DERIVATION AND IMPLICATIONS OF THE NAVY SHOCK ANALYSIS METHOD

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The working equations of the Belsheim-O'Hara dynamic analysis method are developed nonrigorously, with a minimum of mathematics, on the basis of physical and logical reasoning.

The expressions for modal equivalent (or effective) mass, equivalent displacement, and shock response of system masses are derived here using no postulates other than (a) a definition of dynamic equivalence between vibratory modes of different systems, (b) the assumption that system modes of equal frequencies and equal equivalent masses will respond to the same shock condition at levels consistent with dynamic equivalence, and (c) the requirement that the responses of arbitrary single-degree-of-freedom systems be known, as from test data. The concept of equivalent displacement ratios is preferred to that of the more commonly used participation factors; the former are shown to be fixed quantities for a given system mode, whereas the latter are not.

This approach reveals clearly that the method is not predicated upon any particular shape or nature of the shock input to the system (such as a step velocity input) and can yield no information about the initial phase relationships between the various modes.

Since in a real system the responses also decay rapidly, it follows that there is an inherent difficulty, in this method, of summing the effects of the various modes without being overly conservative.

INTRODUCTION

In the analysis of complex structures subjected to shock excitation, particularly in naval shipboard equipment, the so-called dynamic analysis method of Belsheim and O'Hara [1] must often be used. The procedure involves, as a first step, reducing the actual structure to a lumped-parameter model of a finite number of degrees of freedom and calculating by any appropriate method the normal modes of vibration of this system, as represented by their frequencies and mode shapes (normalized displacements of the masses). From this information two suitably defined quantities, equivalent mass M_e and participation factor P, must be computed for each mode. Next, a shock velocity input V_s, the design spectrum value, is prescribed for each mode and is a function of modal frequency ω and equivalent mass M_e. These inputs are empirical values which have been deduced from the observed responses of single-degree-of-freedom oscillators of various masses and frequencies, placed in actual vessels subjected to shock tests.

The final expression for the equivalent static force on mass k, in the particular mode, can be stated as follows:

$$F_k = M_k X_k P V_s \omega . \tag{1}$$

This expression is not derived in Ref. [1], although undoubtedly rigorous derivations can be found elsewhere.

This paper shows how Eq. (1) can be developed nonrigorously, using a minimum of mathematics and a maximum of physical-logical reasoning, with the thought that this may help to clarify what assumptions are implicit in the method and what the physical significance is of some of the terms involved.

PHYSICAL ASSUMPTIONS AND LOGICAL ARGUMENT

In the more usual rigorous approach to a normal-mode analysis, the complete response of a multi-degree-of-freedom system is analyzed for some assumed complete base motion or excitation. Then that response is broken down into normal modes, and the modal equivalent mass is deduced by noting the similarity between the form of the modal response equation and the single-degree-of-freedom equation. In this nonrigorous approach, we shall regard each mode as an entirely independent entity from the beginning. We shall make no assumptions concerning the total system excitation, but merely assume that each mode responds analogously to a dynamically equivalent single-degree-of-freedom system subjected to the same shock environment.

First let us consider two systems in steady-state vibration. We shall define as "dynamically equivalent" two systems vibrating at the same frequency, if the net momentum and the total energy of the two systems are the same. Since this statement is quite general, each system may represent a multi-degree-of-freedom system vibrating in a particular mode, or a single-degree-of-freedom system vibrating at its natural frequency. (The equality of net momentums also implies that the two systems apply equal forces to the "fixed base" at their points of attachment.) We shall show that, if one of the systems is indeed a single-mass-spring oscillator, then dynamic equivalence requires a definite relationship between the mass of the latter, which we can now refer to as equivalent mass M_e, and the masses and mode shape of the other system. We shall also show that dynamic equivalence requires a definite relationship between the vibratory displacement amplitude of the single-mass system, which we can now refer to as equivalent displacement X_e, and the masses, mode shape, and excitation level of the other system.

It should be clearly noted — though one would hardly expect anything otherwise — that the equivalent mass is independent of the excitation level of the multimass system, whereas the equivalent displacement is proportional to the excitation level of the multimass system. This implies, however, that the ratio of the displacement of any one of the masses in the multimass system to the equivalent displacement is once again independent of the excitation level, and, like the equivalent mass, can be considered a property of the given system vibrating in the given mode. Let us call such ratios equivalent displacement ratios.

To apply the foregoing to shock analysis, we must make the physical assumption that if two systems of the same natural frequency and the same equivalent mass are subjected to the same transient disturbance at their base, their steady-state responses will be such that they are dynamically equivalent as defined previously. Hence, the vibratory amplitudes of the two systems can be related to each other by means of the equivalent displacement ratios.

Furthermore, we assume that the total response of a multi-degree-of-freedom system to a shock input is a superposition of the independent responses in each normal mode and that the response of each mode can be computed by means of the previous argument from the response of a single-mass oscillator of the same frequency and equivalent mass. The responses of single-degree-of-freedom oscillators to typical shock inputs, as a function of their mass and frequency, are assumed known from empirical data.

Note what assumptions do not have to be made; in particular, the nature of the shock disturbance, i.e., the displacement-time history of the base, is not prescribed or defined in any way, and, therefore, it also follows that no phase relationships can be adduced for the responses in the various modes. Furthermore, in real systems with damping the responses represent maximum rather than steady-state vibratory amplitudes.

MATHEMATICAL DERIVATION

Consider a unidirectional, multi-degree-of-freedom system vibrating in one of its normal modes, the angular frequency of which is ω_a. All displacements are assumed to occur in the x direction. The subscript a identifies the particular mode under consideration; the subscript i identifies the particular mass.

The net or algebraic momentum K_a associated with a mode of harmonic oscillation in a multi-degree-of-freedom system is

$$K_a = \omega_a \sum_i M_i X_{ia} . \tag{2}$$

The corresponding kinetic energy is

$$E_a = \frac{1}{2}\omega_a^2 \sum_i M_i X_{ia}^2 . \tag{3}$$

The X_{ia} values here represent any arbitrary set of displacement amplitudes. Their relative magnitudes are of course determined by the mode shape or eigenvector of the mode under consideration.

If the mass and displacement, respectively, of the dynamically equivalent single-degree-system are M_{ea} and X_{ea}, its momentum and energy are expressed by

$$K_{ea} = \omega_a M_{ea} X_{ea} \tag{4}$$

and

$$E_{ea} = \frac{1}{2}\omega_a^2 M_{ea} X_{ea}^2 . \tag{5}$$

Since by "dynamic equivalence" we mean that $K_{ea} = K_a$ and $E_{ea} = E_a$, we can equate Eqs. (2) with (4), and (3) with (5) to obtain

$$M_{ea} X_{ea} = \sum_i M_i X_{ia} \tag{6}$$

and

$$M_{ea} X_{ea}^2 = \sum_i M_i X_{ia}^2 . \tag{7}$$

Dividing Eq. (7) by (6) we obtain the equivalent displacement, i.e., the displacement amplitude of the dynamically equivalent system:

$$X_{ea} = \frac{\sum_i M_i X_{ia}^2}{\sum_i M_i X_{ia}} . \tag{8}$$

Substituting Eq. (8) back into (6) results in the expression for the equivalent mass:

$$M_{ea} = \frac{\left[\sum_i M_i X_{ia}\right]^2}{\sum_i M_i X_{ia}^2} . \tag{9}$$

It is important now to note that equivalent mass M_{ea} is independent of the absolute values of the X_{ia}'s, as long as they bear the proper relationships to one another as prescribed by the mode shape of the mode considered. Any arbitrary or arbitrarily normalized set of displacement amplitudes describing that mode can be used to compute M_{ea}. On the other hand, equivalent displacement X_{ea} does depend on the excitation level; if all X_{ia}'s are multiplied by some given factor, the corresponding X_{ea} is also multiplied by that factor. This means, simply, that if the amplitudes of a multi-degree-of-freedom system are, say, doubled, then the amplitude required for dynamic equivalence, in the equivalent-mass single-degree system, is also doubled.

We can now draw our attention to any particular mass k in the system, and define its equivalent displacement ratio R_{ka} as follows:

$$R_{ka} \equiv \frac{X_{ka}}{X_{ea}} = \frac{X_{ka} \sum_i M_i X_{ia}}{\sum_i M_i X_{ia}^2} . \tag{10}$$

The R_{ka} values, which represent the ratios of the displacements of the individual masses in the multimass system to the displacement of the dynamically equivalent single-mass system, are once again functions of the masses and mode shape only and can be computed from arbitrarily normalized X_{ia} values.

We are now ready to make use of the foregoing relationships to deduce the response of our multimass system under a typical shock environment. We suppose that enough single-degree systems have been observed under typical shock environments to predict from these measurements the displacement response of any single-degree-of-freedom system of arbitrary mass and frequency.

Therefore, for each normal mode a, we can compute the frequency ω_a and equivalent mass M_{ea}, and then from the empirical data predict what actual shock displacement would be recorded by a single-mass system of that frequency and mass. Let this be denoted by X_{eas}. We then assert that the response of the multimass system under the same environment would be such as to satisfy the complete conditions of dynamic equivalence as previously defined. This implies that ratios of the shock displacements of the multimass system to that of the equivalent single-mass system would once again be given by the values R_{ka} which have been computed. Thus, the shock displacement for mass k, in mode a, is given by:

$$X_{kas} = R_{ka} X_{eas} . \tag{11}$$

The inertia force on mass k, which we intend to use as an equivalent static loading in the computation of shock stresses, is

$$F_{ka} = M_k X_{kas} \omega_a^2 = M_k R_{ka} X_{eas} \omega_a^2 . \tag{12}$$

If the shock response of the equivalent system is specified in terms of the velocity amplitude V_{eas} rather than displacement X_{eas}, we can substitute the relationship $X_{eas} = V_{eas}/\omega_a$ to obtain

$$F_{ka} = M_k R_{ka} V_{eas} \omega_a . \tag{13}$$

The physical significance of Eq. (13) is evident: the force is the product of the mass (M_k) being considered, the shock acceleration of the equivalent single-mass system ($V_{eas}\omega_a$), and the ratio R_{ka} which relates the acceleration, velocity, or displacement of mass k to that of the dynamically equivalent single-mass system. Equation (13) can be shown to be identical to Eq. (1), and it will be seen that Belsheim and O'Hara's participation factor P_a is nothing more than the reciprocal of our equivalent displacement. Hence $R_{ka} = X_{ka} P_a$, from Eq. (10). The writer feels that the ratios R_{ka}, which are invariants for a given system and mode, are more meaningful than a participation factor whose magnitude depends on how the displacement values have been normalized. In fact, the R_{ka} values could be regarded as rationally normalized displacements for the particular system and mode. Note that the equivalent mass can be expressed in terms of the R_{ka} values simply as follows:

$$M_{ea} = \sum_i M_i R_{ia} . \qquad (14)$$

This can readily be seen by comparing Eqs. (9) and (10).

The physical arguments which we have advanced make it easy to see how the equations must be modified when a multidirectional system (one which, when shock excited in a given direction, will exhibit responses in all directions) is considered. Due attention must then be given to the fact that momentum is a vector quantity whereas energy is scalar. The component of momentum which is significant is that in the shock direction; therefore, in carrying out the summation of Eq. (2), only the displacements in the shock direction should be included. On the other hand, in computing the system energy by Eq. (3), the total energy as determined by all response directions must be included.

SUMMARY AND CONCLUSIONS

The expressions for modal equivalent (or effective) mass, participation factor, and shock response of system masses have been derived here using no postulates other than (a) a definition of dynamic equivalence between vibratory modes of different modes of different systems, (b) the assumption that system modes of equal frequencies and equal equivalent masses will respond to the same shock condition at levels consistent with dynamic equivalence, and (c) the requirement that the responses of single-degree systems are known, as from test data. The concept of equivalent displacement ratios is preferred to that of participation factors. Implicit in this approach is the assumption that the modes considered are uncoupled and the response of each is completely unaffected by the existence of other modes.

The term "starting velocity" is sometimes used; this can lead to the erroneous inference that this method is somehow based on the physical assumption of step velocity inputs. This approach shows that the method makes no allegations about the shape or nature of the shock input to the system or to the ship as a whole, or about the initial phase relationship of the modal vibrations.

The prescribed design spectrum values are based on the maximum measured responses of certain instrumented equipment in full-scale and model tests [1, pp. 26-27]. How and when the buildup to the maximum response level occurs will differ for different modes, depending on the relationship of the frequency of vibration to the velocity-time profile of the shock input at the hull or foundation, and on the time required for propagation of the shock or stress waves through the system. The latter is not an insignificant effect, since the duration of the shock pulse is on the order of milliseconds, and in a millisecond a stress wave in steel travels only 16 ft. Thus, design spectrum values obtained from compact equipment close to the hull may not be valid for more extensive or distant systems where the time delay and attenuation inherent in the propagation process would be expected to reduce the maximum response in a given mode. The damping in a complex system could be expected to be greater than in a simple mass-spring oscillator.

There still follows the question of how the effects (i.e., stresses or loads) of the various modes are to be combined, which is one of the outstanding problems in the application of this procedure. Some of the remarks above, together with the known fact that shock-induced vibrations decay rather rapidly from their peak values, will suggest why phase coincidence of many modes at their peak values is most unlikely. Yet the normal-mode method is inherently incapable of deriving, on a theoretical basis, anything but the very conservative upper bound obtained by direct superposition of the maximum loadings in all modes. This is one of its most serious deficiencies. To obtain hopefully more realistic "rules" for combining modes and predicting maximum stresses, recourse must be had to empirical data once again, or, perhaps, in systems with very many degrees of freedom, to statistical methods. Also, it will be found that the predicted maximum stresses can be very sensitive to the manner in which the structure is broken down into discrete masses and flexibilities to form the analytic model. The guidelines proposed to date for dealing with these problems have seemed rather arbitrary, and more work, or more englightenment, in this area is to be encouraged.

REFERENCE

1. R. O. Belsheim and G. J. O'Hara, "Shock Design of Shipboard Equipment - Part I: Dynamic Analysis Method," Bureau of Ships, Navy Department, NAVSHIPS 250-423-30, May 1961.

Voice: Could this method apply to all systems? Could it be extended to very large systems?

Mr. Heymann: The method is no different from the DDAM. It is just one way of seeing how the equations are arrived at. This method does specify the velocity input and frequency for any mode of a large system and equivalent mass. You are making the analogy between one mode of a large system and the response of a single degree of freedom system. I think what I am trying to show is that you can't tell from this analysis what the maximum response is going to be when you superimpose modes. It can't tell you anything about the time interval between the arrival of a shock front at the hull and the maximum response for different single degree of freedom systems. You don't know what the phase relationship is going to be, so you have to look elsewhere for the rationale of some of the modes.

Mr. Forkois (Naval Research Lab.): How do you determine the equivalent rotational modes of vibration from that single mass? You show a three mass system which obviously has some unbalance. How do you determine the equivalent modes on a single mass?

Mr. Heymann: In this case I don't. This is simply a one-dimensional model to show the principle. I have indicated in my paper that the analysis can be extended to three dimensions. I don't know about rotation. If the dynamic equivalence is defined in terms of the energies and momenta, when you have response in three directions, you would only take the sum of the momenta in the shock direction. Since energy is a scalar quantity you would take the energies involved in all three response directions. I think maybe that you could extend that to include rotational response, but I really haven't thought it out.

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