Directed Distance-Based Formation Control of Nonlinear Heterogeneous Agents in 3-D Space

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Abstract—This paper studies distance-based formation control of a set of nonlinear multi-agent systems over directed graphs. We propose a distributed, distance-based formation control scheme for a set of heterogeneous, nonlinear agents over a particular class of minimally, structurally persistent, directed graphs in a 3-D space, namely directed trilateral Laman graphs. The responsibility of controlling each directed edge is assigned to only one of the adjacent agents. The state-dependent Riccati equation is used to design the control method for nonlinear agents. Based on the mathematical induction and stability theory of cascade interconnected systems, we rigorously prove the asymptotic stability of the overall formation. A combination of signed area and volume constraints is used to prevent agents from converging to the flip-ambiguous framework in 3-D space. The proposed control law assures collision avoidance between the neighboring pairs of agents. Simulation results are provided to verify the theoretical results.

Index Terms—Multi-agent systems, distance-based formation control, nonlinear dynamics, heterogeneous systems, directed graphs, collision avoidance.

I. INTRODUCTION

The concept of formation control stems from nature, where various formation examples can be observed. Results have shown that when multiple agents are assigned a particular task, the overall performance and efficiency improve if agents form a certain geometric shape [1]. For instance, it is well-known that birds flying in a “V” shape formation save energy and extend their flight range up to approximately 70 percent compared to the flight of a single bird [2].

Distance-based formation control of multi-agent systems has recently attracted a significant research interest in the control systems community [3], [4]. There is an increasing focus on distance-based formation control due to its theoretical challenges and vast applications [5]. In a distance-based formation control, the desired formation is specified in terms of intra-agent distances, where agents aim to reach the desired shape via controlling the distances between the pairs of neighbors [6].

Distance-based formation control problems can be modeled by undirected or directed graphs, where the responsibility of controlling a distance between neighboring agents is assigned to both adjacent agents or just one of them [7]. It was shown that the undirected distance-based formation could face some critical challenges if there is a measurement disagreement between the neighboring agents [8]. This is not an issue in the case of a directed graph topology since each edge is controlled by only one agent. Also, the reduction in the communication and control complexities of the directed case compared to the undirected formation was demonstrated in [9].

Due to mathematical complexity of distance-based formation control, the majority of the research has been focused mostly on single-integrator models, to name a few [10], [11], [12], [13], [14], and few considered the double-integrator model, e.g., [15], [16], [17]. Some research also studied the robustness of distance-based formation control, where the linear models are affected by disturbances. For instance, distance-based formation control of single-integrators with additive exogenous disturbances over undirected graphs was studied in [18], [19], [20], and over directed graphs in [21]. A flocking and distance-based formation control for a set of second-order agents with parametric uncertainty over directed topologies was considered in [22]. However, as a new study, this paper studies the distance-based formation control of heterogeneous, affine, nonlinear multi-agent systems.

Most of the work that has been done on the formation of nonlinear agents considered displacement-based approach, e.g., [23], [24]. Displacement-based formation control of a set of nonlinear aircrafts is studied in [25], and a set of nonlinear quadrotors in [26]. Formation control for linear agents with state constraints is studied in [27]. The reinforcement learning-based approach is proposed in [28] for displacement-based formation control of uncertain, heterogeneous, linear multi-agent systems. Authors in [29] proposed an adaptive, event-triggered approach for robust consensus-based formation control of heterogeneous, nonlinear, multi-agent systems.

There is a plethora of work on distance-based formation control of linear systems. However, in comparison, there is no much work done on the distance-based formation control of nonlinear systems. Distance-based formation of unicycles was studied in [30], [31], [32]. References [33], [34] considered distance-based formation of agents with nonlinear, second-order dynamics where the uniform ultimate boundedness of the formation is shown. The multi-layered formation of autonomous marine vehicles is studied for a particular class of nonlinear, double-integrator models in [35].

The main contribution of this paper is three-fold. First, we developed a distributed, distance-based formation control for agents with general, affine nonlinear models over directed trilateral Laman graphs in 3-D space, without restriction on the number of the agents. The proposed control scheme can deal with nonlinear dynamics effectively by utilizing the state-dependent Riccati equation method. Second, we extended the solution to a set of heterogeneous agents (the agents’ dynamics are not required to be identical), considering that the heterogeneous agents are a more realistic and practical scenario than...
the homogeneous one. To the best of the authors’ knowledge, there are no research results that consider heterogeneous, nonlinear agents over directed graph topologies in a distance-based formation control setting. Third, the method prevents the agents from converging to the undesired, flip-ambiguous, equilibrium points using the barrier function method.

A shorter version of this manuscript was submitted to a conference [36]. Compared to [36], where the planar formation of homogeneous agents was studied, this paper has three main novelties: (i) directed distance-based formation problem was studied in 3-D space that requires structural persistency of the desired framework; (ii) we studied a set of heterogeneous, nonlinear agents in a formation and provided a stability analysis for such multi-agent formation control problem; and (iii) adopting the method introduced in [37], we applied a combination of signed area and volume constraints to prevent the framework’s flip-ambiguity in 3-D and to ensure collision avoidance among neighboring agents.

This paper differs from our previous results. In [17], we studied a centralized, undirected, distance-based formation of linear multi-agent systems. In [38], we developed a directed, distance-based formation control for linear, single- and double-integrator agents over triangular topologies in 2-D space. Furthermore, references [33], [34] proposed neural network-based adaptive controllers for specific classes of nonlinear second-order agents that only showed boundedness of the error signals. However, in this paper, we present a distributed, distance-based formation control of heterogeneous, affine, nonlinear multi-agent systems over directed trilateral Laman graphs in 3-D space within an optimal control framework.

II. PRELIMINARIES

A. Notation

The following notation is used throughout this paper. Let \( \mathbb{R} \) denote the real numbers set. Let \( \mathbb{R}^n \) denote an \( n \)-dimensional Euclidean space and \( \mathbb{R}^{n \times m} \) denote \( n \times m \) real matrices. Let \( I_n \) denote \( n \times n \) identity matrix. A vector of all-ones with \( n \) elements is \( \mathbf{1}_n \), and zero vector of size \( n \) is \( \mathbf{0}_n \). With \( \text{diag}(a_i) \), we denote a block diagonal matrix with matrices \( a_i \) on its diagonal. By \( \| \cdot \| \) we denote the Euclidean norm. The cardinality of a set \( V \) is denoted by \( |V| \).

B. Graph Theory

A desired distance-based formation of \( N \) agents can be represented by a directed graph (also known as a digraph) \( G = (V, \mathcal{E}) \), where \( V = \{v_1, \ldots, v_N\} \) and \( \mathcal{E} \subset V \times V \) are vertex set and edge set, respectively. Note that \( (i, j) \in \mathcal{E} \) means that the agent \( i \) can measure the relative position of the agent \( j \) with respect to its own coordinate frame. The edge \( e_{ij} \) is shown by an arrow with the head in vertex \( j \) (sink) and the tail in vertex \( i \) (source). A function that maps the vertices of the graph to a set of points in a Euclidean space is called a realization. Also, a realization of \( G \) at given points \( p = [p_1^T, \ldots, p_N^T]^T \) is called a framework and denoted by \( F(G, p) \). Two frameworks, \( F(G, p) \) and \( F(G, q) \), are said to be equivalent (resp. congruent) if \( \|p_i - p_j\| = \|q_i - q_j\| \) for all \( (i, j) \in \mathcal{E} \) (resp. \( (i, j) \in \mathcal{V} \)). If two frameworks are equivalent, but not congruent, they are said to be flip-ambiguous. The neighboring set of the agent \( i \), denoted by \( N_i \), is a set of agents whose relative positions can be sensed by the agent \( i \), [39].

Definition 2.1 ([40]): Henneberg directed vertex addition in 3-D space, also known as directed trilateration, is adding a vertex with three directed edges to an existing directed graph, provided that the added vertex is the source of the added edges.

In order to have a feasible formation, a directed graph in 3-D space needs to be structurally persistent. Figure 1 shows structural persistency of directed graphs in 3-D. The graph shown in Figure 1(a) is not structurally persistent because agents \#1 and \#5 can freely move, and other agents cannot preserve their constraints. However, the graph in Figure 1(b) is structurally persistent. A minimally, structurally persistent graph is a structurally persistent graph that loses its persistency by removing any of its edges.

Lemma 1 ([41]): A directed graph in \( \mathbb{R}^3 \), that is obtained via directed vertex addition operation starting from a primitive leader-first-follower (LFF) graph, is minimally, structurally, persistent.

Definition 2.2 ([37]): A minimally, structurally, persistent graph in 3-D, obtained via directed vertex addition sequences, is called a directed trilateral Laman graph. Figure 2(a) shows the primitive triangle, which is called the LFF structure. Note that in the rest of the paper, we label the leader as agent \#1, the first-follower as agent \#2, and followers as agents \#3, \#4, ..., \( N \). Figure 2(b) shows the process of constructing a directed trilateral Laman graph using the directed vertex addition procedure.
C. SDRE Method

Suppose that an affine, nonlinear system is described by

\[ \dot{x} = f(x) + B(x)u, \]  

where \( x \in \mathbb{R}^q \) and \( u \in \mathbb{R}^p \) are state and input vectors, \( f(x) \) is a nonlinear mapping and \( B(x) \) is a continuous, matrix-valued function. Provided that \( f(0) = 0 \) and \( f(x) \in C^1 \), the system (1) can be written in the linear-like form

\[ \dot{x} = A(x)x + B(x)u, \]

where \( A(x) \), known as state-dependent coefficient (SDC) matrix, is a continuous matrix-valued function. Note that \( A(x) \) is non-unique for non-scalar systems, and can be obtained by mathematical factorization. An interested reader may refer to [42] for more detailed discussions.

Remark 1: If \( A(x) \) is an SDC representation of the nonlinear system (1), such that \( f(x) = A(x)x \), then \( A(x) = A(x) + \dot{E}(x) \) is also an SDC representation of the system (3), for every \( E(x) \) that satisfies \( E(x)x = 0 \).

Suppose a quadratic-like cost functional is given by

\[ J = \frac{1}{2} \int_0^\infty \{x(t)^TQ(x)x(t) + u(t)^TR(x)u(t)\}dt, \]  

where \( Q \geq 0 \) and \( R > 0 \) are state-dependent weight matrices to be selected. The nonlinear optimal control problem is to find a control input \( u(t) \) that minimizes the cost functional (3), with respect to the system dynamics (2) and that guarantees the asymptotic stability of the closed-loop system.

Assumption 1: The pairs \( \{A(x), B(x)\} \) and \( \{Q^{1/2}(x), A(x)\} \) are point-wise stabilizable and detectable, in the linear sense, for all \( x \), in some nonempty neighborhood of the origin \( \Omega \).

Remark 2: For the general system (1), Assumption 1 is a common assumption in the literature that guarantees the SDRE equation’s solvability, uniqueness, and positive-definiteness of the solution [43]. While positive definiteness of \( Q(x) \) guarantees the detectability condition, the stabilizability of the system depends on the selection of matrix \( A(x) \). An interested reader may refer to [42], [44] for more details on optimal selection of SDC matrices.

Lemma 2 ([45]): For system (3), there exist an SDC representation that satisfies Condition 1, unless \( (x, f(x)) \) are linearly dependent and \( Q^{1/2}(x)x = 0 \).

The following lemma presents the main results in SDRE theory.

Lemma 3 ([43]): Given Assumption 1, the control law

\[ u(x) = -R^{-1}(x)B^T(x)S(x)x(t), \]  

where \( S(x) \) is a unique, symmetric, positive-definite solution of the Riccati equation

\[ Q(x) + A^T(x)S(x) + S(x)A(x) \]
\[ -S(x)B(x)R^{-1}(x)B^T(x)S(x) = 0, \]

asymptotically minimizes the cost functional (3) and guarantees local asymptotic stability of the closed-loop system.

D. Stability of Interconnected Systems

Here we present two main lemmas, which will be used later to prove the stability of the overall formation.

Definition 2.3 ([46]): The system

\[ \dot{x} = f(x, u), \]

where \( f(.) \) is a Lipschitz nonlinear mapping, is input-to-state stable (ISS) if there exist class \( KL \) and class \( K \) functions \( \sigma \) and \( \psi \), respectively, such that the following is satisfied

\[ ||x(t)|| \leq \sigma(||x(0)||, t) + \psi\left(\sup_{0 \leq \tau \leq t} ||u(\tau)||\right), \]

for any bounded input \( u \) and initial condition \( x(0) \). The results are local if the condition holds only in some neighborhood of the origin.

Lemma 4 ([47]): The origin of system

\[ \dot{x} = g(x, u), \]

where \( g \in C^1 \), is locally input-to-state stable if the origin of the unforced system

\[ \dot{x} = g(x, 0), \]

is asymptotically stable.

Lemma 5 ([48]): The origin of the system

\[ \begin{align*}
\dot{x}_1 &= f_1(x_1) \\
\dot{x}_2 &= f_2(x_2, x_1) \\
&\vdots \\
\dot{x}_i &= f_i(x_i, x_{i-1}, \ldots, x_1),
\end{align*} \]

is locally asymptotically stable equilibrium point, if the origin of the system (10a) is locally asymptotically stable, and all subsystems \( \dot{x}_j = f_j(.) \), \( 1 < j \leq i \), are locally ISS with respect to \( x_k \), for \( k < j \).

E. Signed Area and Signed Volume

A signed area of a triangle, formed by three agents \( i, j, k \), which are located at \( p_i, p_j, p_k \in \mathbb{R}^2 \), respectively, is given by [12]:

\[ \kappa = \frac{1}{2} \det \begin{bmatrix} p_j - p_i & p_k - p_i \end{bmatrix}. \]

The sign of \( \kappa \) depends on the clockwise or counterclockwise order of agents.

Suppose that a tuple \( (i, j, k, l) \) of agents is deployed in the 3-D space. The signed volume of the tetrahedron, that is formed between the agents located at \( p_i, p_j, p_k, p_l \in \mathbb{R}^3 \), is given by [49]:

\[ V = \frac{1}{6} \det \begin{bmatrix} x_i - x_l & y_i - y_l & z_i - z_l \\
 x_j - x_l & y_j - y_l & z_j - z_l \\
 x_k - x_l & y_k - y_l & z_k - z_l \end{bmatrix}. \]

The sign of \( V \) depends on the two possible formation realizations. It is shown that for any rotation and translation, the signed area and volume remain unchanged [12]. Figure 3 depicts the flip-ambiguity of a formation in 3-D space, with the desired formation shown in Figure 3(a). While agent #4 meets the distance constraints in Figure 3(b), the formation is reflected compared with the desired formation.
In a directed trilateral Laman topology, the leader is stationary since it has no constraints to satisfy. The first follower (agent #2) is assigned the edge \((e_{21})\) to control. Thus, the dynamics of the edge associated with agent #2 is

\[
\dot{e}_{21} = \frac{p_{21}^T}{\|p_{21}\|} v_2.
\]  

(19)

Let us define the normalizing operator, also known as the relative bearing, as \(\eta(p_{ij}) = \frac{p_{ij}^T}{\|p_{ij}\|}\). The aggregate error vector for agent #2 is

\[
e_2 = [e_{21}, v_2^T, w_2^T]^T.
\]  

(20)

The time-derivative of (20) is given by

\[
\dot{e}_{21} = \eta(p_{21}) v_2 \quad (21a)
\]

\[
\dot{v}_2 = \dot{f}_2(v_2, w_2) + \dot{h}_2(v_2, w_2) u_2 \quad (21b)
\]

\[
\dot{w}_2 = \dot{f}_2(v_2, w_2) + \dot{h}_2(v_2, w_2) u_2. \quad (21c)
\]

We can write (21) in a linear-like form as

\[
\dot{e}_2 = A_2(e_2) e_2 + B_2(e_2) u_2. \quad (22)
\]

Then, the associated local optimal control problem with agent #2 is given by

\[
J_2 = \min \frac{1}{2} \int_0^\infty \{e_2^T Q_2 e_2 + u_2^T R_2 u_2\} dt
\]

s.t.

\[
\dot{e}_2 = A_2 e_2 + B_2 u_2 \quad (23)
\]

\[
Q_2 \geq 0
\]

\[
R_2 > 0.
\]

The following result provides the suboptimal SDRE control law that minimizes (23) and ensures that \(e_{21}\) asymptotically converges to zero.

**Theorem 3.1:** For agent #2, described by the nonlinear affine model (14), under Assumption 1, the control law

\[
u_2 = -R_2^{-1} B_2^T S_2 e_2, \quad (24)
\]

where \(S_2\) is the solution of the following Riccati equation

\[
Q_2 + A_2^T S_2 + S_2 A_2 - S_2 B_2 R_2^{-1} B_2^T S_2 = 0, \quad (25)
\]

renders the origin of the closed-loop system asymptotically stable and steers agent #2 to the desired position; hence, \(e_{21}\) converges to zero.

**Proof:** Given Assumption 1, the proof of the theorem is a direct result of Lemma 3. \(\square\)

Note that, based on Lemma 2, selecting a positive definite \(Q(x)\) guarantees the detectability of the representation and assures existence of a stabilizable representation of the system.

**Remark 3:** Substituting the control law (24) in (22), the closed-loop dynamics is

\[
\Delta_2 : \quad \dot{e}_2 = (A_2 - B_2 R_2^{-1} B_2^T S_2) e_2. \quad (26)
\]

For agent #3, we define \(e_3 = [e_{31}, e_{32}, v_3^T, w_3^T]^T\). Thus, one has

\[
\dot{e}_{31} = \eta(p_{31}) v_3 \quad (27a)
\]

\[
\dot{e}_{32} = \eta(p_{32}) (v_3 - v_2) \quad (27b)
\]

\[
\dot{e}_3 = \dot{f}_3(v_3, w_3) + \dot{h}_3(v_3, w_3) u_3 \quad (27c)
\]

\[
\dot{w}_3 = \dot{f}_3(v_3, w_3) + \dot{h}_3(v_3, w_3). \quad (27d)
\]
Equation (27) can be written in the matrix form
\[ \Xi_3 : \dot{e}_3 = A_3(e_3)e_3 + \hat{B}_3(e_3)u_3, \]  
(28)
where \( u_3 = [v_2, v_3] \). Suppose that the system is unforced with respect to \( v_2 \), then the nominal system, where \( v_2 = 0 \), can be written in a linear-like form as
\[ \Sigma_3 : \dot{e}_3 = A_3(e_3)e_3 + B_3(e_3)u_3, \]  
(29)
where \( A_3(e_3) \) and \( B_3(e_3) \) are matrix functions obtained from (27). One can write the associated, nominal optimal control problem for agent #3 as
\[ J_3 = \min \frac{1}{2} \int_0^\infty \{ e_3^T Q_3 e_3 + u_3^T R_3 u_3 \} dt \]
\[ \text{s.t.} \]
\[ \dot{e}_3 = A_3(e_3)e_3 + B_3(e_3)u_3 \]
\[ Q_3 \geq 0 \]
\[ R_3 > 0. \]
(30)
The following theorem provides the control law that guarantees asymptotic stability of the closed-loop system’s origin and, consequently, the asymptotic convergence of \( e_3 \) to zero.

**Theorem 3.2:** For agent #3, described by the nonlinear affine model (14), under Assumption 1, the control law
\[ u_3 = -R_3^{-1}B_3^T S_3 e_3, \]  
(31)
where \( S_3 \) is the positive definite solution of the following state-dependent Riccati equation
\[ Q_3 + A_3^T S_3 + S_3 A_3 - S_3 B_3 R_3^{-1} B_3^T S_3 = 0, \]  
(32)
achieves local asymptotic stability of the origin of the closed-loop system, and hence, the desired distance-based formation.

**Proof:** The origin of the system \( \Delta_2 : \dot{e}_2 = f_2(e_2) \) is asymptotically stable. Given Assumption 1, the control law (31) results in the asymptotic stability of the system \( \Xi_3 : \dot{e}_3 = f_3(e_3, v_2) \). Thus, based on Lemma 4, the system \( \Sigma_3 : \dot{e}_3 = f_3(e_3, v_2) \) is locally ISS. Since \( v_2 = [0, I_3, 0] e_2 \), where \( m \) is the size of \( w_2 \), we conclude the local input-to-state stability of \( \Xi_3 : \dot{e}_3 = f_3(e_3, e_2) \). Therefore, based on Lemma 5, it follows that the origin of the interconnected system
\[ \Delta_3 : \begin{cases} \dot{e}_2 = f_2(e_2) \\ \dot{e}_3 = f_3(e_3, e_2), \end{cases} \]  
(33)
is locally asymptotically stable. \( \square \)

To address the collision and reflection avoidance issues, we formulate the problem in the form of an optimal control problem with hard constraints. There are few methods in the literature to solve the constrained SDRE problem [42]. We apply here the method proposed by Friedland [50], which is based on the barrier function method, to solve an SDRE problem with state constraints.

**Corollary 3.2.1:** Select
\[ Q_3 = \kappa_3 \text{diag}[q_{31}, q_{32}, q_{w3}, q_{w3}] + I, \]  
(34)
where
\[ \kappa_3 = \frac{\mathcal{K}^* - \mathcal{K}}{\mathcal{K}^* + \mathcal{K}}, \]  
(35)
and \( \mathcal{K}^* \) and \( \mathcal{K} \) are desired and actual signed areas of the triangle formed by agents \( 1, 2, 3 \), and
\[ q_{3m} = \nu_{3m} + \mu_{3m}, \quad m \in \{1, 2\}, \]  
(36)
where \( \nu_{3m} \) is a positive constant and \( \mu_{3m} \) is a positive barrier multiplier given by
\[ \mu_{3m} = \left( \frac{d_{3m}^2}{d_{3m} - r_{d3m}} \right)^\epsilon, \]  
(37)
for suitable \( \epsilon \geq 1 \). Let \( r_{d3m} \) be a safe distance between pair of agents to prevent the collision, and \( q_{w3}, q_{w3} \) are appropriate positive definite matrices. Then, by using the weighting factor (34), the proposed SDRE control law (31) guarantees an inter-agent collision avoidance and prevents flip-ambiguity of agent #3 in the directed, distance-based formation.

**Proof:** For agent #3, the collision avoidance constraint can be written as
\[ d_{3m} > r_{d3m}, \quad m \in \mathcal{N}_3. \]  
(38)
The reflection prevention can formulated as the following constraint
\[ \mathcal{K}(T_f) = \mathcal{K}^*, \]  
(39)
where \( \mathcal{K}(T_f) \) is a signed area of the triangle at final time \( T_f \to \infty \). Constraints (38) and (39) are added to the nonlinear optimal control problem (30). We use the inverse barrier function method, proposed in [50], to solve the obtained constrained SDRE problem.

The proposed weighting matrix \( Q_3 \) approaches infinity if the collision avoidance or reflection prevention conditions are violated. Thus, based on the method proposed in [50], we conclude that the SDRE control law prevents collision of agent #3 with the leader and the first-follower. Moreover, it also prevents agent #3 from converging to a reflected position. \( \square \)

For agent #4, the aggregate error vector is \( e_4 = [e_{41}, e_{42}, e_{43}, v_{43}T, w_4T] \). The formation error dynamics can be written as
\[ \begin{align*}
\dot{e}_{41} &= \eta(p_{41}) v_4 \\
\dot{e}_{42} &= \eta(p_{42})(v_4 - v_2) \\
\dot{e}_{43} &= \eta(p_{43})(v_4 - v_3) \\
\tilde{v}_4 &= \hat{f}_4(v_4, w_4) + \hat{h}_4(v_4, w_4) u_4 \\
\tilde{w}_4 &= \hat{f}_4(v_4, w_4) + \hat{h}_4(v_4, w_4) u_4.
\end{align*} \]  
(40a-40e)
It can also be written in a matrix form as
\[ \Xi_4 : \dot{e}_4 = A_4(e_4)e_4 + \hat{B}_4(e_4)u_4, \]  
(41)
where \( u_4 = [v_2T, v_3T, v_4T]T \). The corresponding, unforced, nominal system in which \( v_2 \) and \( v_3 \) are zero, can be written in the linear-like form as
\[ \Sigma_4 : \dot{e}_4 = A_4(e_4)e_4 + B_4(e_4)u_4, \]  
(42)
where $B_4(e_4)$ is obtained from (40). The associated, nominal optimal control problem can be formulated as

$$J_4 = \min \frac{1}{2} \int_0^\infty \{e_4^T Q_4 e_4 + u_4^T R_4 u_4\} dt$$

s.t.

$$\dot{e}_4 = A_4 e_4 + B_4 u_4$$

$$Q_4 \geq 0$$

$$R_4 > 0.$$  

The following theorem specifies the control law for agent #4 that guarantees the convergence to the desired formation.

**Theorem 3.3:** For agent #4, described by the nonlinear model (14), under Assumption 1, the distributed control law

$$u_4 = -R_4^{-1} B_4^T S_4 e_4,$$  

where $S_4$ is the unique solution of the equation

$$Q_4 + A_4^T S_4 + S_4 A_4 - S_4 B_4 R_4^{-1} B_4^T S_4 = 0,$$  

achieves local asymptotic stability of the origin for the closed-loop, directed, distance-based formation.

**Proof:** The control law (44) is the stabilizing control of the nominal system $\Sigma_4 : \dot{e}_4 = f_4(e_4, 0, 0)$ provided that Assumption 1 is satisfied. Thus, according to Lemma 4, the system $\Xi_4 : \dot{e}_4 = f_4(e_4, e_3, e_2)$ is input-to-state stable with respect to $(e_3, e_2)$. We have shown that the system $\Delta_3$ is asymptotically stable. Thus, based on Lemma 5, the origin of the interconnected system

$$\Delta_4 : \begin{align*}
\dot{e}_2 &= f_2(e_2) \\
\dot{e}_3 &= f_3(e_3, e_2) \\
\dot{e}_4 &= f_4(e_4, e_3, e_2),
\end{align*}$$

is locally asymptotically stable. □

**Corollary 3.1:** Select

$$Q_4 = \kappa_4 \text{diag} \{q_{41}, q_{42}, q_{43}, q_{44}, q_{45}\} + I,$$  

where

$$\kappa_4 = \frac{V_4^* - V_4}{V_4^* + V_4},$$

and $V_4^*$ and $V_4$ are desired and actual signed volumes of the tetrahedron between the tuple $(1, 2, 3, 4)$ and $q_{4m} = \nu_{4m} + \mu_{4m}$, $m \in \{1, 2, 3\}$,

$$\mu_{4m} = (\frac{d_{4m}^*}{d_{4m}^* - r_{4m}})^\epsilon,$$  

for suitable $\epsilon \geq 1$. A safe distance between pair of agents that prevents collision is $r_{4m}$, and $q_{4c}, q_{u4}$ are the appropriate weighting matrices. By using the weighting factor (47), the proposed SDRE control law (44) guarantees an inter-agent collision avoidance and prevents flip-ambiguity of the directed, distance-based formation.

**Proof:** For agent #4, the collision avoidance is formulated as the following constraint

$$d_{4m} > r_{4m}, \quad m \in \mathcal{N}_4.$$  

The reflection prevention in 3-D can be formulated as the following constraint

$$\forall_i(T_f) = V_i^*,$$  

where $V_i(T_f)$ is a signed volume of the tetrahedron at final time $T_f \to \infty$. The reflection prevention (52) and collision prevention (51) constraints can be added to the optimal control problem (43). The proposed inverse barrier function is

$$\Phi_4 = e_4^T (Q_4 - I)e_4,$$  

where $Q_4$ is defined in (47). Adding the proposed inverse barrier function (53) to the cost functional of the corresponding optimal control problem yields the corresponding unconstrained problem. If conditions for collision avoidance or reflection avoidance have been violated, then the proposed barrier function (53) will approach infinity, and based on [50], will prevent collisions between neighboring agents. □

Since the topology of the desired formation is assumed to be a directed trilateral Laman graph, the desired formation can be constructed via a sequence of Henneberg directed vertex addition operations. Consequently, for the next follower, say agent $i$, which is added to an existing directed trilateral Laman graph via directed trilateration sequence, the error vector is $e_i = [e_{ij}, e_{il}, e_{ik}, v_{ij}^T, w_{ij}^T]^T$. The error dynamics can be written as

$$\dot{e}_{ij} = \eta(p_{ij})(v_i - v_j)$$

$$\dot{e}_{il} = \eta(p_{il})(v_i - v_l)$$

$$\dot{e}_{ik} = \eta(p_{ik})(v_i - v_k)$$

$$\dot{v}_i = \dot{f}_i(v_i, v_l) + \dot{h}_i(v_i, w_i)$$

$$\dot{w}_i = \dot{f}_i(v_i, v_l) + \dot{h}_i(v_i, w_i) u_i.$$  

The equation (54) can be written in the matrix form as

$$\Xi_i : \dot{e}_i = A_i(e_i) e_i + B_i(e_i) u_i,$$  

where $u_i = [v_j^T, v_l^T, v_k^T, u_j^T]^T$. The corresponding unforced nominal system, in which $v_j = v_l = v_k = 0$, is

$$\Sigma_i : \dot{e}_i = A_i(e_i) e_i + B_i(e_i) u_i,$$  

where similarly $B_i(e_i)$ can be obtained from (54). The associated, nominal optimal control problem is given by

$$J_i = \min \frac{1}{2} \int_0^\infty \{e_i^T Q_i e_i + u_i^T R_i u_i\} dt$$

s.t.

$$\dot{e}_i = A_i e_i + B_i u_i$$

$$Q_i \geq 0$$

$$R_i > 0.$$  

The following theorem offers the associated, distributed control law for the agent $i$, which guarantees the convergence to the desired formation.

**Theorem 3.4:** For the agent $i$, described by the nonlinear model (14), under Assumption 1, the distributed control law

$$u_i = -R_i^{-1} B_i^T S_i e_i,$$  

where $S_i$ is obtained from (40). The associated, nominal optimal control problem can be formulated as the following constraint

$$\forall_i(T_f) = V_i^*,$$  

where $V_i(T_f)$ is a signed volume of the tetrahedron at final time $T_f \to \infty$. The reflection prevention (52) and collision prevention (51) constraints can be added to the optimal control problem (43). The proposed inverse barrier function is

$$\Phi_i = e_i^T (Q_i - I)e_i,$$  

where $Q_i$ is defined in (47). Adding the proposed inverse barrier function (53) to the cost functional of the corresponding optimal control problem yields the corresponding unconstrained problem. If conditions for collision avoidance or reflection avoidance have been violated, then the proposed barrier function (53) will approach infinity, and based on [50], will prevent collisions between neighboring agents. □
where \( S_i \) is the unique solution of the equation
\[
Q_i + A_i^T S_i + S_i A_i - S_i B_i R_i^{-1} B_i^T S_i = 0, \tag{59}
\]
achieves the local asymptotic stability of the closed-loop, directed, distance-based formation.

**Proof:** The control law (58) is the stabilizing control of the nominal system \( \Sigma_i \) under Assumption 1; thus, the system \( \Sigma_i \) is input-to-state stable with respect to \((e_1, e_2, e_3)\), according to Lemma 4. The stability analysis is based on the mathematical induction and is similar to the proof of Theorem 3.3, with similar reasoning using the stability analysis of interconnected systems. Using the mathematical induction, one can show that the system \( \Delta_{i-1} \), which is the interconnected system of \( i - 1 \) agents, is asymptotically stable. Thus, based on Lemma 5, the origin of the interconnected system
\[
\Delta_i : \begin{cases} 
\dot{e}_2 = f_2(e_2) \\
\dot{e}_3 = f_3(e_3, e_2) \\
\vdots \\
\dot{e}_i = f_i(e_i, \ldots, e_2),
\end{cases} \tag{60}
\]
is locally asymptotically stable. \(\square\)

**Remark 4:** For the agent \( i \), the error vector \( e_i \) consists of associated edge errors that are being measured in the agent’s local coordinate frame and the agent’s states, except its global position \( p_i \). The agent’s velocity can be measured by onboard sensors.

**Remark 5:** Although the global asymptotic stability of the formation is demonstrated in some recent results for single-integrator dynamics, e.g., [51], [52], the global stability is still an open problem for more complex dynamics. To the best of our knowledge, this paper is the first available result on the distance-based formation control of nonlinear system (14). Actually, the proposed method in this paper guarantees the global asymptotic stability of the formation for the single-integrator case, as shown in [37].

**Corollary 3.4.1:** Select
\[
Q_i = \kappa_i \text{diag}[q_{ij}, q_{ik}, q_{il}, q_{iv}, q_{vi}] + I, \quad i \geq 4 \tag{61}
\]
where
\[
\kappa_i = \frac{V_i^* - V_i}{V_i^* + V_i}, \tag{62}
\]
and \( V_i^* \) and \( V_i \) are desired and actual signed volumes of the tetrahedron between the clique \((i, j, l, k)\), and
\[
q_{im} = \nu_{im} + \mu_{im}, \quad m \in \{j, l, k\}, \tag{63}
\]
where \( \nu_{im} > 0 \) is a constant, and \( \mu_{im} \) is a positive barrier multiplier defined by
\[
\mu_{im} = \left( \frac{d_{im}^*}{d_{im} - r_{d_{im}}} \right)^\epsilon, \tag{64}
\]
for suitable \( \epsilon \geq 1 \). The scalar \( r_{d_{im}} \) is the safe distance between a pair of agents that prevents collision and \( q_{iv}, q_{vi} \) are the appropriate weighting matrices. By using the weighting factor (61), the proposed SDRE control law (58) guarantees an inter-agent collision avoidance and prevents the flip-ambiguity of the directed, distance-based formation.

**Proof:** The proof is similar to the proof of the Corollary 3.3.1, and thus, is omitted here. \(\square\)

**Remark 6:** Even though the stability results of the proposed method are local, extensive simulation results showed that the region of attraction (ROA) includes reflected configurations. In fact, by proper selection of SDC matrices, the ROC can be enlarged. There are methods in literature to estimate the ROA of the controller [53]. Therefore, using the proposed reflection prevention method has both practical and theoretical significance.

**IV. Simulation Results**

This section presents simulation results of the proposed distance-based formation control over directed trilateral Laman graphs. Note that the analytical solution of the SDRE equation is extremely challenging except for very simple scalar systems, as shown in [42], [43]. However, there are very effective numerical methods for solving SDRE proposed in the literature [43].

Figure 3(a) shows the desired, distance-based formation in 3-D space. The desired configuration is in the form of a directed trilateral Laman graph. Note that the leader is not responsible for any edges; thus, it is stationary. The simulation results for a set of heterogeneous, nonlinear agents are provided. Since this article studies multi-agents with heterogeneous, nonlinear dynamics, we chose a set of complex, nonlinear models that satisfy the SDRE feasibility conditions to verify the theoretical results. The agents’ models are as follows.

The leader is placed at \( p_1 = [20, 20, 30] \). Agent #2 (first follower) is modeled as
\[
\begin{aligned}
\dot{x}_2 &= v_{2x} \\
\dot{y}_2 &= v_{2y} \\
\dot{z}_2 &= v_{2z}
\end{aligned}
\]
and the initial position is \( p_2 = [50, 20, 30] \). Agent #3 is modeled as
\[
\begin{aligned}
\dot{x}_3 &= v_{3x} \\
\dot{y}_3 &= v_{3y} \\
\dot{z}_3 &= v_{3z}
\end{aligned}
\]
and the initial position is \( p_3 = [50, -30, 20] \). Agent #4 is modeled as
\[
\begin{aligned}
\dot{x}_4 &= v_{4x} \\
\dot{y}_4 &= v_{4y} \\
\dot{z}_4 &= v_{4z}
\end{aligned}
\]
and the initial position is \( p_4 = [-10, 20, -10] \).

Figure 4 shows trajectories for a set of \( N = 4 \) agents in 3-D space using the proposed controller. Figure 5 shows the formation errors corresponding to the formation shown in Figure 4.
Next, we added agent #5 to the desired formation using the directed vertex addition operation. The new desired formation is shown in Figure 1(b). Agent #5 is modeled as

\begin{align*}
\dot{x}_5 &= v_5 \quad \dot{v}_x = v_5 v_5 + u_5 \\
\dot{y}_5 &= v_5 \quad \dot{v}_y = v_5 + u_5 \\
\dot{z}_5 &= v_5 \quad \dot{v}_z = v_5 v_5 + u_5,
\end{align*}

(68)

and the initial position of agent #5 was selected as \( p_5 = [10, 60, 10]^T \).

To validate the results of the proposed reflection and collision prevention method, we ran a set of simulations. We first simulated the proposed controller without signed volume constraints. The results show the flip-ambiguity of the formation. Afterward, the signed area constraints are added. Figure 6 (top) shows the controller performance without the signed volume constraints where \( Q_5 \) matrix was selected as constant identity matrix. All other parameters remained unchanged. It shows that agent #5 moved to the reflected position, making the configuration flip-ambiguous. Figure 6 (bottom) shows the simulation results while the proposed weighting matrix in Corollary 3.4.1 is utilized. The simulation results show that the proposed controller, with signed volume and collision avoidance constraints, prevented the convergence of agent #5 to the reflected position and prevented collision with other agents.

V. CONCLUSIONS

This paper studies the distance-based formation of a set of heterogeneous, affine, nonlinear agents. We presented a novel distributed, distance-based formation control for a group of heterogeneous nonlinear agents. We considered the distance-based formation problem over the directed trilateral Laman graphs. In directed topologies, only one of the neighboring pairs is designated to preserve the desired distance. We formulated a problem for affine, nonlinear agent models and provided a distributed solution using the stability theory of interconnected systems and mathematical induction. We proposed a state-dependent Riccati equation (SDRE) method-based controller that assures the desired formation’s asymptotic stability. Furthermore, to solve the problem of reflection of desired formations, the proposed control law uses signed area and volume constraints to prevent the formation flip-ambiguity. The proposed method also guarantees collision avoidance between neighboring agents.

REFERENCES


