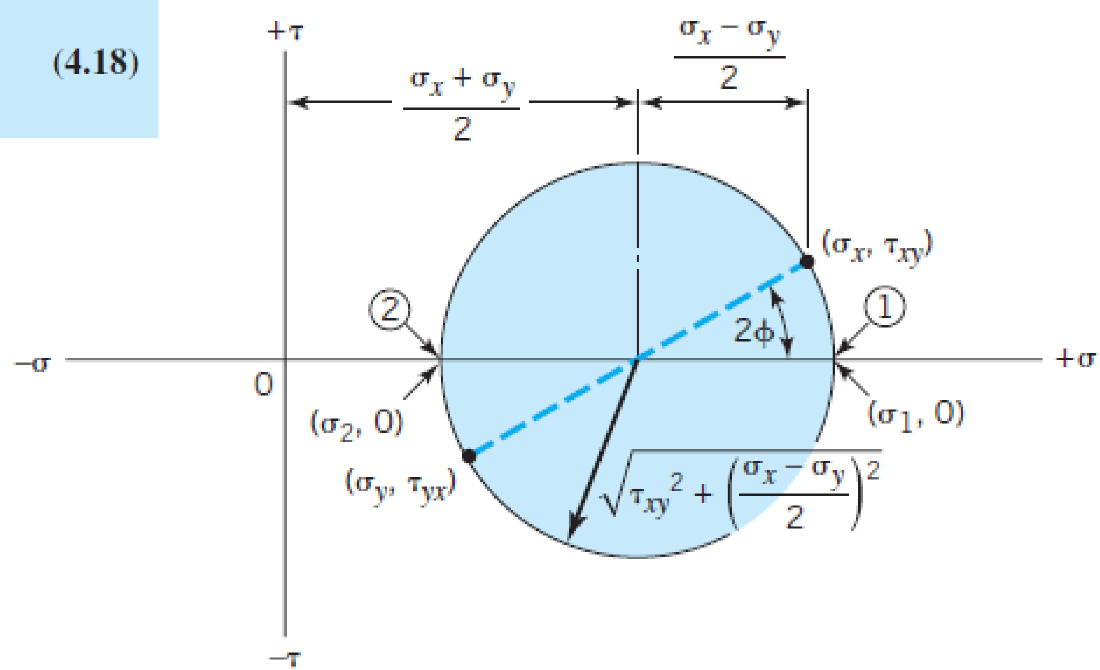
Comment: In support of neglecting the transverse shear stress in step 1, it is of interest to note that its maximum value at the neutral bending axis of the 1-in.-diameter shaft is $4V/3A = (4)(2000 \text{ lb})/[(3)(\pi)(1 \text{ in.})^2/4] = 3.4$ ksi

# 4.10 Stress Equations Related to Mohr's Circle

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2} \qquad (4.16)$$

$$2\phi = \tan^{-1} \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad (4.17)$$

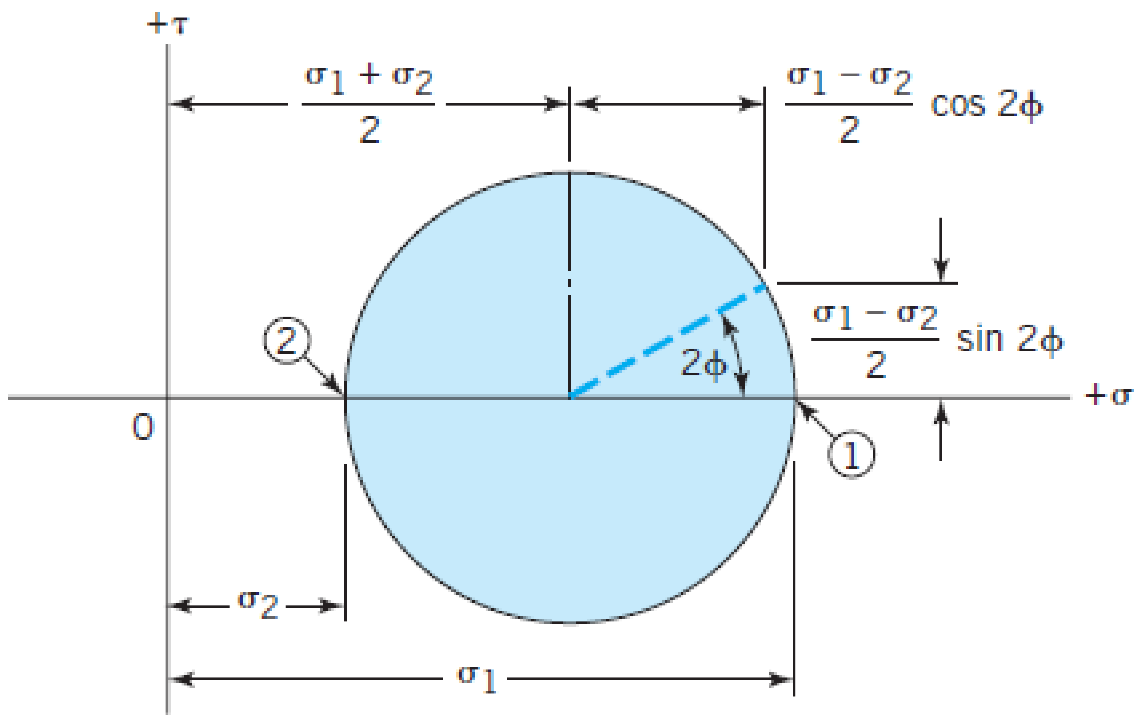
$$\tau_{max} = \pm \sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2} \qquad (4.18)$$



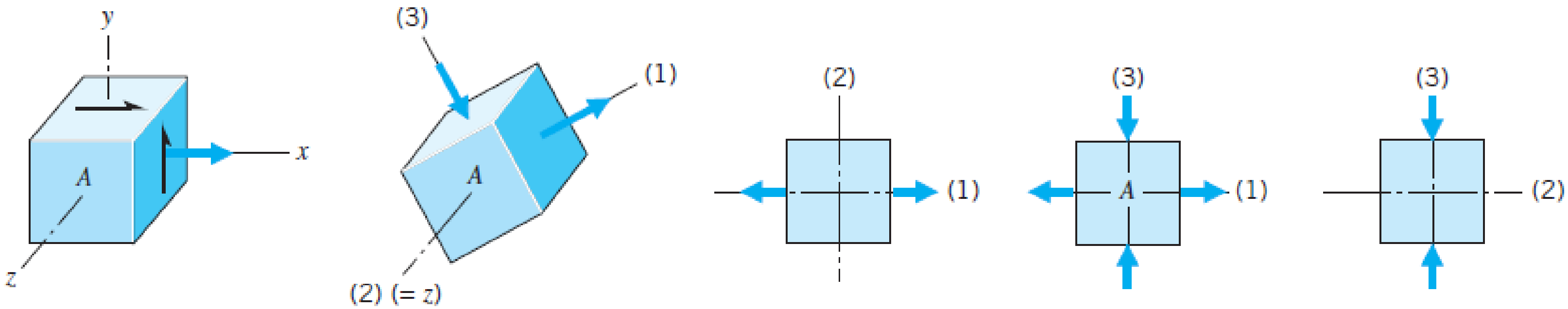
When the principal stresses are known and it is desired to determine the stresses acting on a plane oriented at any angle $\phi$ from the #1 principal plane, the equations are

$$\sigma_\phi = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\phi \quad (4.19)$$

$$\tau_\phi = \frac{\sigma_1 - \sigma_2}{2} \sin 2\phi \quad (4.20)$$



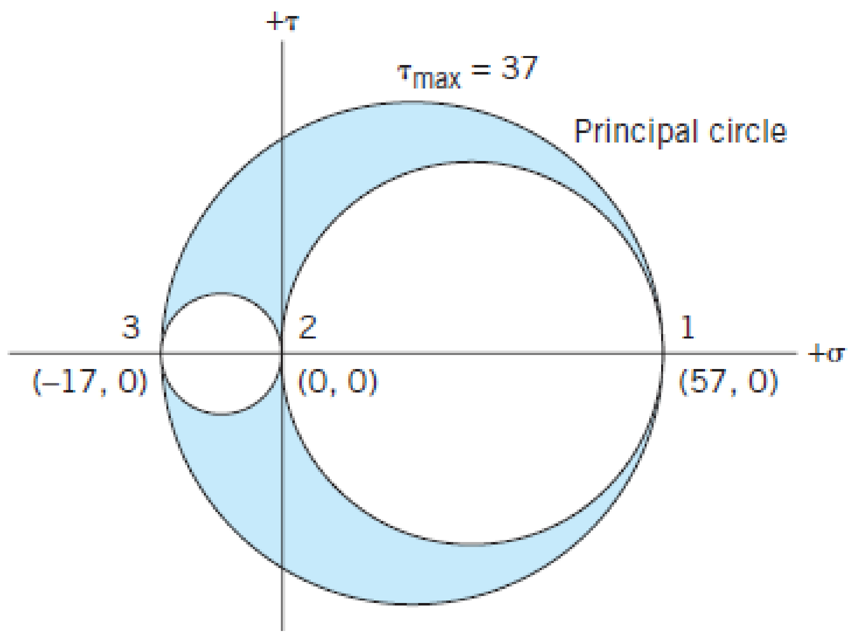
# 4.11 Three-Dimensional Stresses




(*a*) Original element (*b*) Principal element (*c*) 1–2 plane (*d*) 1–3 plane (*e*) 2–3 plane

**FIGURE 4.32**
Elements representing the state of stress at point A.

- Since the largest of the three Mohr circles always represents the maximum shear stress as well as the two extreme values of normal stress, Mohr called this the principal circle.



# 4.11 Three-Dimensional Stresses

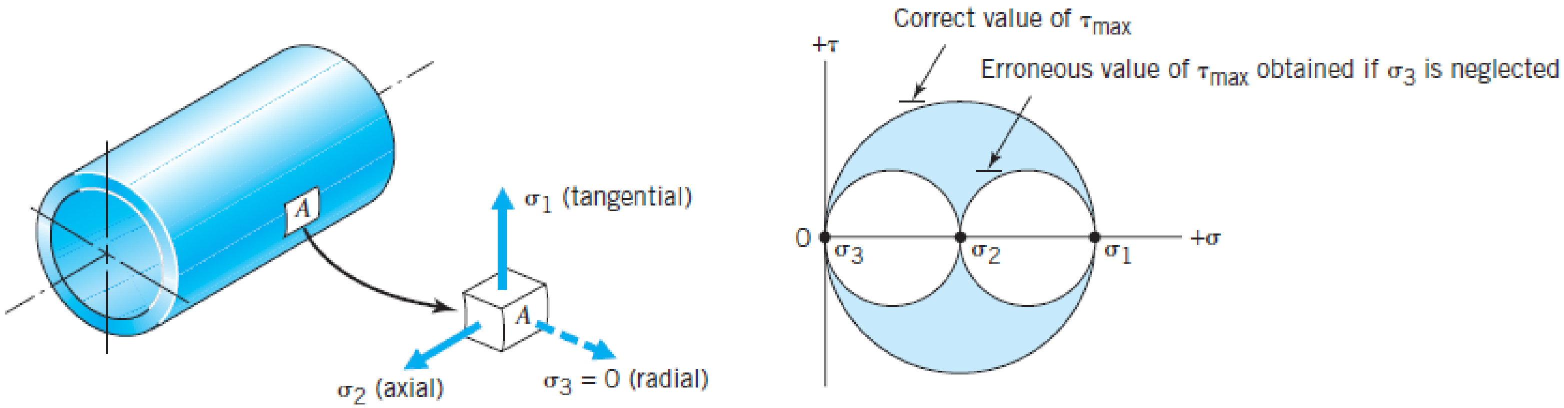



**FIGURE 4.33b**

Example of biaxial stress where correct determination of $\tau_{max}$ requires taking $\sigma_3$ into consideration. Internally pressurized cylinder illustrates biaxial stress states where correct determination of $\tau_{max}$ requires taking $\sigma_3$ into account. Note that (1) for an element on the inside surface, $\sigma_3$ is negative and numerically equal to the internal fluid pressure and (2) for thin-wall cylinders $\sigma_2 \approx \sigma_1/2$.

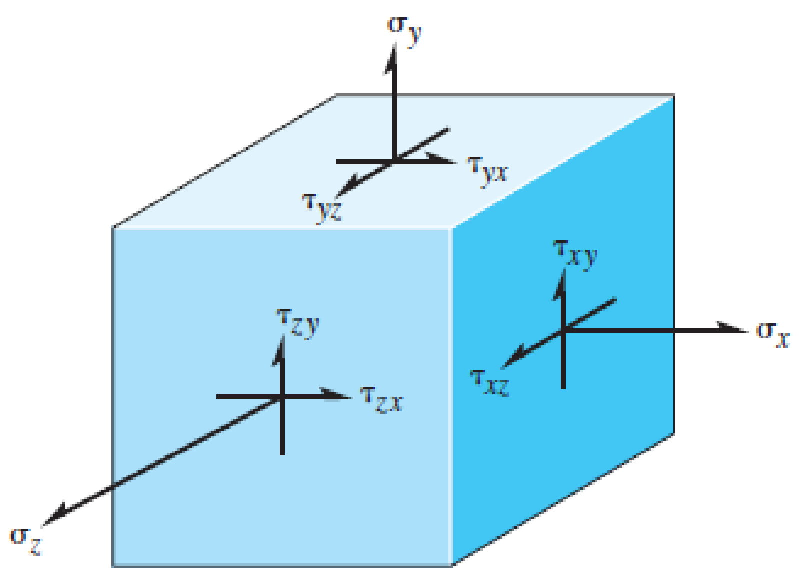
- A common example in which the maximum shear stress would be missed if we failed to include the zero principal stress in the Mohr plot is the outer surface of a pressurized cylinder.
- Here, the axial and tangential stresses are tensile principal stresses, and the unloaded outer surface ensures that the third principal stress is zero.
- Figure illustrates both the correct value of maximum shear stress and the incorrect value obtained from a simple two-dimensional analysis.
- The same situation exists at the inner surface of the cylinder, except that the third principal stress (which acts on the surface) is not zero but a negative value numerically equal to the internal fluid pressure.

## 4.11 Three-Dimensional Stresses

**Known:** A member has a location of critical three-dimensional stress.

**Find:** Determine the principal normal stresses, the maximum shear stress, and draw the three Mohr circles.

**Schematic and Given Data:**




$\sigma_x = 60{,}000$  $\tau_{xy} = 10{,}000$

$\sigma_y = 40{,}000$  $\tau_{yz} = 20{,}000$

$\sigma_z = -20{,}000$  $\tau_{zx} = -15{,}000$ psi

**FIGURE 4.34a**
Element at critical point showing state of stress.

## Assumptions:

1. The stress is completely defined by the normal and shear stresses given.
2. The member behaves as a continuum.

## Analysis:

1. The three principal stresses are found by finding the roots of the *characteristic equation*:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \tag{a}$$

where the *first, second, and third stress invariants*, $I_1$, $I_2$, and $I_3$ are given as

$$I_1 = \sigma_x + \sigma_y + \sigma_z \tag{b}$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \tag{c}$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 \tag{d}$$

2. The characteristic equation is solved for the *principal normal stresses.*

$$\sigma_1, \sigma_2 \text{ and } \sigma_3, \text{ where } \sigma_1 > \sigma_2 > \sigma_3.$$

3. The principal shear stresses are then computed as $\tau_{13}$, $\tau_{21}$, and $\tau_{12}$ where

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2}$$

$$\tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2}$$

$$\tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2}$$

**4.** We start by computing the first, second, and third stress invariants:

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 60{,}000 + 40{,}000 - 20{,}000 = 80{,}000$$

$$I_2 = \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$= (60{,}000)(40{,}000) + (40{,}000)(-20{,}000) + (60{,}000)(-20{,}000)$$

$$= -(-10{,}000)^2 - (2{,}000)^2 - (-15{,}000)^2 = -3.25E^8$$

$$I_3 = \sigma_x \sigma_y \ \sigma_z + 2\,\tau_{xy}\,\tau_{yz}\,\tau_{zx} - \sigma_z \ \tau_{xz}^2 - \sigma_y \ \tau_{zx}^2 - \sigma_z \ \tau_{xy}^2$$

$$= (60{,}000)(40{,}000)\ (-20{,}000) + 2(-10{,}000)(20{,}000)(-15{,}000)$$

$$- 60{,}000(20{,}000)^2 - (40{,}000)(-15{,}000)^2$$

$$- (-20{,}000)(-10{,}000)^2 = -7.3E^{13}$$

**5.** Next we substitute values for the stress invariants into the characteristic equation and solve for the principal normal stresses:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma + I_3 = 0$$

$$\sigma^3 - 80{,}000\sigma^2 - 3.25E^8\sigma + 7.3E^{13} = 0$$

$$\sigma_1 = 69{,}600;\ \sigma_2 = 38{,}001;\ \sigma_3 = -27{,}601 \text{ psi}$$

6. The principal shear stresses can then be computed as $\tau_{13}$, $\tau_{21}$ and $\tau_{32}$, using

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2} = \frac{|69{,}600-(-27{,}601)|}{2} = 48.600$$

$$\tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2} = \frac{|38{,}001-69{,}600|}{2} = 15{,}799$$

$$\tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2} = \frac{|-27{,}601-(38{,}001)|}{2} = 3{,}280 \text{ psi}$$

Comment: The maximum shear stress, since $\sigma_1 > \sigma_2 > \sigma_3$, is $\tau_{max} = \tau_{13}$. A Mohr's three-circle diagram is shown below.

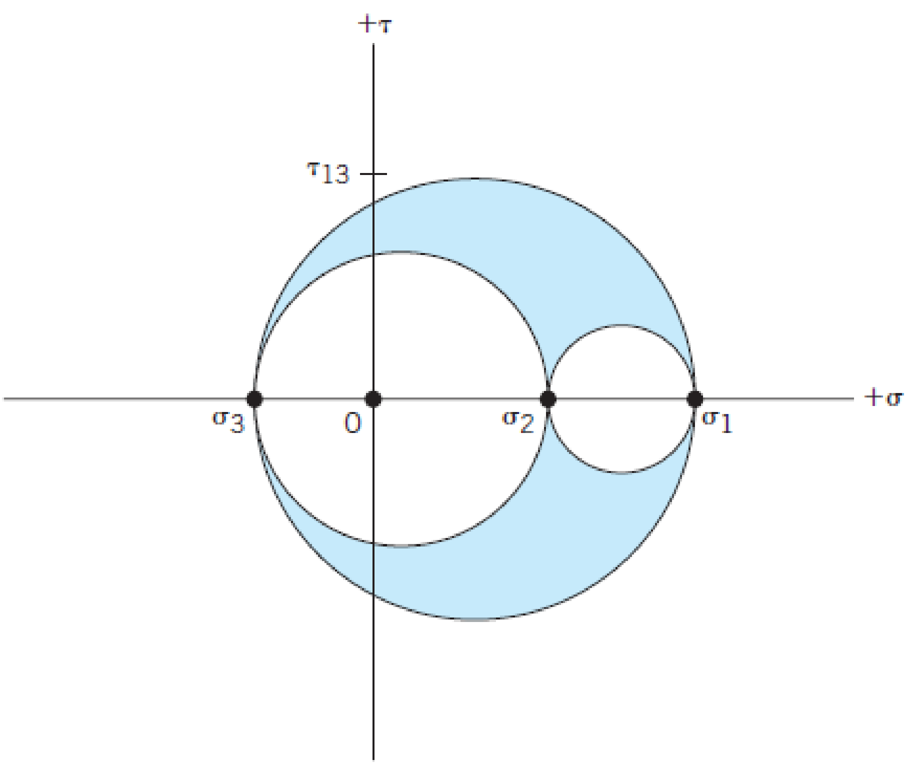
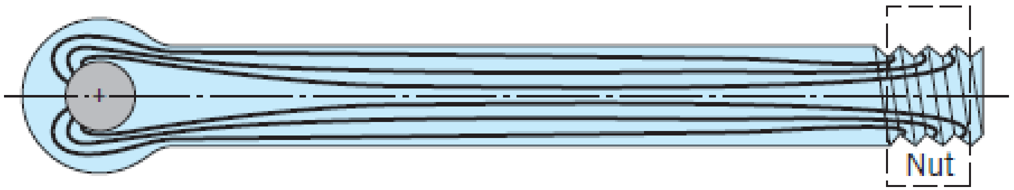



**FIGURE 4.34*b***
Sample Problem 4.4. Mohr circle repesentation of the stress state at a critical point for Figure 4.34a

# 4.12 Stress Concentration Factors, $K_t$




(e)

- Figure indicates lines of force flow through a tensile link.
- Uniform distribution of these lines exist in regions away from the ends.
- At the ends, the force flow lines indicate stress concentration near outer surface.
- We need to evaluate the stress concentration associated with various geometric configurations to determine maximum stresses existing in a part
- The first mathematical treatments of stress concentration were published after 1900 with experimental methods for measuring highly localized stresses
- In recent years, FEM studies have also been employed. The results of many of these studies are available in the form of published graphs giving values of the theoretical stress concentration factor, $K_t$ (based on a theoretical elastic, homogeneous, isotropic material), for use in the equations

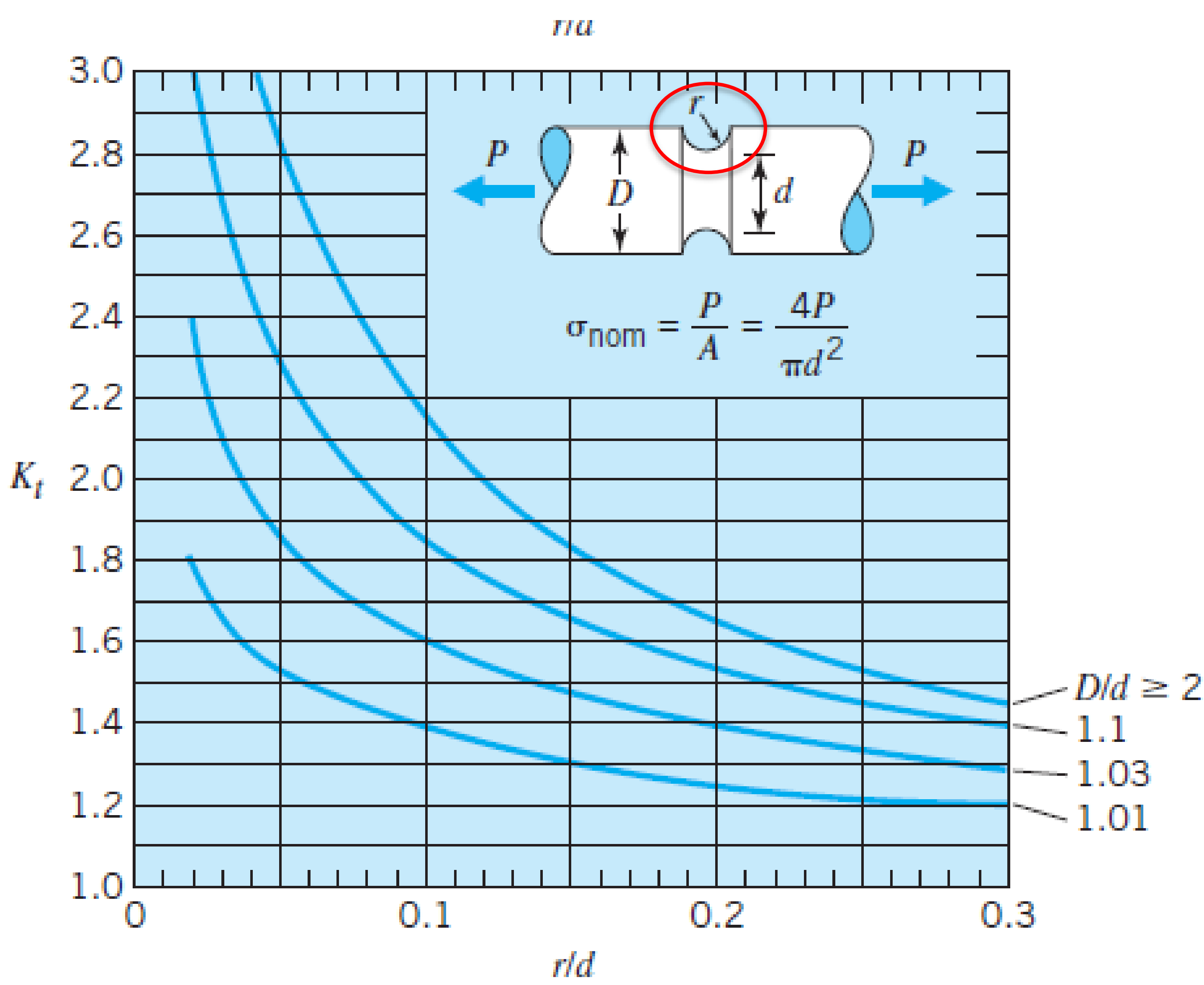
$$\sigma_{max} = K_t \sigma_{nom} \quad \text{and} \quad \tau_{max} = K_t \tau_{nom} \tag{4.21}$$

## 4.12 Stress Concentration Factors, $K_t$

- For eg, the maximum stress for axial loading would be P/A * appropriate $K_t$ .
- Note that the stress concentration graphs are plotted on the basis of dimensionless ratios, indicating that only part shape (not size) is involved. Also note that stress concentration factors are different for axial, bending, and torsional loading.
- In many situations involving notched parts in tension or bending, the notch not only increases the primary stress but also causes one or both of the other principal stresses to take on nonzero values.
- This is referred to as the biaxial or triaxial effect of stress raisers ("stress raiser" is a general term applied to notches, holes, threads, etc.).
- Consider, for example, a soft rubber model of the grooved shaft in tension illustrated in Figure 4.36b. As the tensile load is increased, there will be a tendency for the outer surface to pull into a smooth cylinder.
- This will involve an increase in the diameter and circumference of the section in the plane of the notch. The increased circumference gives rise to a tangential stress, which is a maximum at the surface. The increase in diameter is associated with the creation of radial stresses. (Remember, though, that this radial stress must be zero at the surface because there are no external radial forces acting there.)

FIGURE 4.36

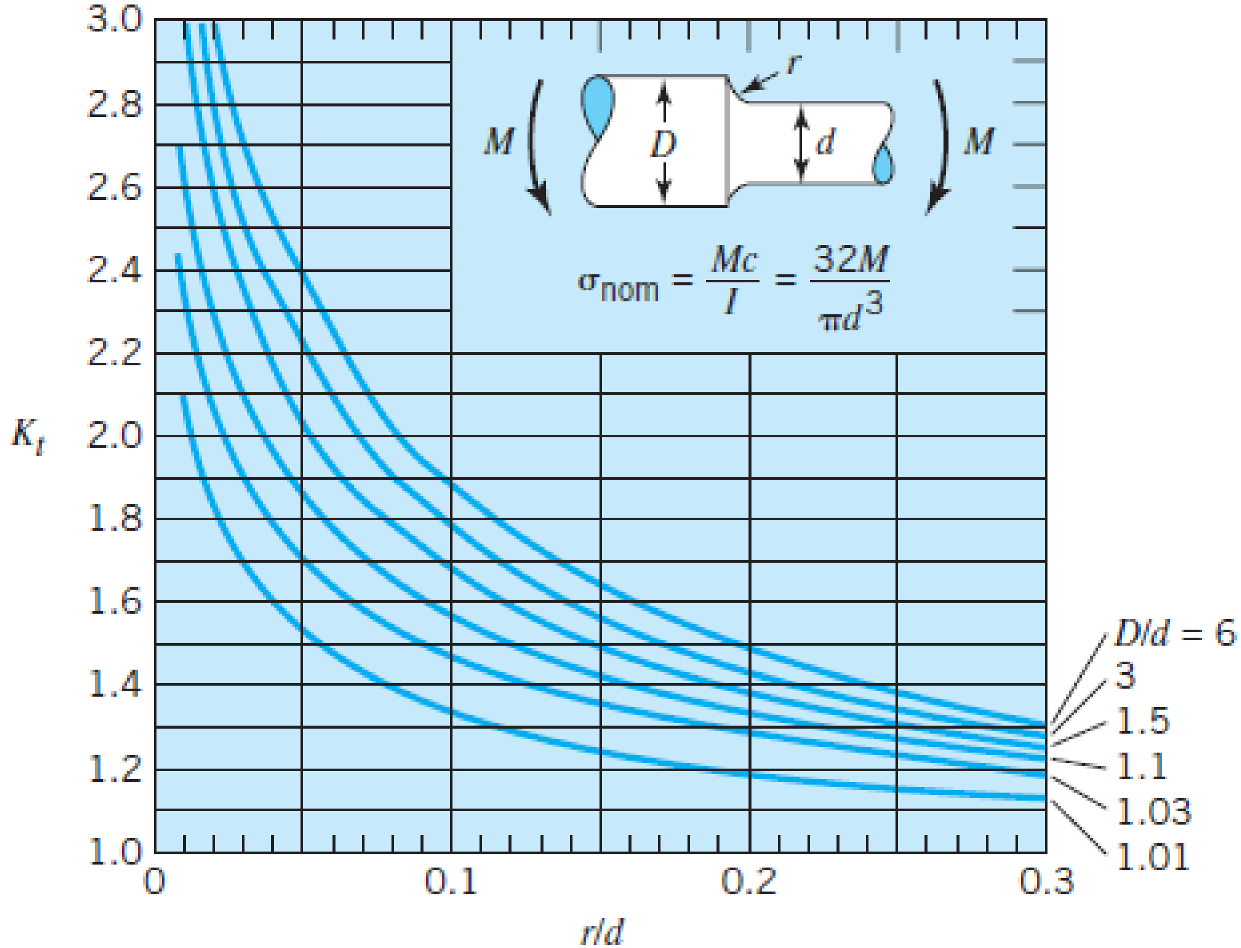
Grooved shaft (*a*) bending; (*b*) axial load; (*c*) torsion [7].




(*b*)

**FIGURE 4.35**

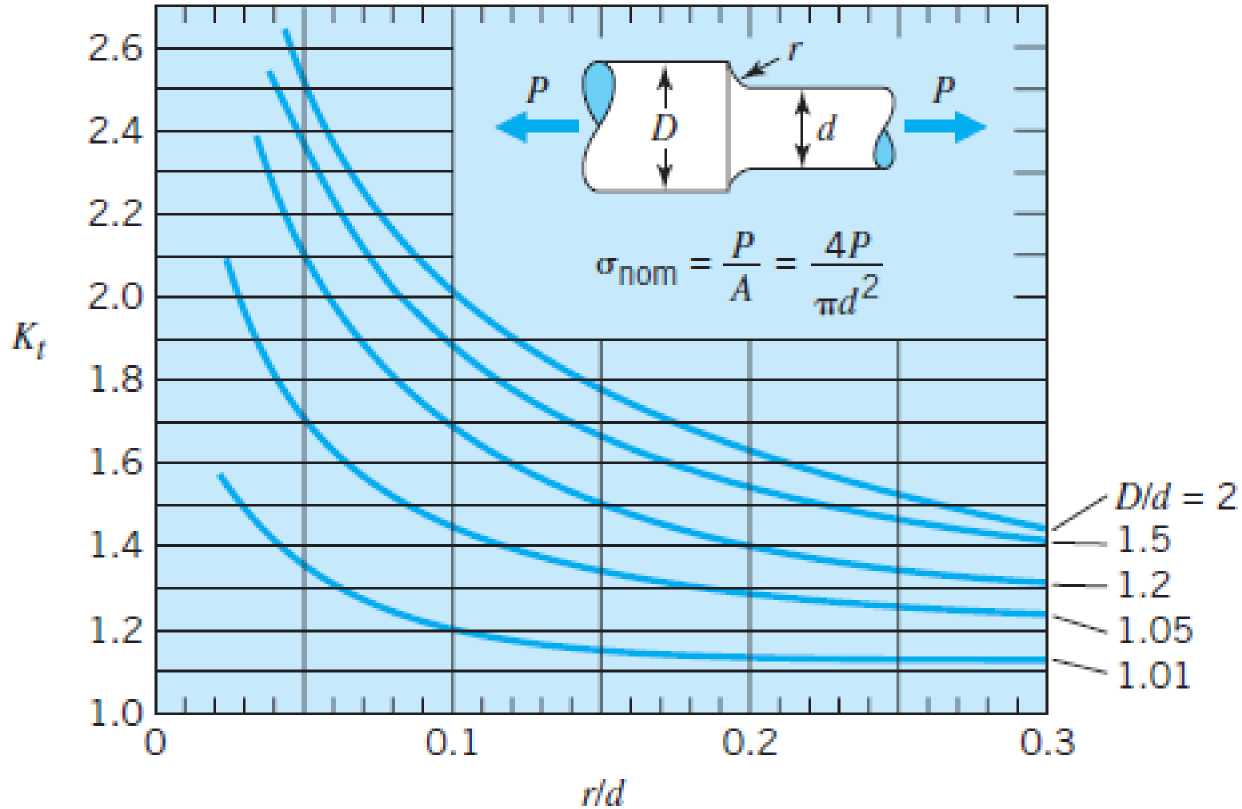
Shaft with fillet (*a*) bending; (*b*) axial load; (*c*) torsion [7].




(*a*)

**FIGURE 4.35**

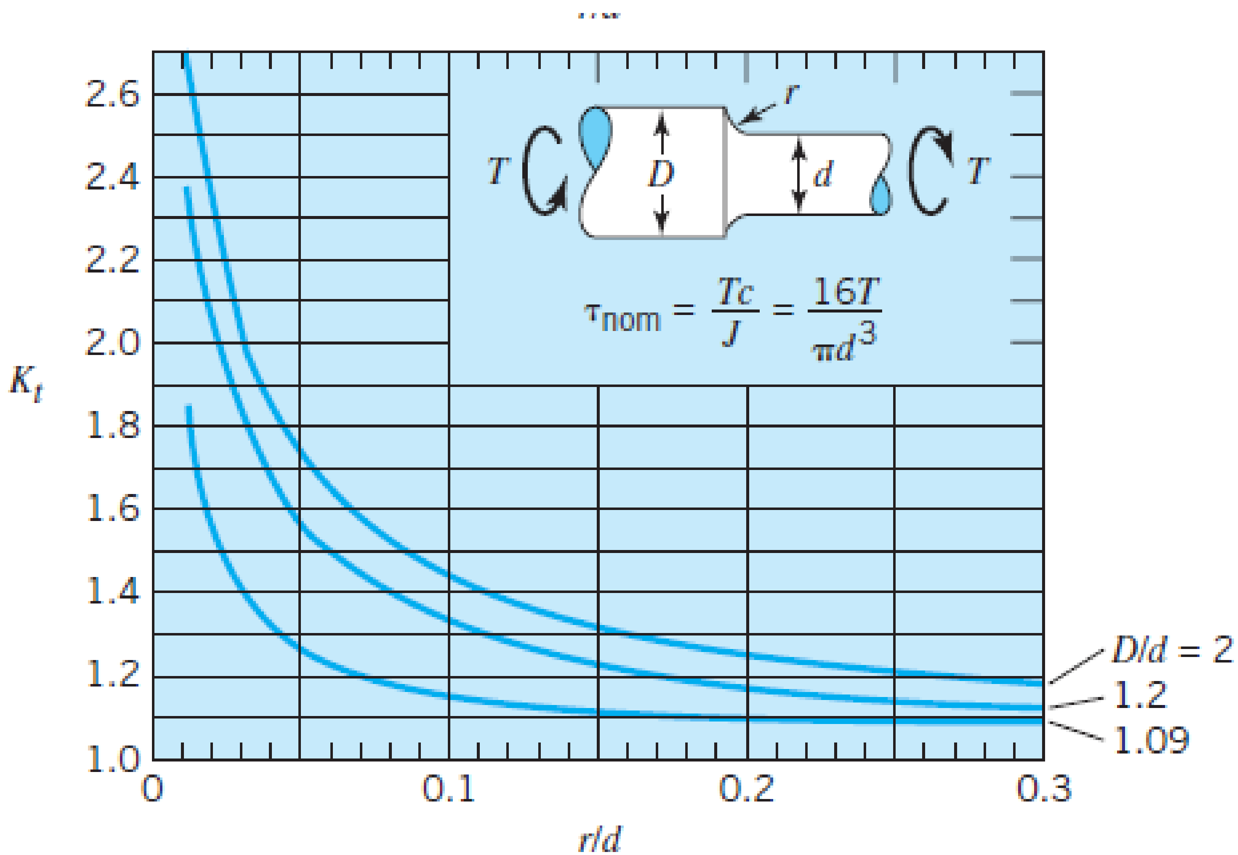
Shaft with fillet (*a*) bending; (*b*) axial load; (*c*) torsion [7].




(*b*)

**FIGURE 4.35**

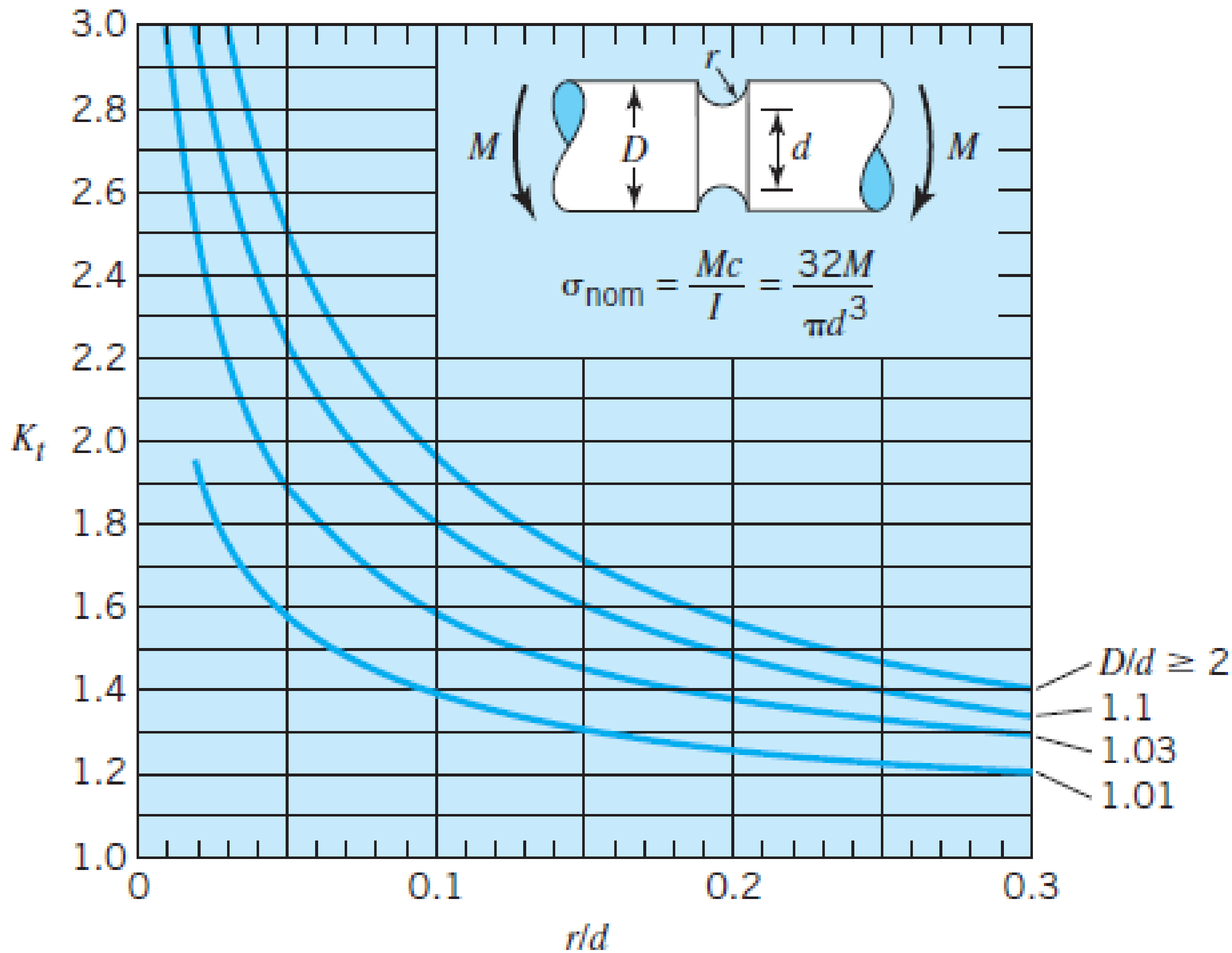
**Shaft with fillet (*a*) bending; (*b*) axial load; (*c*) torsion [7].**




(*c*)

**FIGURE 4.36**

Grooved shaft (*a*) bending; (*b*) axial load; (*c*) torsion [7].




(*a*)

**FIGURE 4.36**

Grooved shaft (*a*) bending; (*b*) axial load; (*c*) torsion [7].

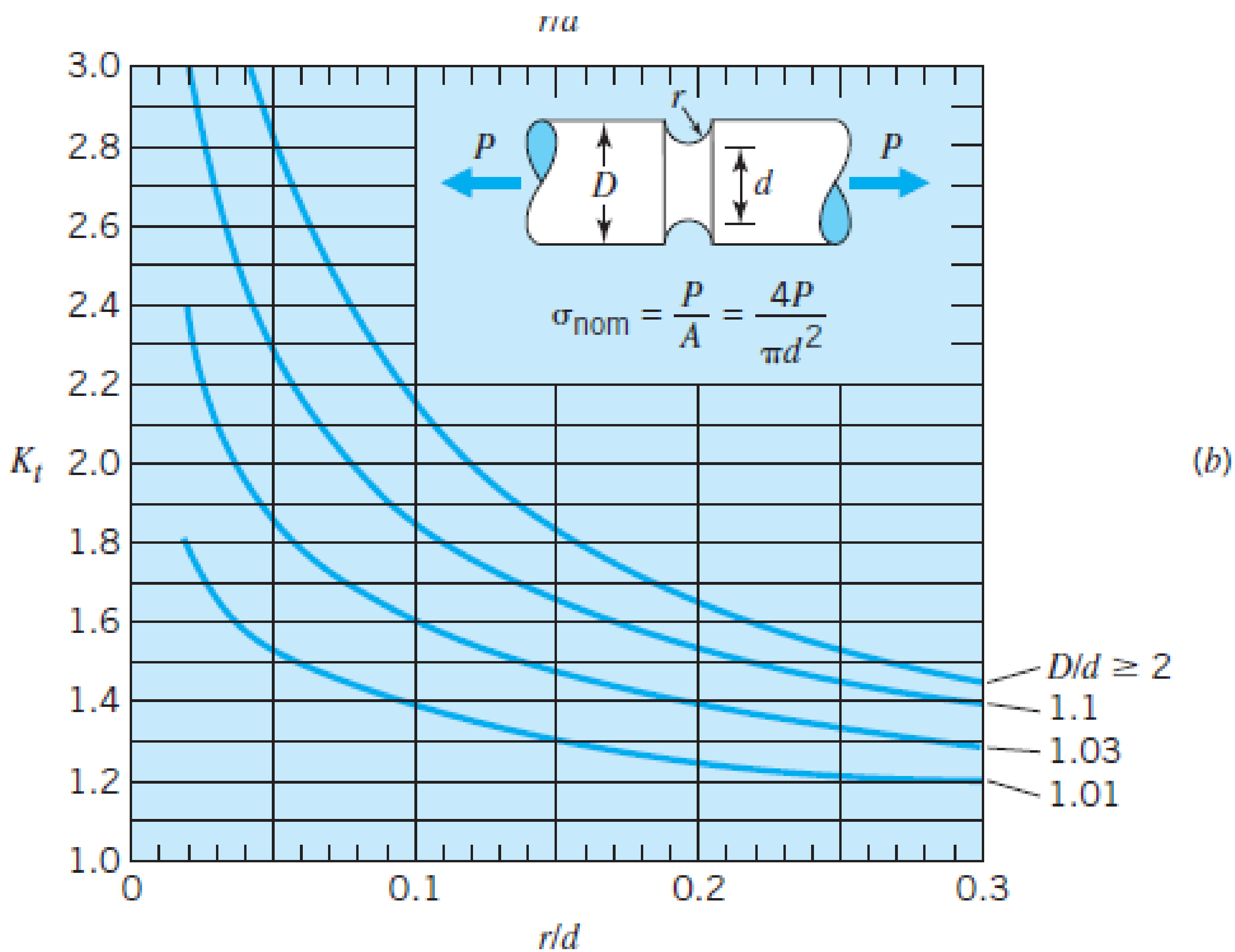
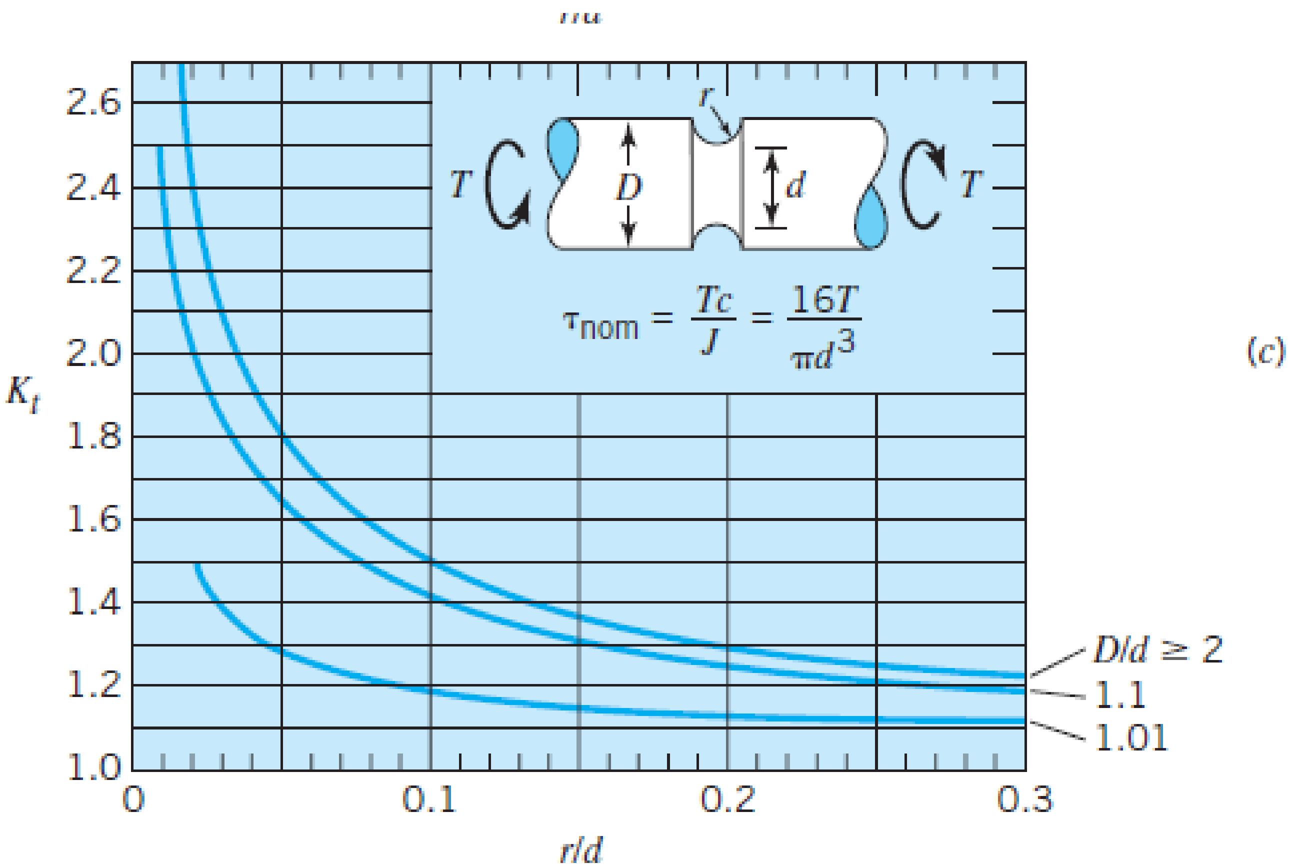
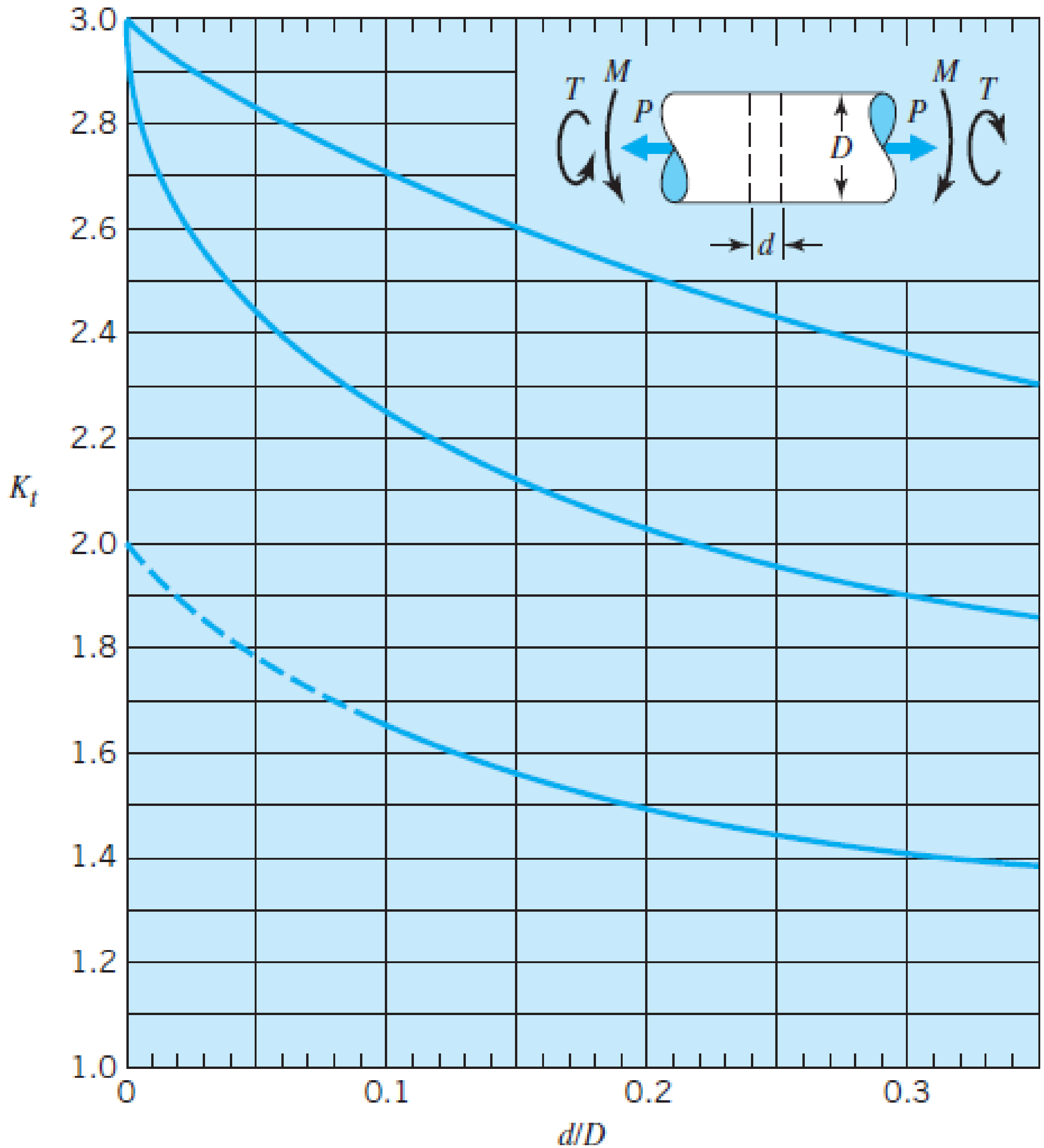


**FIGURE 4.36**
Grooved shaft (*a*) bending; (*b*) axial load; (*c*) torsion [7].






Axial load:

$$\sigma_{\text{nom}} = \frac{P}{A} \approx \frac{P}{(\pi D^2/4) - Dd}$$

Bending (in this plane):

$$\sigma_{\text{nom}} = \frac{Mc}{I} \approx \frac{M}{(\pi D^3/32) - (dD^2/6)}$$

Torsion:

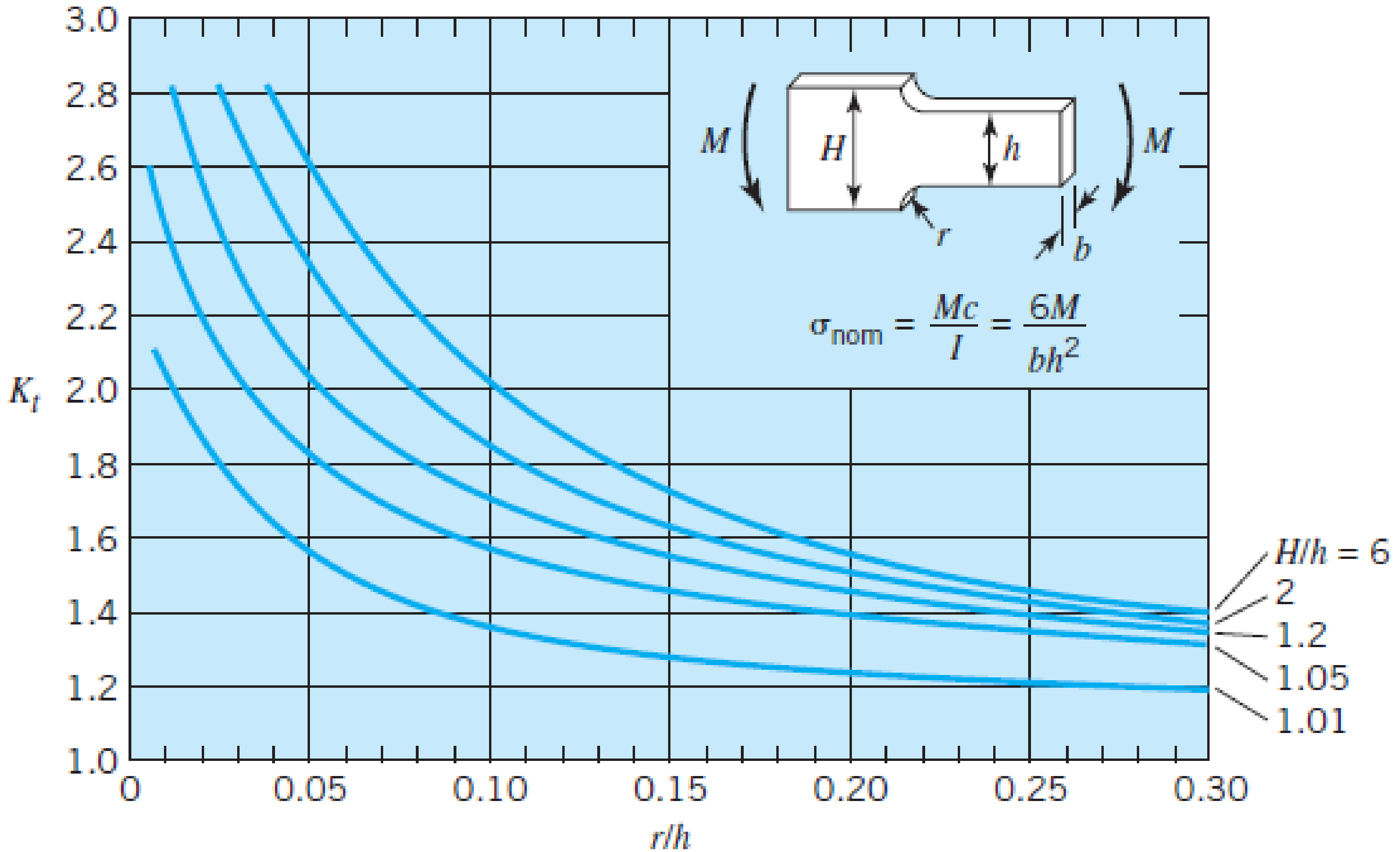
$$\tau_{\text{nom}} = \frac{Tc}{J} \approx \frac{T}{(\pi D^3/16) - (dD^2/6)}$$

**FIGURE 4.37**

Shaft with radius hole [7].

**FIGURE 4.38**

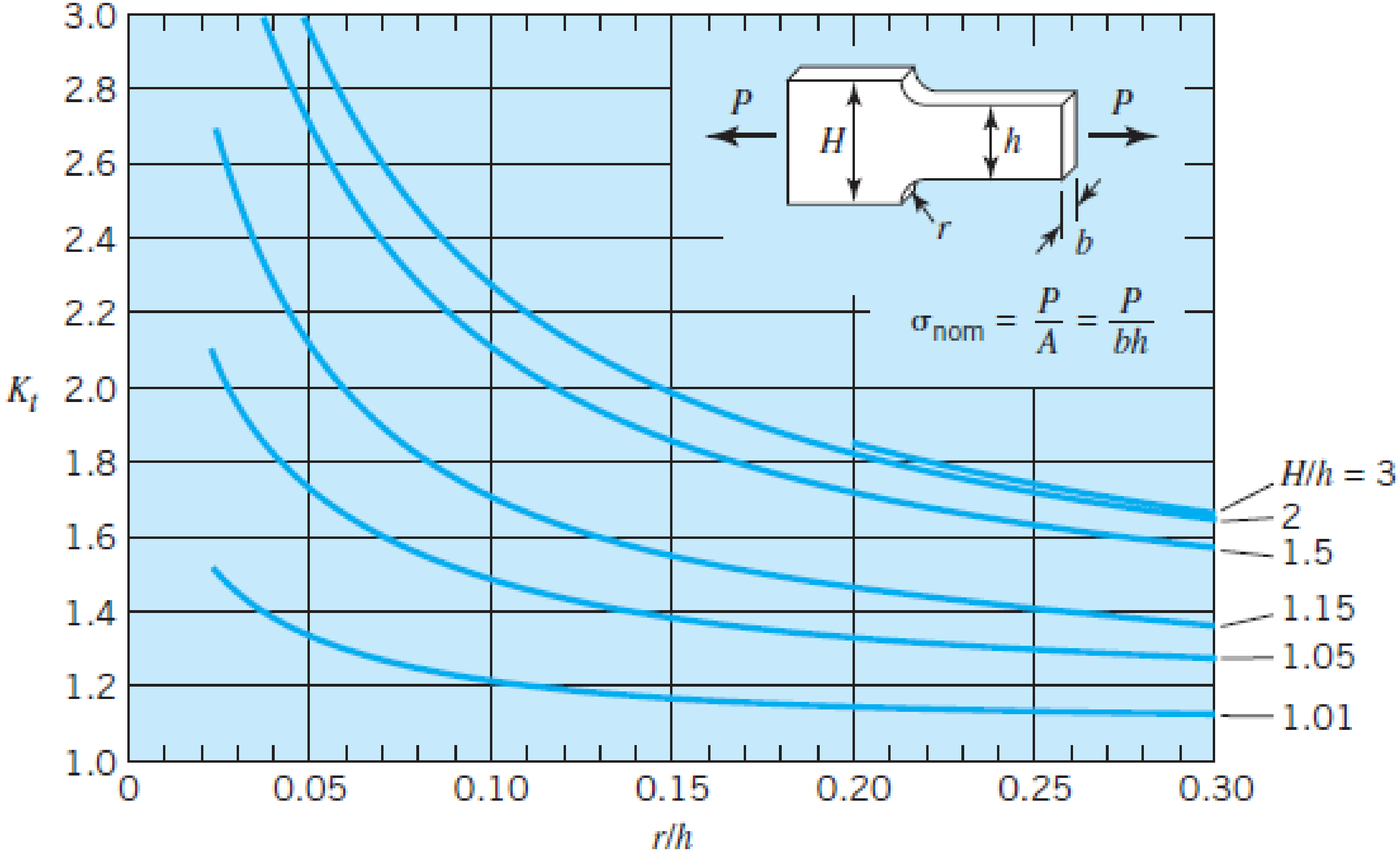
**Bar with shoulder fillet (*a*) bending; (*b*) axial load [7].**




(*a*)

**FIGURE 4.38**

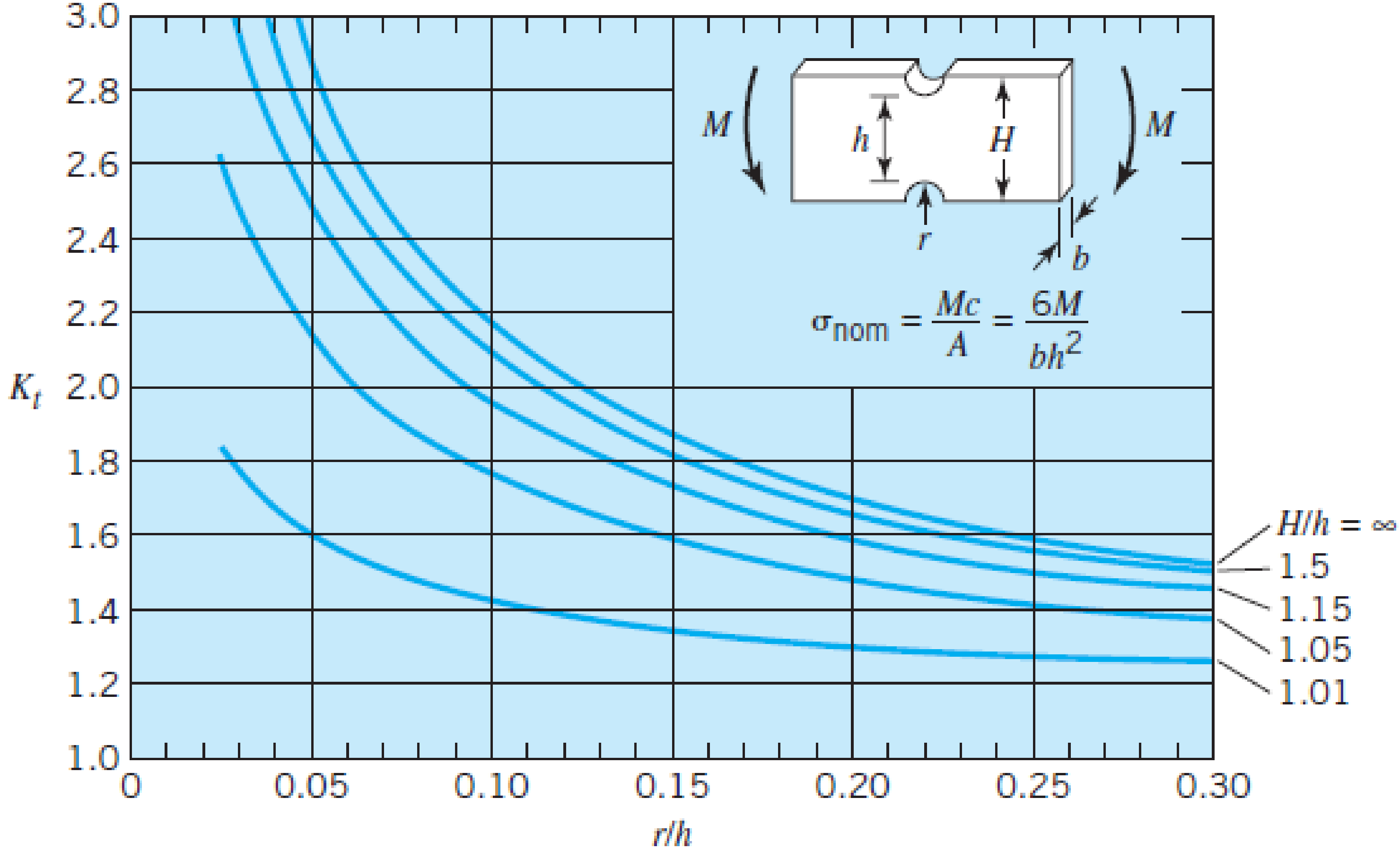
Bar with shoulder fillet ($a$) bending; ($b$) axial load [7].




($b$)

**Figure 4.39**

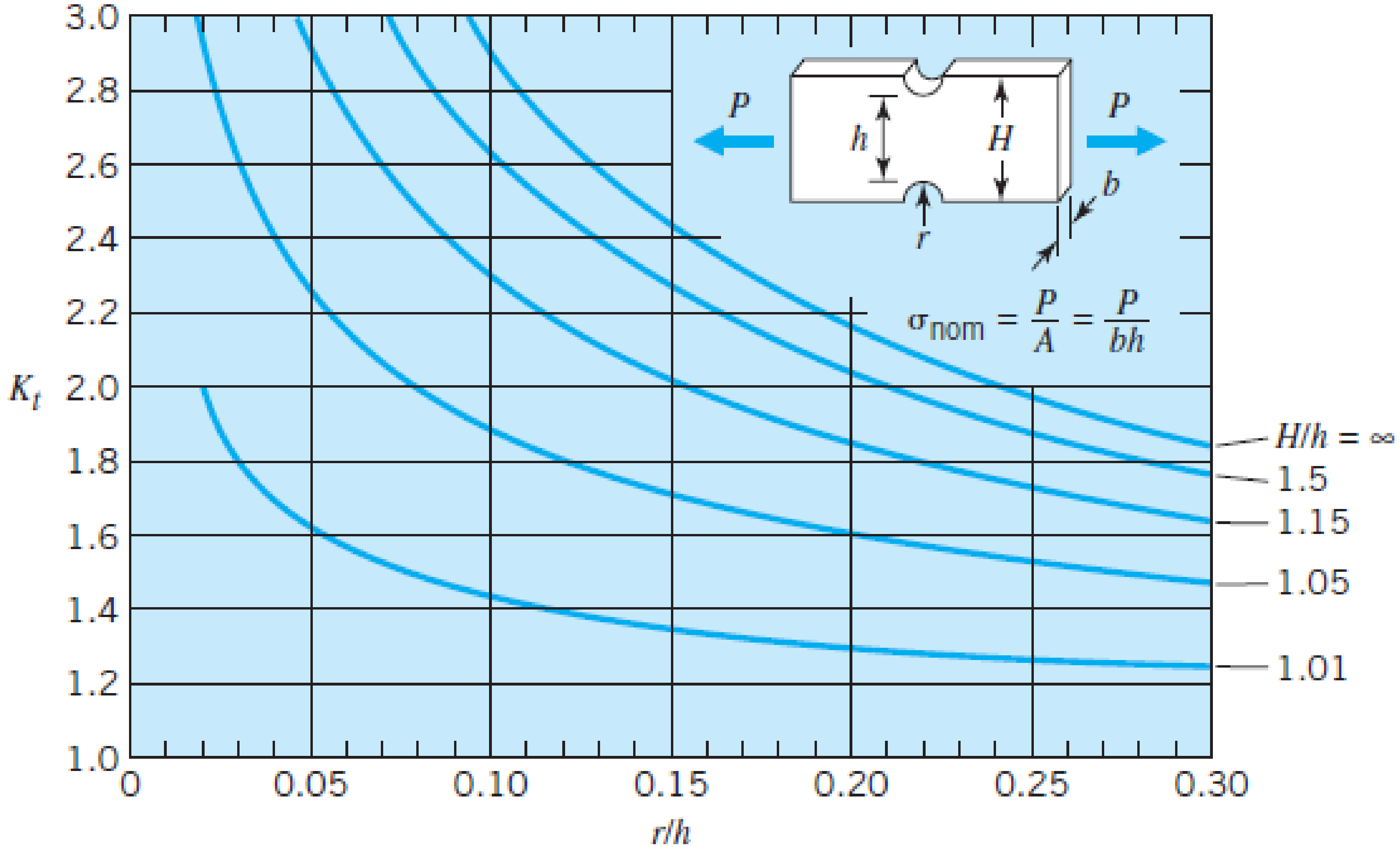
Notched flat bar (*a*) bending; (*b*) tension [7].




(*a*)

**FIGURE 4.39**

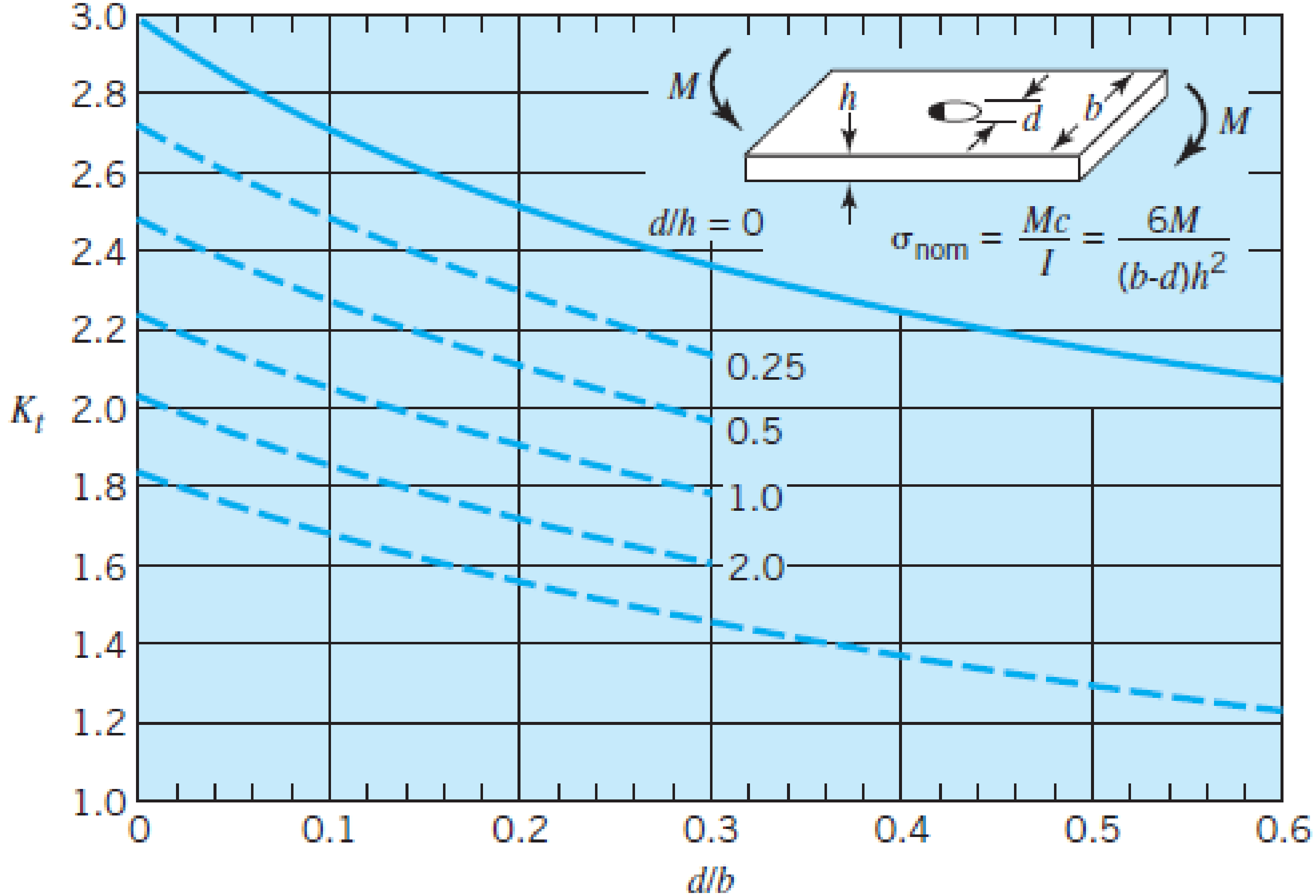
Notched flat bar (*a*) bending; (*b*) tension [7].




(*b*)

**FIGURE 4.40**

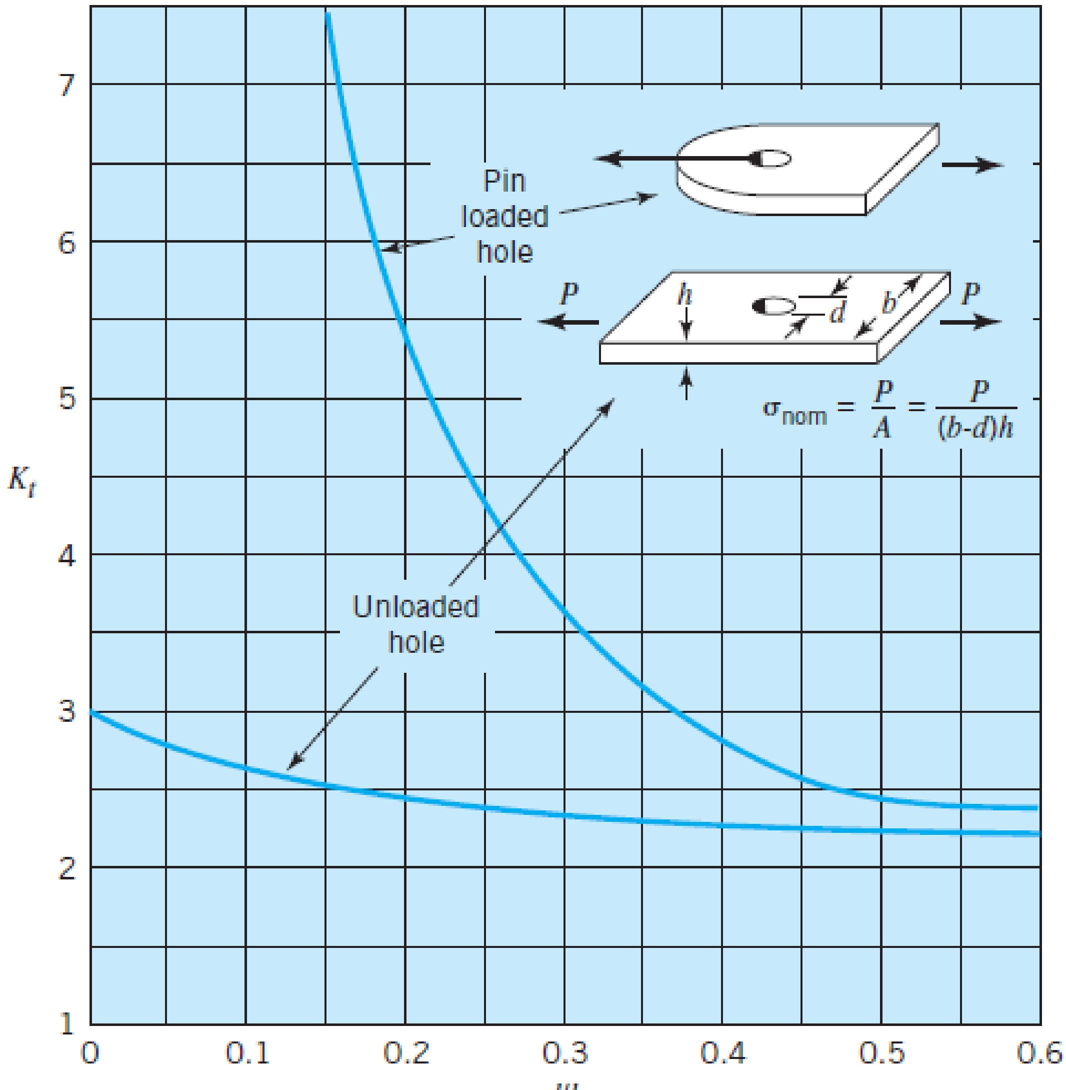
**Plate with central hole (*a*) bending [7]; (*b*) axial hole [10].**




(*a*)

FIGURE 4.40

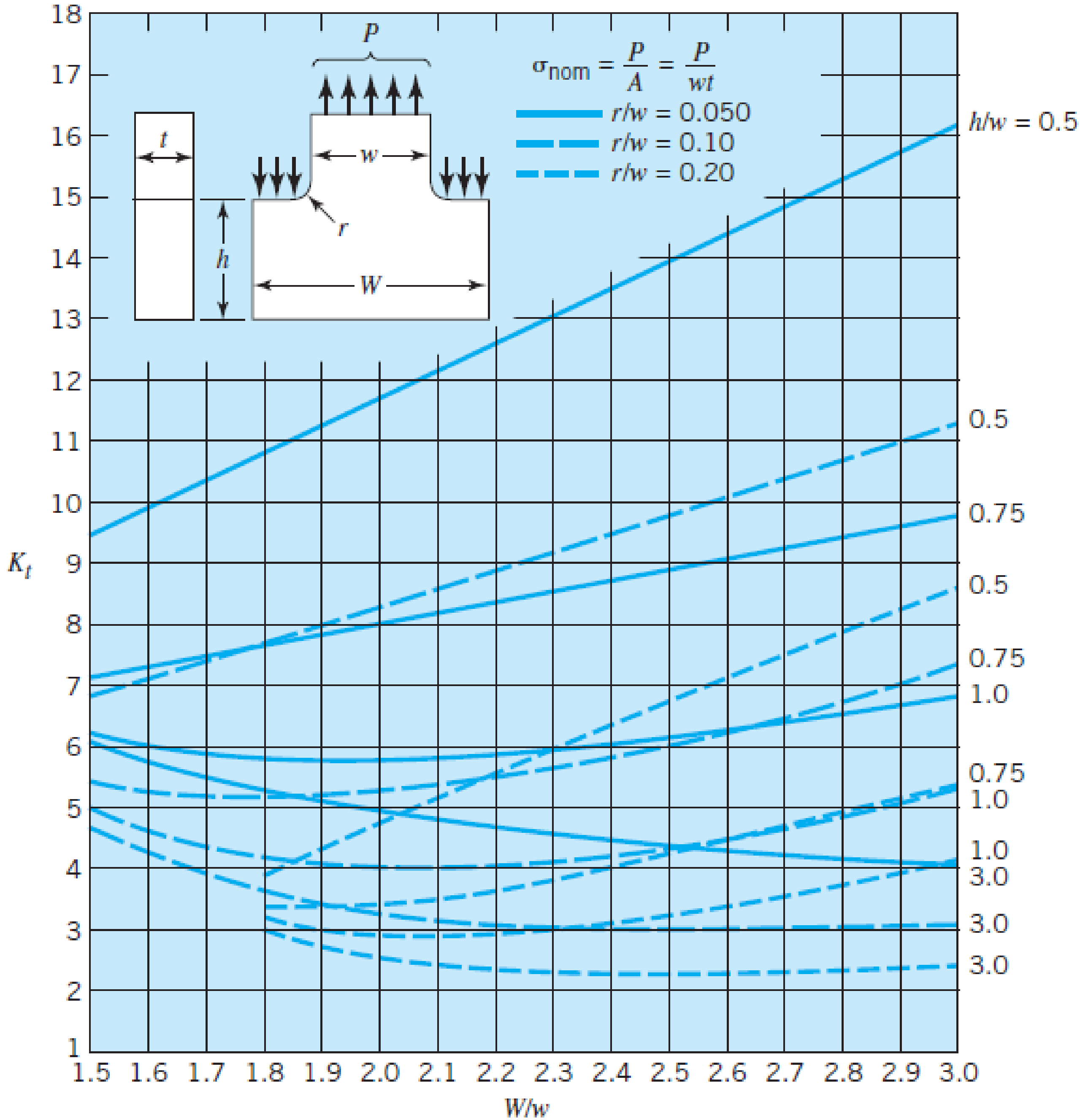
Plate with central hole (*a*) bending [7]; (*b*) axial hole [10].



(*b*)

**FIGURE 4.41**

T-head member with an axial load [7].



# 4.13 Importance of Stress Concentration

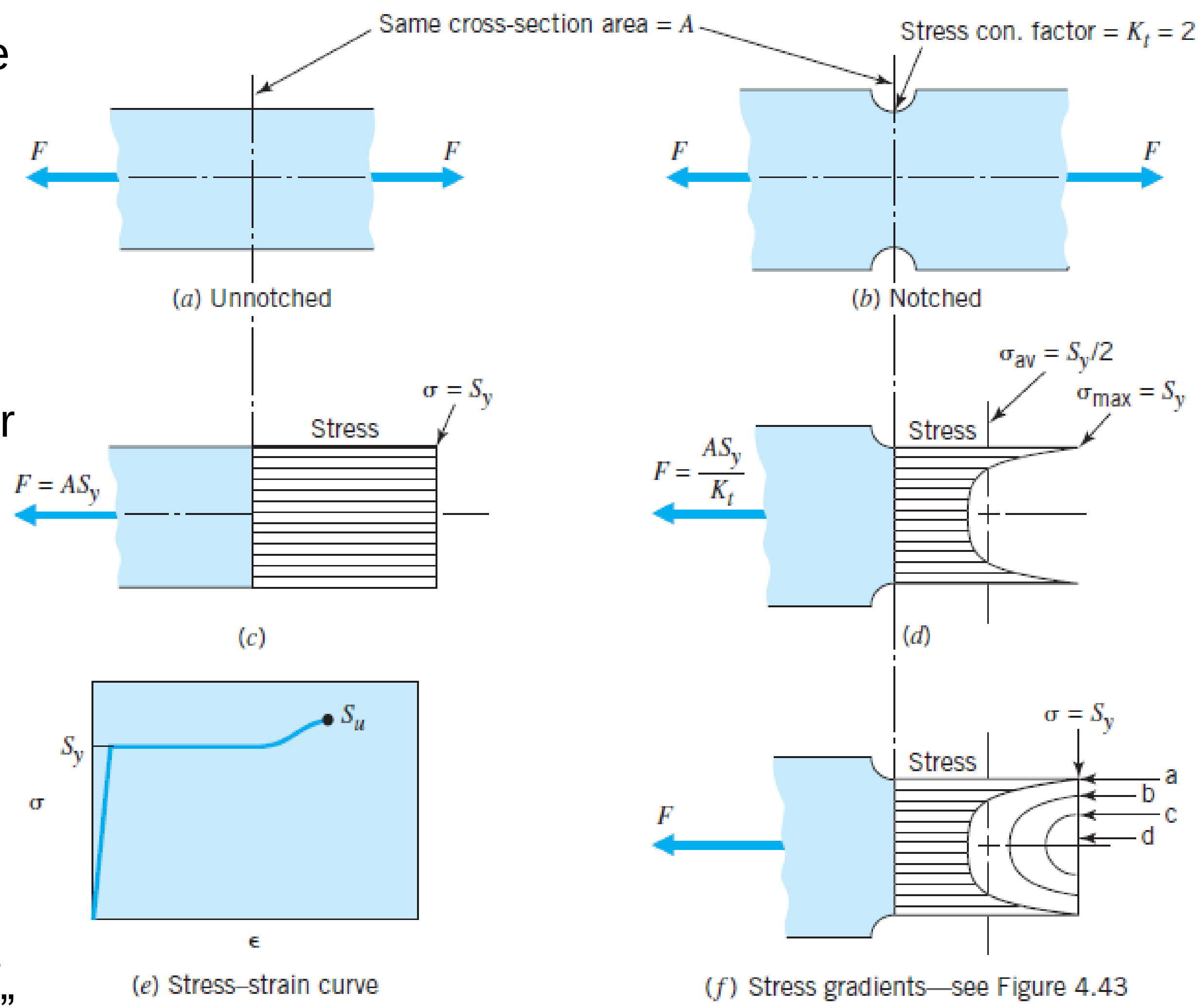
- $K_t$ factors given in the graphs are theoretical (hence, the subscript t) or geometric factors based on a theoretical homogeneous, isotropic, & elastic mat'l
- Real materials have microscopic irregularities causing a certain nonuniformity of microscopic stress distribution, even in notch-free parts.
- Hence, the introduction of a stress raiser may not cause as much additional damage as indicated by the theoretical factor.
- Moreover, real parts—even if free of stress raisers—have surface irregularities (from processing and use) that can be considered as extremely small notches.
- The extent to which we must take stress concentration into account depends on
  - (1) the extent to which the real material deviates from the theoretical and
  - (2) whether the loading is static, or fatigue
- For materials permeated with internal discontinuities, such as gray cast iron, stress raisers usually have little effect, regardless of the nature of loading, as geometric irregularities affect less than the internal irregularities.

# 4.13 Importance of Stress Concentration

- For fatigue & impact loading of engineering materials, $K_t$ must be considered
- For the case of static loading, $K_t$ is important only with unusual materials that are both brittle and relatively homogeneous;
  - When tearing a package wrapped in clear plastic film, a sharp notch in the edge is most helpful!
- or for normally ductile materials that, under special conditions, behave in a brittle manner
- For the usual engineering materials having some ductility (and under conditions such that they behave as ductile), it is customary to ignore $K_t$ for static loads.

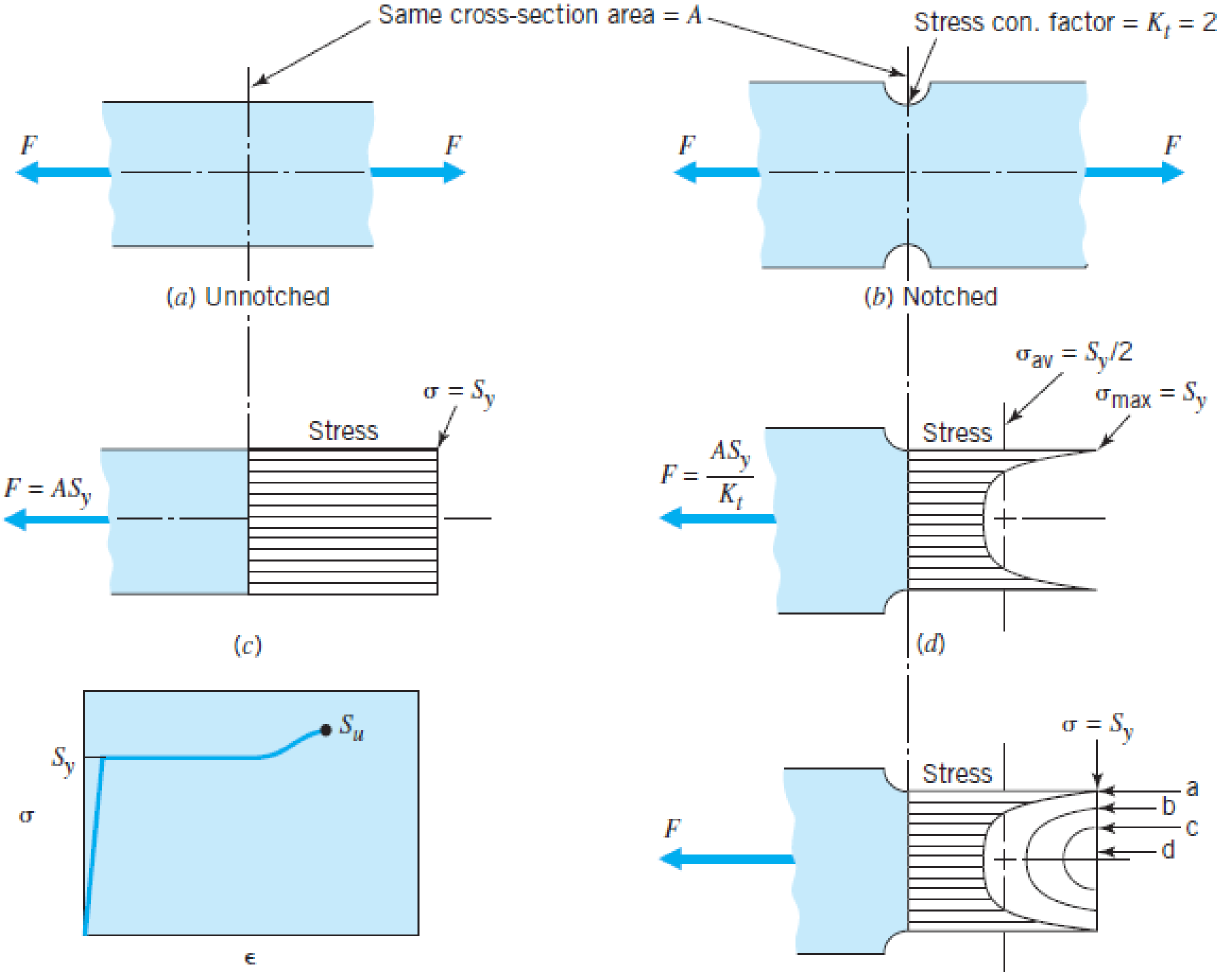
# 4.13 Importance of Stress Concentration

- Figure show two flat tensile bars each having a minimum CSA of A, each a ductile material having the “idealized” stress–strain curve shown in e.
- The load on the unnotched bar can be increased to the product of area times yield strength before failure occurs as shown in c.
- Since the grooved bar in b has a $K_t$ of 2, yielding will begin at only half the load, as shown in d.
- This is repeated as curve “a” of f. As the load is increased, the stress distribution (shown in f) becomes “b,” “c,” and finally “d.”



# 4.13 Importance of Stress Concentration

- These curves reflect a continuous deepening of local yielding, which began at the root; but gross yielding involving the entire CS begins only if “d” is reached.
- Note that the load associated with curve “d” is identical to the unnotched load capacity
- Also note that curve “d” can be achieved without significant stretching of the part.
- The part as a whole cannot be significantly elongated without yielding the entire CS.
- So, for practical purposes, the grooved bar will carry the same static load as the ungrooved bar.



(e) Stress–strain curve

(f) Stress gradients—see Figure 4.43

# 4.14 Residual Stresses Caused by Yielding—Axial Loading

- When a part is yielded nonuniformly throughout a CS, residual stresses remain in this CS after the external load is removed.
- For eg, the 4 levels of loading of the notched tensile bar shown in f.
- This same bar and the 4 levels of loading are represented in the left column
- Note that only slight yielding is involved—not major yielding such as often occurs in processing.
- The middle column shows the change in stress when the load is removed.

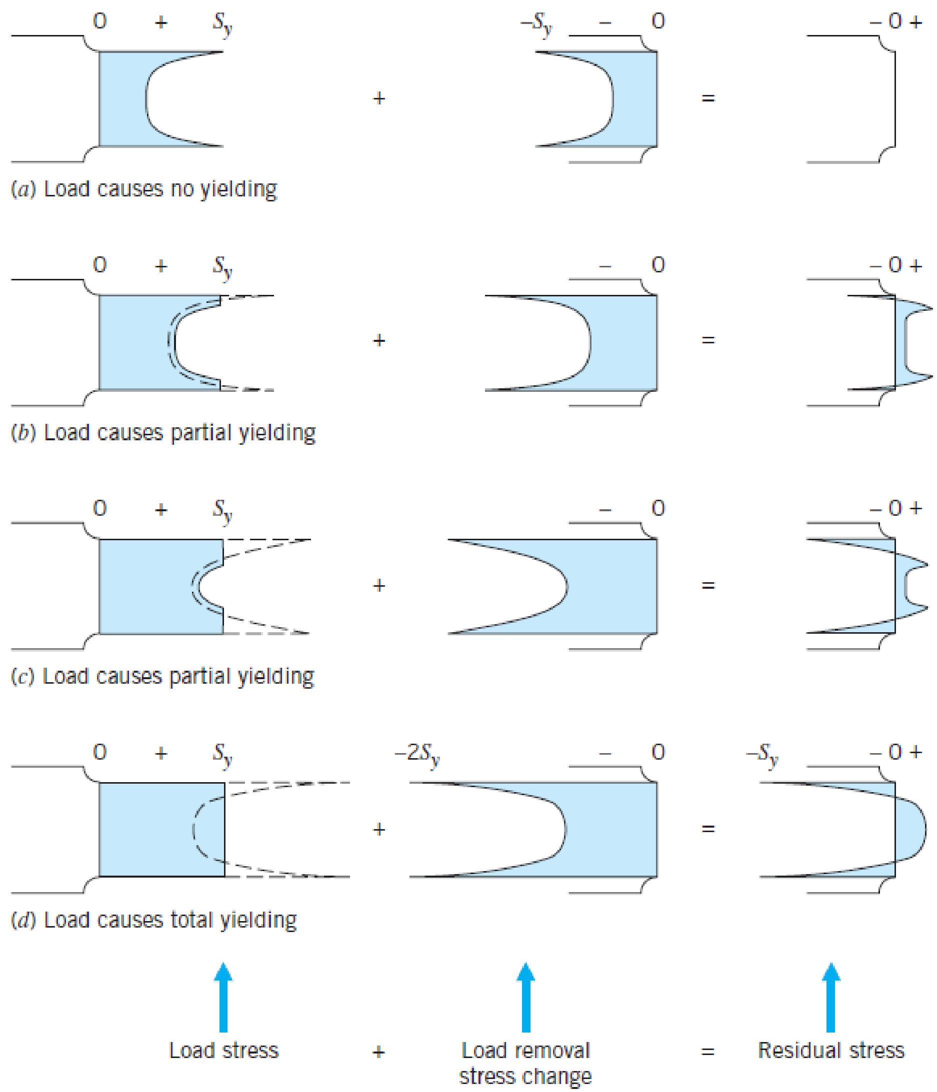



**FIGURE 4.43**
Residual stresses caused by yielding of a notched tensile bar of $K_t = 2$ for stress gradients $a$ to $d$ in Figure 4.42$f$.

# 4.14 Residual Stresses Caused by Yielding—Axial Loading

- Except for a, where the did not cause yielding at the notch root, the stress change when the load is removed does not exactly cancel the stresses by applying the load.
- Hence, residual stresses remain after the load is removed. These are shown in the right column
- In each case, the stress change caused by removing the load is elastic.
- It must be remembered, too, that this development of residual stress curves was based on assuming that the material conforms to the idealized stress–strain curve
- So the residual stress curves can only be good approximations.

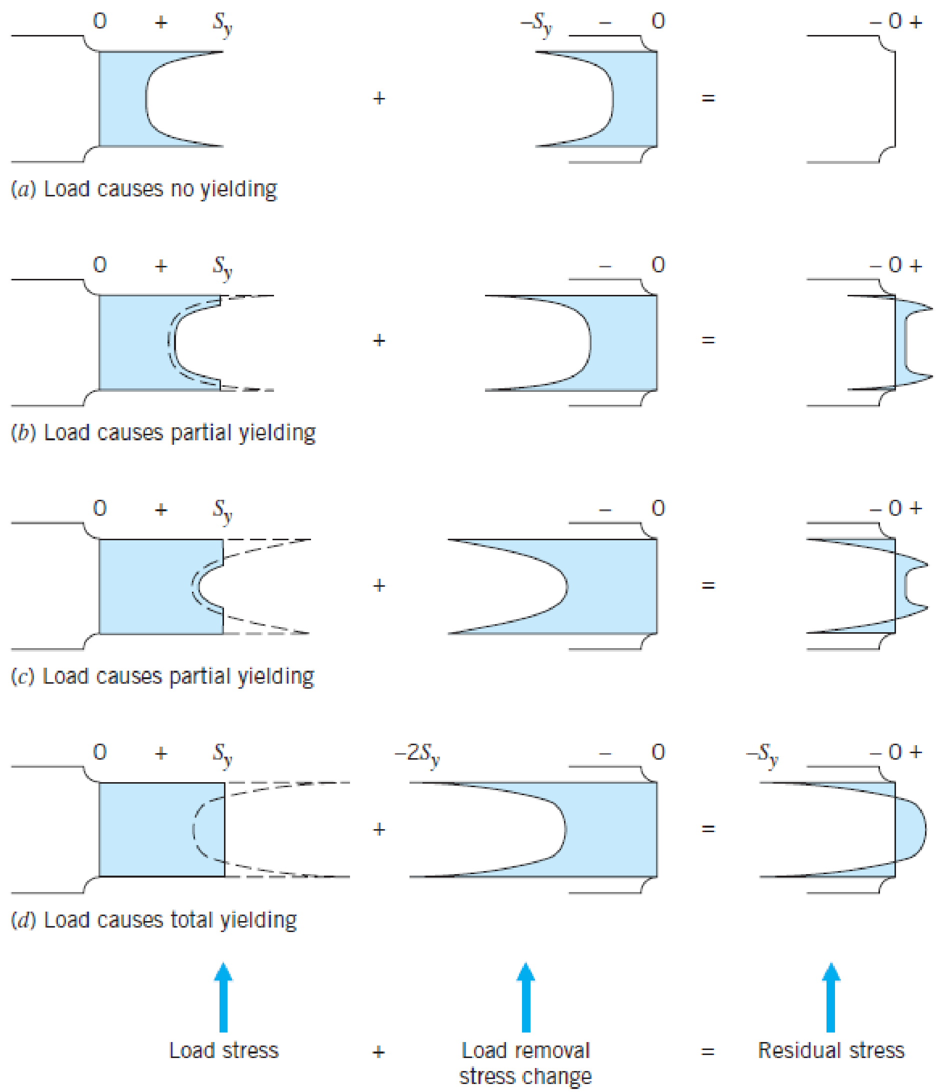



**FIGURE 4.43**
Residual stresses caused by yielding of a notched tensile bar of $K_t = 2$ for stress gradients $a$ to $d$ in Figure 4.42$f$.

# 4.15 Residual Stresses Caused by Yielding—Bending and Torsional Loading

- Figure illustrates residual stresses caused by the bending of an unnotched 25 * 50-mm rectangular beam.
- made of steel having an idealized stress–strain curve with Sy = 300 MPa.
- Unknown moment $M_1$ produces the stress distribution shown in Figure, with yielding to a depth of 10 mm. Let us first determine the magnitude of moment M1
- If the distributed stress pattern is replaced with concentrated forces F1 and F2 at the centroids of the rectangular and triangular portions of the pattern, respectively,
- M1 is equal to the sum of the couples produced by F1 and F2. The magnitude of F1 is equal to the product of the average stress (300 MPa) times the area over which it acts (10 mm * 25 mm).

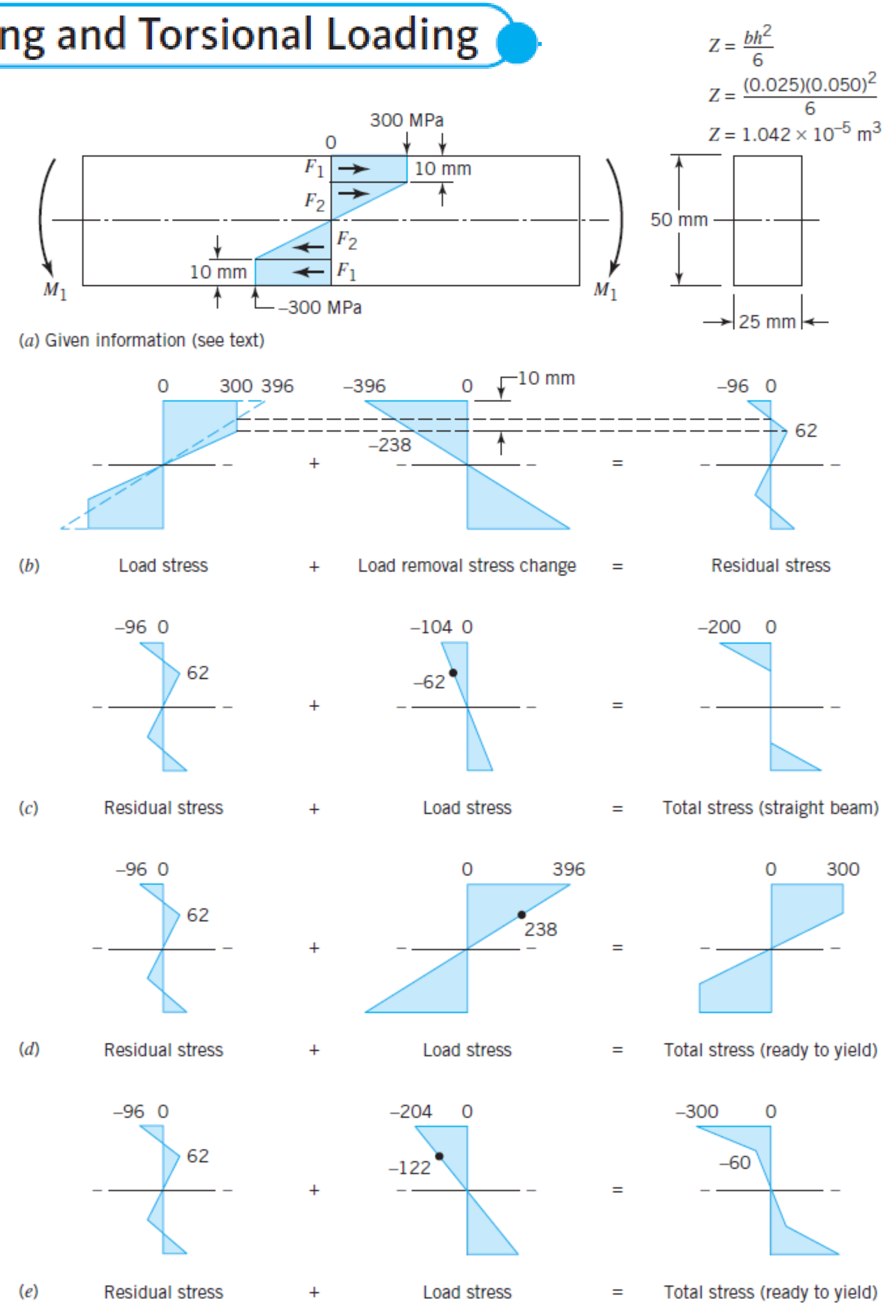



**FIGURE 4.44**
Residual stresses in an unnotched rectangular beam.

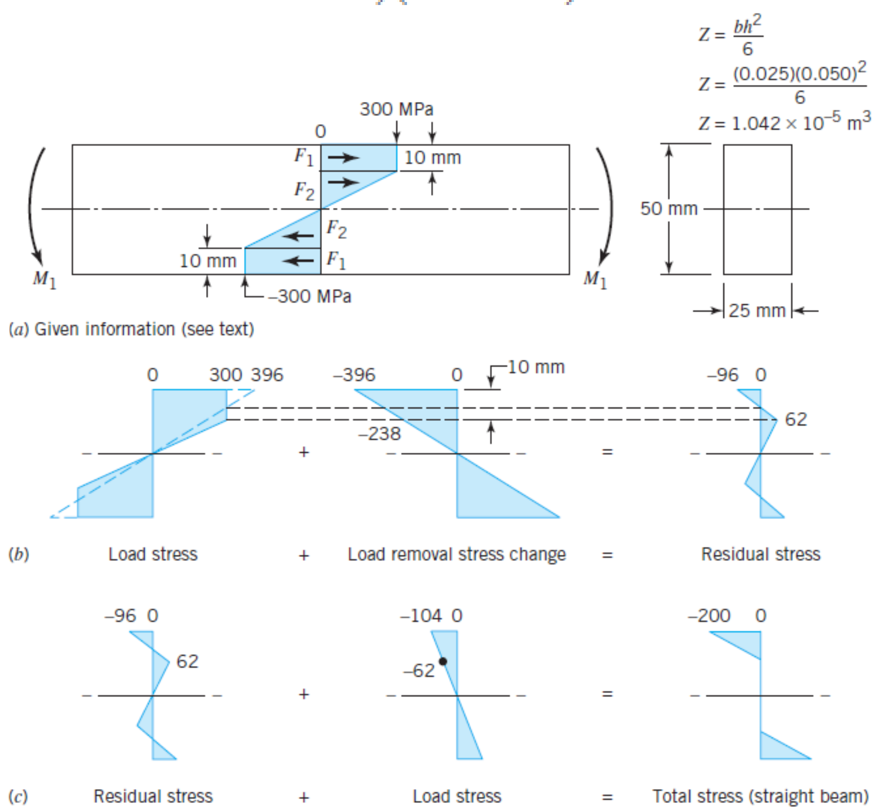
- Similarly, F2 is equal to an average stress of 150 MPa times an area of 15 mm * 25 mm. The moment arms of the couples are 40 mm and 20 mm, respectively

$$M_1 = (300 \text{ MPa} \times 250 \text{ mm}^2)(0.040 \text{ m}) + (150 \text{ MPa} \times 375 \text{ mm}^2)(0.020 \text{ m})$$
$$= 4125 \text{ N} \cdot \text{m}$$

- After $M_1$ is removed

$$\sigma = M/Z = 4125 \text{ N} \cdot \text{m}/(1.042 \times 10^{-5} \text{ m}^3)$$
$$= 3.96 \times 10^8 \text{ Pa} = 396 \text{ MPa}$$

- Note that at this point the beam is slightly bent
- Figure 4.44c shows that the desired center portion stress-free condition requires superimposing a load that develops a compressive stress of 62 MPa, 10 mm below the surface



## 4.15 Residual Stresses Caused by Yielding—Bending and Torsional Loading

- With this load in place, total stresses are as shown.
- Since center portion stresses are zero, the beam is indeed straight.
- Stress 396 MPa is due to moment of 4125 Nm.
- By simple proportion, a stress of 104 MPa requires a moment of 1083 N # m.
- Let us now determine the elastic bending moment capacity of the beam after the residual stresses have been established.
- A moment in the same direction as $M_1$ can be added that superimposes a surface stress of +396 MPa without yielding. It is of 4125 Nm.
- The release of original moment $M_1$ = 4125 Nm caused no yielding; so, it can be reapplied.

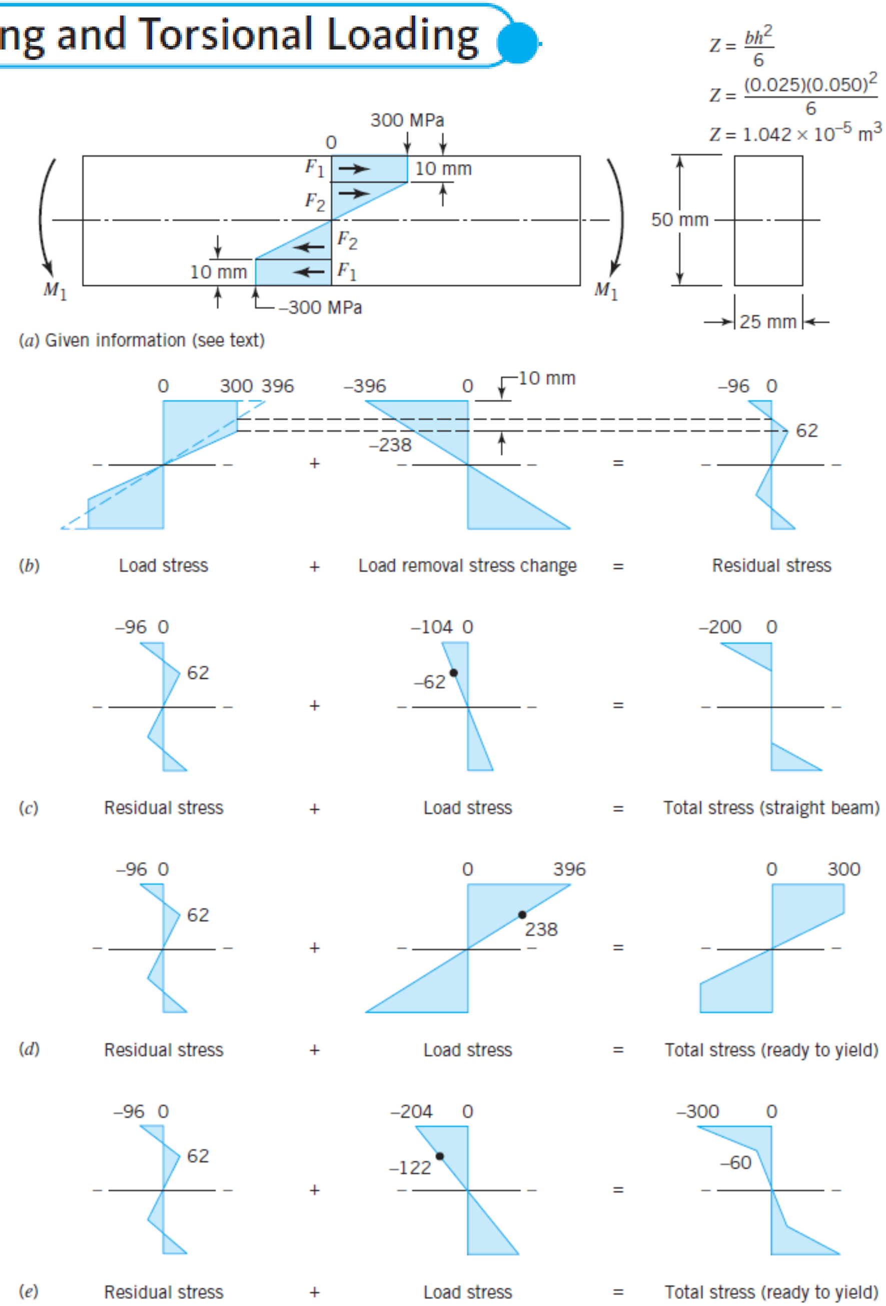



**FIGURE 4.44**
Residual stresses in an unnotched rectangular beam.

# 4.15 Residual Stresses Caused by Yielding—Bending and Torsional Loading

- Figure e shows that in the direction opposite the original moment $M_1$, a moment giving a surface stress of 204 MPa is all that can be elastically withstood.
- This corresponds to moment of 2125Nm.
- *An overload causing yielding produces residual stresses that are favorable to future loads in the same direction and unfavorable to future loads in the opposite direction.*
- Furthermore, on the basis of the idealized stress–strain curve, the increase in load capacity in one direction is exactly equal to the decrease in load capacity in the opposite direction.

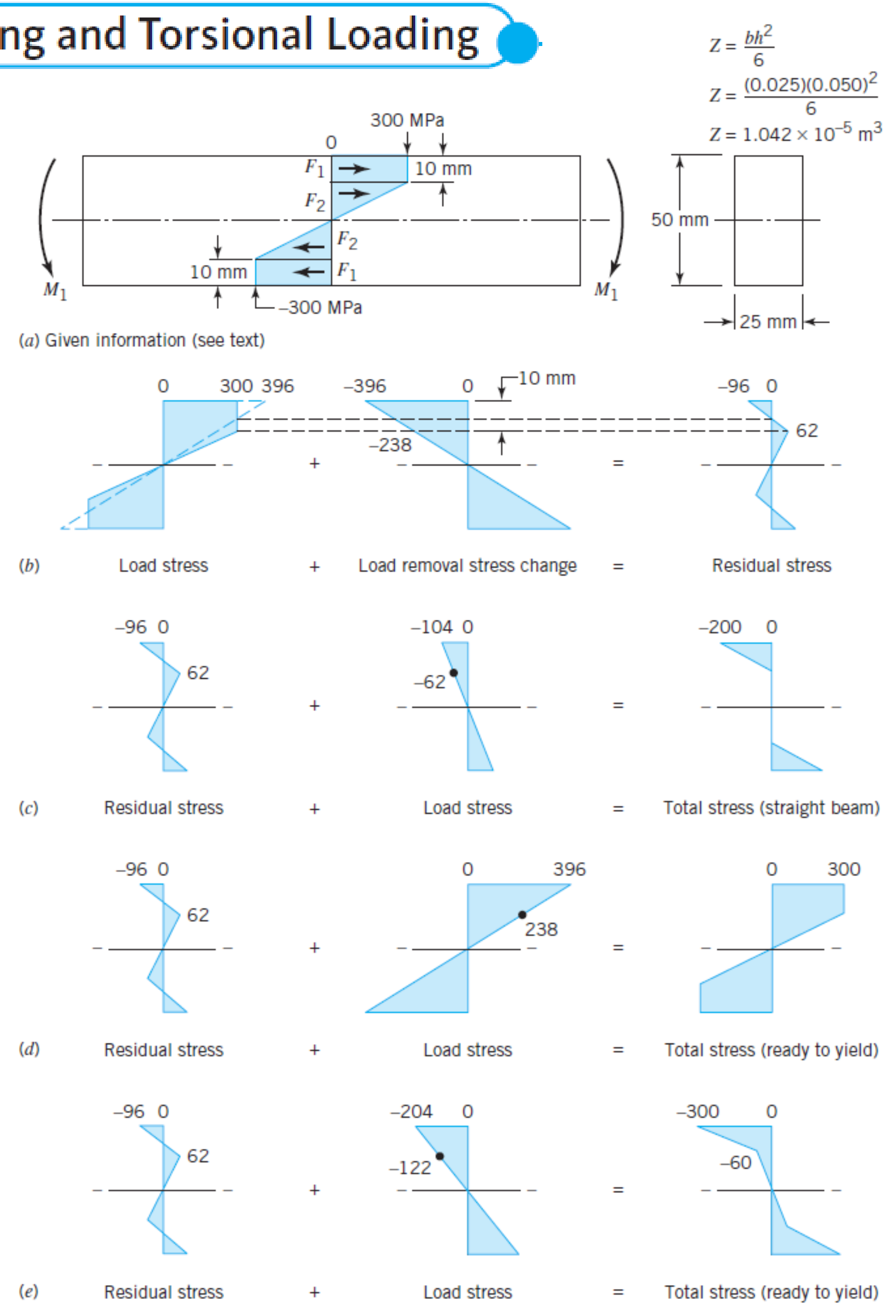



**Figure 4.44**
Residual stresses in an unnotched rectangular beam.

# 4.16 Thermal Stresses

- We ve seen stresses caused by external loads. Stresses can also be caused by expansion and contraction due either to temperature changes or to a material phase change.
- It is important to become familiar with the basic principles. When the temperature of an unrestrained homogeneous, isotropic body is uniformly changed, it expands (or contracts) uniformly in all directions, according to the relationship

$$\epsilon = \alpha \Delta T$$

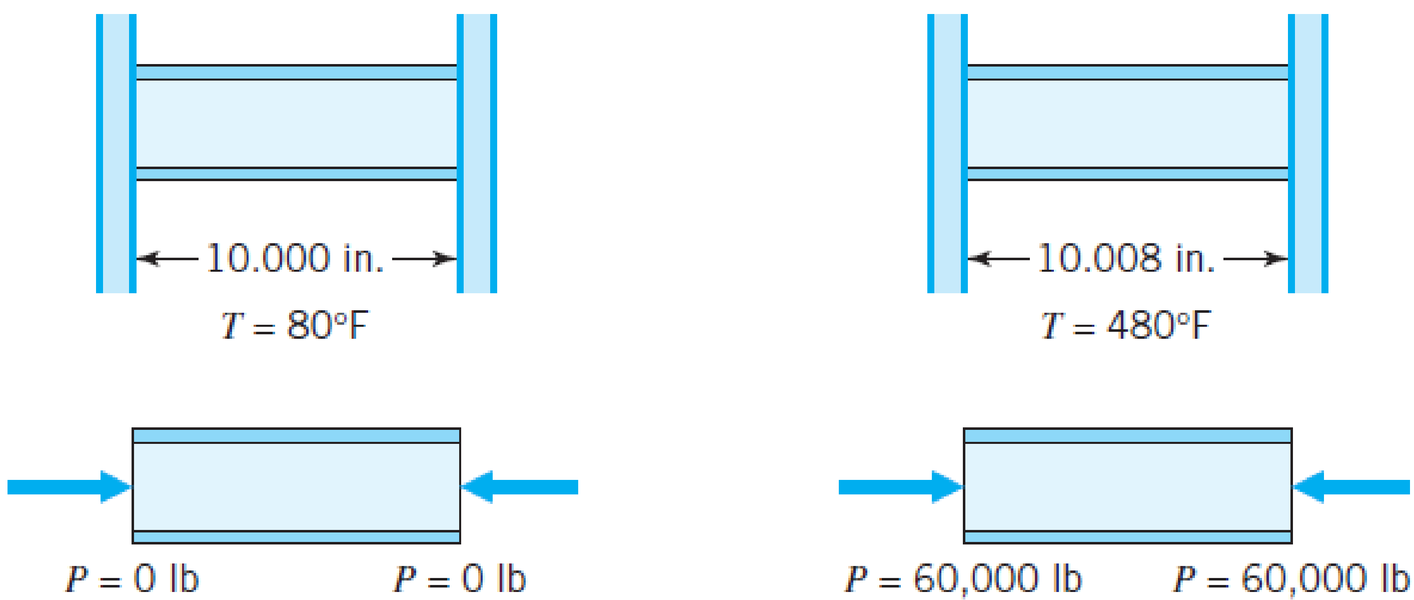
- where $\varepsilon$ is the strain, $\alpha$ is the thermal expansion coef and $\Delta T$ is the temperature change. Values of $\alpha$ for several common metals are given in Appendix C-1.
- If restraints are placed on the member during the temperature change, the resulting stresses can be determined by
- (1) computing the dimensional changes that would take place in the absence of constraints,
- (2) determining the restraining loads necessary to enforce the restrained dimensional changes, and
- (3) computing the stresses associated with these restraining loads.

## SAMPLE PROBLEM 4.5 Thermal Stresses in a Tube

- We ve seen stresses caused by external loads. Stresses can also be caused by expansion and contraction due either to temperature changes or to a material phase change. A 10-in. L steel tube ($E = 30 \times 10^6$ psi and $\alpha = 7 \times 10^{-6}$ per ° F) having a CSA of 1 in$^2$ is installed with “fixed” ends so that it is stress-free at 80° F. In operation, the tube is heated throughout to a uniform 480° F. Careful measurements indicate that the fixed ends separate by 0.008 in. What loads are exerted on the ends of the tube, and what are the resultant stresses?

- **Known**: A given length of steel tubing with a known CSA expands 0.008 in. from a stress-free condition at 80° F when the tube is heated to a uniform 480° F

Schematic and Given Data:



## Assumptions:

1. The tube material is homogeneous and isotropic.
2. The material stresses remain within the elastic range.

## Analysis:

1. For the unrestrained tube

$$\epsilon = \alpha \Delta T = (7 \times 10^{-6})(400) = 2.8 \times 10^{-3}$$
$$\Delta L = L\epsilon = 10 \text{ in.} (2.8 \times 10^{-3}) = 0.028 \text{ in.}$$

2. Since the measured expansion was only 0.008 in., the constraints must apply forces sufficient to produce a deflection of 0.020 in. From the relationship

$$\delta = \frac{PL}{AE}$$

which is from elementary elastic theory, and reviewed in Chapter 5,

$$0.020 = \frac{P(10)}{(1)(30 \times 10^6)}, \qquad \text{or} \quad P = 60{,}000 \ lb$$

3. Because the area is unity, $\sigma = 60\ ksi$.

**Comment:** Since these answers are based on elastic relationships, they are valid only if the material has a yield strength of at least 60 ksi at $480°F$.

- If stresses caused by temperature change are undesirably large, the best solution is often to reduce the constraint. - using expansion joints, loops, or telescopic joints
- Thermal stresses also result due to *temperature gradients* - if a thick metal plate is heated in the center of one face with a torch, the hot surface is restrained from expanding by the cooler surrounding material; it is in a state of compression.
- Then the remote cooler metal is forced to expand, causing tensile stresses.
- If the forces and moments do not balance for the original geometry, it will distort or warp to bring about internal equilibrium.
- If stresses are within the elastic limit, the part will revert to its original geometry when the initial temperature conditions are restored.
- If some portion of the part yields, this portion will not tend to revert to the initial geometry, and there will be warpage and internal (residual) stresses when initial temperature conditions are restored. This must be taken into account in the design

- Residual stresses are added to any subsequent load stresses in order to obtain the total stresses.
- If a part with residual stresses is machined, the removal of residually stressed material causes the part to warp or distort. As this upsets the internal equilibrium.
- A common (destructive) method for determining the residual stress in a particular zone of a part is to remove very carefully material from the zone and then to make a precision measurement of the resulting change in geometry. (Hole drilling method)
- Residual stresses are often removed by annealing. The unrestrained part is uniformly heated (to a sufficiently high temperature and for a sufficiently long period of time) to cause virtually complete relief of the internal stresses by localized yielding.
- The subsequent slow cooling operation introduces no yielding. Hence, the part reaches room temperature in a virtually stress-free state.

# 4.17 Importance of Residual Stresses

- In general, residual stresses are important in situations in which stress concentration is important.
- These include brittle materials involving all loading types, and the fatigue and impact loading of ductile as well as brittle materials.
- For the static loading of ductile materials, harmless local yielding can usually occur to relieve local high stresses resulting from either (or both) stress concentration or superimposed residual stress.
- It is easy to overlook residual stresses because they involve nothing that ordinarily brings them to the attention of the senses. When one holds an unloaded machine part, for example, there is normally no way of knowing whether the stresses are all zero or whether large residual stresses are present.
- a reasonable qualitative estimate can often be made by considering the thermal and mechanical loading history of the part.
- An interesting example shows that residual stresses remain in a part as long as heat or external loading does not remove them by yielding.

# 4.17 Importance of Residual Stresses

- The Liberty Bell, cast in 1753, has residual tensile stresses in the outer surface because the casting cooled most rapidly from the inside surface.
- After 75 years of satisfactory service, the bell cracked, probably as a result of fatigue from superimposed vibratory stresses caused by ringing the bell.
- Holes were drilled at the ends to keep the crack from growing, but the crack subsequently extended itself.
- Almen and Black cite this as proof that residual stresses are still present in the bell.