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Cardinality-based inference control in data cubes *

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This paper addresses the inference problem in on-line analytical processing (OLAP) systems. The inference problem occurs when the exact values of sensitive attributes can be determined through answers to OLAP queries. Most existing inference control methods are computationally expensive for OLAP systems, because they ignore the special structures of OLAP queries. By exploiting such structures, we derive cardinality-based sufficient conditions for safe OLAP data cubes. Specifically, data cubes are safe from inferences if their core cuboids are dense enough, in the sense that the number of known values is under a tight bound. We then apply the sufficient conditions on the basis of a three-tier inference control model. The model introduces an aggregation tier between data and queries. The aggregation tier represents a collection of safe data cubes that are pre-computed over a partition of the data using the proposed sufficient conditions. The aggregation tier is then used to provide users with inference-free queries. Our approach mitigates the performance penalty of inference control, because partitioning the data yields smaller input to inference control algorithms, pre-computing the aggregation tier reduces on-line delay, and using cardinality-based conditions guarantees linear-time complexity.

1. Introduction

Decision support systems such as On-line Analytical Processing (OLAP) are becoming increasingly important in industry. OLAP systems assist users in exploring trends and patterns in large amount of data by providing them with interactive results of statistical aggregations. Contrary to this initial objective, inappropriate disclosure of sensitive data stored in the underlying data warehouses results in the breach of individual's privacy and jeopardizes the organization's interest. It is well known that access control alone is insufficient in controlling information disclosure, because information not released directly may be inferred indirectly by manipulating legitimate queries about aggregated information, which is known as the *inference control* problem in databases. OLAP systems are especially vulnerable to such unwanted inferences, because of the aggregations used in OLAP queries. Providing inference-free answers to data cube style OLAP queries while not adversely impacting the response time of an OLAP system is the subject of this paper.

The inference problem has been investigated since 70's with many inference control methods proposed, especially for statistical databases. However, most of those

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methods become computationally infeasible if directly applied to OLAP systems. Most OLAP applications demand instant responses to interactive queries, although those queries usually aggregate a large amount of data [19,27]. Most existing inference control algorithms have run times proportional to the size of the queries or data set. Furthermore, those algorithms are enforced only after queries arrive, which makes it difficult to shift the computational complexity to off-line processing. The performance penalty renders them impractical for OLAP systems.

It has been pointed out that one way to make inference control practical is to restrict users to statistically meaningful queries [9]. Unlike in statistical databases where ad-hoc queries are quite common, queries having rich structures are usually more meaningful to OLAP users. In this paper we consider queries composed of the data cube operator introduced in [26]. The data cube operator generalizes many common OLAP operations such as group-by, cross-tab and sub-totals. As we shall show in this paper, efficient inference control is possible by exploiting special structures of the data cube operator.

Table 1 shows a data cube represented by four *cross tabulations* [26]. Each cross tabulation corresponds to a quarter of the year. The two *dimensions* are month and employee. Each *internal cell* of a cross tabulation contains the monthly commission of an employee. Assume that individual commissions are sensitive and should be hidden from users, and therefore have been replaced with "?". An empty internal cell

Table 1
An example data cube

Quarter	Month/Employee	Alice	Bob	Jim	Mary	Sub total
1	January	?	?	?	?	5500
	February	?	?	?	?	5500
	March	?	?	?	?	5500
	Sub total	3000	3000	4500	6000	
2	April	?	?	?	?	6100
	May	?		?	?	6100
	June	?	?	?	?	4100
	Sub total	4500	3300	4500	4000	
3	July	?	?	?	?	6100
	August	?	?	?	?	6100
	September				?	2000
	Sub total	3500	2200	2500	6000	
4	October	?	?	?		7100
	November		?	?		4100
	December	?			?	4100
*	Bonus	?			?	6000
	Sub total	7000	4300	3000	7000	

indicates that the employee is on leave and does not have a commission in the month. Each *external cell* of a cross tabulation contains either the subtotal commission of the four employees in a month, or the subtotal commission of an employee in a quarter.

Suppose that a malicious user Mallory wants to learn the hidden values represented by "?" in Table 1. Mallory can obtain knowledge about the table through two different channels: queries and the external knowledge (that is, the knowledge obtained through channels other than queries [17]). Firstly, she can ask legitimate queries about the subtotals in Table 1. Secondly, she knows the positions of empty cells in the table, because she works in the same company and knows which employee is on leave in which month. Now the inference problem occurs if any of the hidden values represented by "?" can be determined by Mallory. The following observations are relevant in this respect:

1. In the first and second quarter, no hidden value can be determined by Mallory, because infinitely many choices exist for any of them with all the subtotals satisfied. To illustrate, consider the first quarter. Suppose Mallory can determine a unique value x_1 for Alice's commission in January. Then this value should not change with the choices of other values. Now let S be a set of values satisfying the subtotals, in which Alice's commission in February is x_2, and Bob's commission in January and February are y_1 and y_2 respectively, as shown in the upper cross tabulation in Table 2. Now we can derive a different set of values S' from S by replacing x_1 with $x_1 - 100$, x_2 with $x_2 + 100$, y_1 with $y_1 + 100$ and y_2 with $y_2 - 100$, as shown in the lower tabulation in table 2. S' also satisfies all the subtotals in quarter one, which implies that Alice's commission in January cannot be determined by Mallory.
2. For the third quarter, Mary's commission in September can be determined by Mallory as 2000, equal to the subtotal in September, because Mallory knows from external knowledge that Mary is the only employee who works and draws a commission in that month.
3. For the fourth quarter, no hidden value can be determined in the similar way as in the third quarter, because all the subtotals in the fourth quarter are calculated

Table 2
Example of a data cube safe from inference

Quarter	Month / Employee	Alice	Bob	Jim	Mary	Sub total
1	January	x_1	y_1	?	?	5500
	February	x_2	y_2	?	?	5500
	March	?	?	?	?	5500
	Sub total	3000	3000	4500	6000	
1	January	$x_1 - 100$	$y_1 + 100$	?	?	5500
	February	$x_2 + 100$	$y_2 - 100$	?	?	5500
	March	?	?	?	?	5500
	Sub total	3000	3000	4500	6000	

Table 3
Example of a data cube with inference

Quarter	Month/Employee	Alice	Bob	Jim	Mary	Sub total
4	October	x_1	y_1	z_1		7100
	November		y_2	z_2		4100
	December	?			?	4100
*	Bonus	?			?	6000
	Sub total	7000	4300	3000	7000	

from at least two hidden values. However, the following inference is possible. As shown in Table 3, let x_1 be Alice's commission in October; let y_1 and y_2 be Bob's commission in October and November respectively; and let z_1 and z_2 be Jim's commission in October and November respectively. Mallory asks four legitimate queries:

(a) $x_1 + y_1 + z_1 = 7100$ (The subtotal commission in October)
(b) $y_2 + z_2 = 4100$ (The subtotal commission in November)
(c) $y_1 + y_2 = 4300$ (Bob's total commission in this quarter)
(d) $z_1 + z_2 = 3000$ (Jim's total commission in this quarter)

By adding both sides of the first two equations (a) and (b), and then subtracting from the result the last two equations (c) and (d), Mallory gets $x_1 = 3900$, which is Alice's commission in October.

To generalize the above example, unknown variables and their aggregations are used to represent commissions and subtotals of commissions, respectively. Empty cells are used to represent the values that users already learn from external knowledge. A data cube is compromised if the value of any unknown variables can be uniquely determined from the aggregations and the empty cells. In order to efficiently determine if a given data cube is compromised, we derive cardinality-based sufficient conditions for safe data cubes. Specifically, we show that any data cube is safe from compromises if the number of empty cells is below a given bound. The bound is tight because any data cube having more empty cells than the bound is subject to compromises. We apply the sufficient conditions on the basis of a three-tier inference control model. Besides data and queries, the model introduces a new tier, which represents a collection of safe data cubes. Using the proposed sufficient conditions, the safe data cubes are first computed over a partition of the data, and then used to provide users with inference-free queries. The overhead of inference control in terms of response time is mitigated by such an approach, because partitioning the data yields smaller input to inference control algorithms, pre-computing the aggregation tier reduces on-line delay, and using cardinality-based sufficient conditions for computation guarantees linear-time complexity. In this paper we elaborate and justify the preliminary results given in [41], and address various implementation issues and improvements to the algorithm.

The rest of the paper is organized as follows. Section 2 reviews existing inference control methods proposed in statistical databases. Section 3 formalizes sum-only data cubes and the compromises. Section 4 proves cardinality-based sufficient conditions for safe data cubes. Section 5 proposes a three tier inference control model. Section 6 integrates the sufficient conditions of safe data cubes into an inference control algorithm on the basis of the three-tier model. Section 7 concludes the paper.

2. Related work

Inference control has been extensively studied in statistical databases as surveyed in [1,17,18]. Inference control methods proposed in statistical databases are usually classified into two main categories: *restriction based* techniques and *perturbation based* techniques. Restriction based techniques [25] include restricting the size of a *query set* (that is, the values aggregated by a single query), overlap control [21] in query sets, auditing all queries in order to determine when inferences are possible [6,10,13,29,33], suppressing sensitive data in released statistical tables [14,22], partitioning data into blocks and treating each block as a single value in answering queries [11,12,34]. Perturbation based technique includes adding noise to source data [39], outputs [5], database structure [36], or size of query sets (that is, answering queries using sampled data) [16]. Some variations of the inference problem have been studied lately, such as the inference caused by arithmetic constraints [7,8], the inference of approximate values instead of exact values [32] and the inference of small intervals enclosing exact values [30]. Finally, techniques similar to suppression and partitioning are being developed to protect the sensitive information in released census data [14,22,34,37,42,43].

The inference control methods proposed for statistical databases generally sacrifice efficiency for the capability of dealing with ad-hoc queries. However, this is usually not desirable in OLAP systems where instant responses to queries take priority over the generality of answerable queries. Hence most inference control methods proposed for statistical databases become computationally expensive if directly applied to OLAP systems. As an example, Audit Expert [13] models inference problem with a linear system of equations $Ax = b$ and detects the occurrence of inferences after queries arrive by transforming the m by n matrix A (corresponding to m queries and n sensitive values) to its reduced row echelon form. The transformation has a well-known complexity of $O(m^2n)$. This is prohibitive for OLAP systems, because m and n can be as large as a million in those systems.

Our work shares with [21] the similar motivation of controlling inferences with cardinalities of data and queries. Dobkin et al. give an lower bound on the number of ad-hoc sum queries that compromise sensitive values [21]. Under the assumptions that each query sums exactly k sensitive values and no two queries sum more than r values in common, Dobkin et al. prove that at least $1 + (k - 1)/r$ sum queries are required to compromise any sensitive value. Although this result certainly applies to

OLAP queries, it does not take into account the special structures of those queries. The restricted form of queries imply better results (that is, tighter bounds). In this paper we pursue such better results by exploiting the special structures of OLAP queries.

Recently a variation of the inference control problem, namely, *privacy preserving data mining*, has drawn considerable attention as seen in [2,3,20,23,24,35,40]. Similar to perturbation-based techniques used in statistical databases, random perturbation is used to hide sensitive values. The statistical distribution model of those values is preserved during perturbations, such that data mining results such as classifications or association rules can be obtained from the perturbed data. In contrast to data mining, the responsibility of OLAP systems is to provide users with the means (that is, precise answers to OLAP queries), instead of the results (the discovered trends or patterns). Preserving the distribution model of sensitive values is insufficient for OLAP applications, because the precision of answers to OLAP queries is not guaranteed. To our understanding, introducing enough noises to protect sensitive data while avoiding bias or inconsistence in the answers to OLAP queries is still an open problem. Our work is not based on perturbation, but based on the restriction of unsafe queries. Although all queries are not answered as required by inference control, the answers are always precise.

In [41] we present some preliminary results including the cardinality-based sufficient conditions for safe data cubes. In this paper we expand those results with more detailed justification and implementation consideration. We relax the assumptions about missing tuples [41], such that our work can deal with general external knowledge of sensitive values. We clarify the sufficient conditions by providing both underlying intuitions and formal justifications. We also address implementation issues such as incrementally updating the results of inference control algorithms once data are inserted or deleted.

3. Preliminaries

This section defines the basic notations. First in Section 3.1 we define the components of *data cubes*. Next in Section 3.2 we define *aggregation matrix* and *compromisability*. Finally in Section 3.3 we explain the choices made in our design.

3.1. Data cube

A data cube is composed of a *core cuboid* and a collection of *aggregation cuobids*. Corresponding to the discussion in Section 1, the core cuboid essentially captures the notion of sensitive values and empty cells, while the aggregation cuboids correspond to the subtotals.

Definition 1 formalizes the concepts related to the core cuboid. We use closed integer intervals for *dimensions*. The Cartesian product of k dimensions form the *full*

core cuboid. We call each vector in the full core cuboid a *tuple*. Hence the full core cuboid is simply the collection of all possible tuples that can be formed by the k dimensions. A *core cuboid* needs not include all possible tuples, but has to satisfy the property that any integer from any dimension must appear in at least one tuple in the core cuboid. This property ensures that any given core cuboid corresponds to a unique full core cuboid (as well as the k dimensions). The tuples not appearing in a core cuboid are said to be *missing*. By fixing a value for one of the k dimension, we can select tuples from a core cuboid to form a *slice* on that dimension. A slice is *full* if no tuple is missing from it. Clearly a slice can be full even if the core cuboid cannot.

Definition 1 (Core Cuboids)**.**

1. Given k ($k > 1$) integers $d_1, d_2, \ldots, d_k$ satisfying $d_i > 1$ for all $1 \leqslant i \leqslant k$, the ith **dimension**, denoted by D_i, is the closed integer interval $[1, d_i]$.
2. The k-dimensional **full core cuboid**, denoted as C_{full}, is the Cartesian product $\Pi_{i=1}^{k} D_i$. Each vector $t \in C_{full}$ is referred to as a **tuple**.
3. A k-dimensional **core cuboid** is any $C_{core} \subseteq C_{full}$ satisfying that $\forall i \in [1, k]\ \forall x \in D_i, \exists t \in C_{core}\ t[i] = x$. Notice that we use notation $t[i]$ for the ith element of vector t from now on.
4. t is **missing** if $t \in C_{full} \setminus C_{core}$.
5. The jth ($j \in D_i$) **slice** of C_{core} on the ith dimension, denoted by $P_i(C_{core}, j)$, is the set $\{t : t \in C_{core}, t[i] = j\}$. $P_i(C_{core}, j)$ is **full** if $P_i(C_{core}, j) = P_i(C_{full}, j)$.

Table 5 gives an example to the concepts in Definition 1. The example describes a two dimensional core cuboid, with both dimensions as $[1, 4]$ (we use normal font for the first dimension and italic for the second for clarity purpose). The upper left tabulation shows the full core cuboid $[1, 4] \times [1, 4]$. The upper right tabulation shows a core cuboid with nine tuples. As required by Definition 1, any integer from any dimension appears in at least one of the nine tuples. Without this restriction, one could have argued that the first dimension is $[1, 5]$ instead of $[1, 4]$, and hence the full core cuboid contains $5 \times 4 = 20$ tuples instead of 16. This situation must be avoided in order to use the cardinalities. The left lower cross tabulation shows the seven missing tuples. The right lower cross tabulation shows the first slice on the first dimension. The core cuboid in Table 5 is analogous to the fourth cross tabulation in Table 1 in the following sense. The employee names and months in Table 1 are abstracted as integers from one to four. The hidden values represented in "?" in Table 1 become integer vectors, and the empty cells correspond to missing tuples. The commissions in October are now called the first slice on the first dimension.

Definition 2 formalizes the concepts related to aggregation cuboids. The **-value* is a special value unequal to any integers. *-values appearing in a vector are called **-elements*. Adding the *-value into any dimension yields an *augmented dimension*.

The Cartesian product of the k augmented dimensions includes all the vectors whose elements are either integers (from the dimensions), or *-elements. A vector with no *-element is simply a tuple, and a vector with j many *-elements are called a j-* *aggregation vector*. The collection of all the j-* aggregation vectors having *-elements at the same positions is called a j-* *aggregation cuboid*. Any aggregation vector can be used to *match* tuples in the core cuboid, with a *-element matching any integers and an integer matching only itself. The matched tuples form the *aggregation set* of the aggregation vector. A *data cube* is a pair of the core cuboid and the collection of all aggregation cuboids.

Definition 2 (Aggregation Cuboids and Data Cubes)**.**

1. A ***-value**, denoted as *, is a special value unequal to any positive integer. A *-value appearing in a vector is called a ***-element**.
2. Given the ith dimension D_i, the ith **augmented dimension**, denoted as D_i^*, is $[1, d_i] \cup \{*\}$.
3. A j-* **aggregation cuboid** is the maximal subset of the Cartesian product $\Pi_{i=1}^{k} D_i^*$ satisfying,

   (a) $\forall i \in [1,k]\ \forall t, t' \in C_{aggr}, t[i] = * \text{ iff } t'[i] = *$; and
   (b) $\forall t \in C_{aggr}, \mid \{i : t[i] = *\} \mid = j$.

   Each vector $t \in C_{aggr}$ is called a j-* **aggregation vector**.
4. Given C_{core}, the **aggregation set** of any aggregation vector t_{aggr}, denoted as $Q\,set(t_{aggr})$, is the set of tuples: $\{t : t \in C_{core}, \forall i \in [1,k],\ t_{aggr}[i] \neq * \Rightarrow t[i] = t_{aggr}[i]\}$. The aggregation set of a set of aggregation vectors C, denoted as $Q\,set(C)$, is the set of tuples: $\cup_{t_{aggr} \in C} Q\,set(t)$.
5. A k-dimensional **data cube** is a pair $< C_{core}, S_{all} >$, where C_{core} is any core cuboid with dimensions $D_1, D_2, \ldots, D_k$ and S_{all} is the set of all aggregation cuboids with dimensions $D_1, D_2, \ldots, D_k$.

Table 4 gives an example to the concepts in Definition 2. The two augmented dimensions are both $\{1, 2, 3, 4, *\}$. As shown in the left cross tabulation in Table 4, the Cartesian product of the two augmented dimensions yields 25 vectors. There are two 1-* aggregation cuboids: $\{(1,*), (2,*), (3,*), (4,*)\}$ and $\{*,1), (*,2), (*,3), (*,4)\}$, and one 2-* aggregation cuboids $(*,*)$. The right cross tabulation in Table 4 shows a 2-dimensional data cube. Notice that the two augmented dimensions are included for the purpose of clarity, and they are not a part of the data cube. As an example of aggregation set, the tuples composing the aggregation set of $(1,*)$ are underlined. The 1-* aggregation vectors in Table 4 abstract the subtotals in the fourth cross tabulation in Table 1, and the 2-* aggregation vector corresponds to the total commission in the fourth quarter.

Table 4
Illustration of aggregation cuboids and data cube

Cartesian Product {1, 2, 3, 4,*} × {1, 2, 3, 4,*}					
	1	2	3	4	*
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,*)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,*)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,*)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,*)
*	(*,1)	(*,2)	(*,3)	(*,4)	(*,*)

A data cube

	1	2	3	4	*
1	(1,1)	(1,2)	(1,3)		(1,*)
2		(2,2)	(2,3)		(2,*)
3	(3,1)			(3,4)	(3,*)
4	(4,1)			(4,4)	(4,*)
*	(*,1)	(*,2)	(*,3)	(*,4)	(*,*)

3.2. Compromisability

We first define *aggregation matrix*, and then formalize *compromisability*. In order to characterize a data cube, it suffices to know which tuple is a member of the aggregation set of which aggregation vector. Aggregation matrix captures this membership relation in a concise manner.

In order to fix notation, we use the following conventions in our further discussions of sets of vectors. Whenever applicable, we assume the members of a set are sorted according to the orders stated below:

1. Tuples in a core cuboid (or its subset) and aggregation vectors in an aggregation cuboid (or its subset) are in dictionary order (by saying so, we are treating vectors as strings with the leftmost element the most significant). For example, the core cuboid C_{core} in Table 5 is sorted as $\{(1,1),(1,2),(1,3),(2,2),(2,3),(3,1),(3,4),(4,1),(4,4)\}$.
2. Aggregation cuboids in S_{all} or its subsets are sorted first in ascending order according to the number of *-elements in their aggregation vectors, and then in descending order on the index of the *-elements. For example, S_{all} shown in Table 4 is sorted as $\{\{(1,*),(2,*),(3,*),(4,*)\},\{(*,1),(*,2),(*,3),(*,4)\},\{(*,*)\}\}$.
3. We use notation $C[i]$ for the ith member of the sorted set C.

Definition 3 formalizes *aggregation matrix*. Suppose the full core cuboid includes n tuples, and we are given m aggregation vectors, then the aggregation matrix of those aggregation vectors is an m by n matrix M. Each element of the aggregation matrix M is either one or zero, and $M[i,j] = 1$ if and only if the jth tuple in

Table 5
Example of a core cuboid with $k = 2$ and $d_1 = d_2 = 4$

The Full Core Cuboid C_{full}

	1	*2*	*3*	*4*
1	(1,*1*)	(1,2)	(1,*3*)	(1,*4*)
2	(2,*1*)	(2,2)	(2,*3*)	(2,*4*)
3	(3,*1*)	(3,2)	(3,*3*)	(3,*4*)
4	(4,*1*)	(4,2)	(4,*3*)	(4,*4*)

A Core Cuboid C_{core}

	1	*2*	*3*	*4*
1	(1,*1*)	(1,2)	(1,*3*)	
2		(2,2)	(2,*3*)	
3	(3,*1*)			(3,*4*)
4	(4,*1*)			(4,*4*)

Missing Tuples $C_{full} \setminus C_{core}$

	1	*2*	*3*	*4*
1				(1,*4*)
2	(2,*1*)			(2,*4*)
3		(3,2)	(3,*3*)	
4		(4,2)	(4,*3*)	

1st Slice on 1st Dimension $P_1(C_{core}, 1)$

	1	*2*	*3*	*4*
1	(1,*1*)	(1,2)	(1,*3*)	
2				
3				
4				

the full core cuboid is included in the aggregation set of the ith aggregation vector. Intuitively, a column of M stands for a tuple, a row for an aggregation vector, and an element of 1 for a matching between them. An element of 0 could mean two things: either the tuple is missing or it is in the core cuboid but does not match the aggregation vector. The aggregation matrix is unique if the above convention of ordering is followed.

Definition 3 (Aggregation Matrix)**.**

1. In a given data cube $< C_{core}, S_{all} >$, suppose $\mid C_{full} \mid = n$ and let C_{aggr} be any set of m aggregation vectors. The **aggregation matrix** of C_{aggr} is the $(m \times n)$ matrix $M_{C_{core},C_{aggr}}$:

$$M_{C_{core},C_{aggr}}[i,j] = \begin{cases} 1, & \text{if } C_{full}[j] \in Q\,set(C_{aggr}[i]); \\ 0, & \text{otherwise.} \end{cases}$$

2. Given a set of sets of aggregation vectors S (for example, S_{all}), $M_{C_{core},S}$ is the row block matrix with the ith row block as the aggregation matrix of the ith set in S. Specially, we use S_1 for the set of all 1-* aggregation cuboids and M_1 for its aggregation matrix, referred to as the **1-* aggregation matrix**.

Table 6 illustrates the concept of aggregation matrix. The cross tabulation shows the same data cube as in Table 4, with the tuples and aggregation vectors indexed with subscripts according to our order convention. For clarity purpose, normal font are used for the indexes of tuples while italic font for those of aggregation vectors. The 1-* aggregation matrix M_1 is shown in the lower part of Table 6. The rows and columns of the matrix are both indexed accordingly. As an example, the first row of M_1 contains three 1 elements, because the first aggregation vector $(1, *)$ has three tuples $(1, 1)$, $(1, 2)$ and $(1, 3)$ in its aggregation set. The fourth column of M_1 is a zero column because the fourth tuple $(1, 4)$ in the full core cuboid is missing from the core cuboid.

Before we formally define compromisability, we first give the underlying intuitions. With the rows of an aggregation matrix M corresponding to aggregation vectors, the elementary row operations on M captures possible inferences users can

Table 6
An example of aggregation matrix

A Data Cube With Tuples and 1-* Aggregation Vectors Indexed

	1	2	3	4	*
1	$(1,1)_1$	$(1,2)_2$	$(1,3)_3$		$(1,*)_{\mathit{1}}$
2		$(2,2)_6$	$(2,3)_7$		$(2,*)_{\mathit{2}}$
3	$(3,1)_9$			$(3,4)_{12}$	$(3,*)_{\mathit{3}}$
4	$(4,1)_{13}$			$(4,4)_{16}$	$(4,*)_{\mathit{4}}$
*	$(*,1)_{\mathit{5}}$	$(*,2)_{\mathit{6}}$	$(*,3)_{\mathit{7}}$	$(*,4)_{\mathit{8}}$	(*,*)

The Aggregation Matrix M_1 With Indexes

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
5	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0
6	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
7	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1

make with the aggregation vectors. For example, in the discussion of Table 1 in Section 1, the addition of two subtotals can be represented by adding two corresponding rows in the aggregation matrix. The compromise of a tuple can then be captured by a sequence of elementary row operations that leads to a *unit row vector* (that is, a row vector having a single 1 elements and all other elements as 0). This unit row vector represents an aggregation of a single tuple, which is compromised. From linear algebra [28], such a sequence of elementary row operations exists for a matrix if and only if the reduced row echelon form (RREF) of the matrix has at least one unit row vector.

Definition 4 formalizes the *compromisability* based on above intuitions. The compromisability is decidable for any given set of aggregation vectors, because the RREF of any matrix is unique and can be obtained through a finite number of elementary row operations. *Trivial compromises* occur when any of the given aggregation vectors already compromises the core cuboid (that is, it aggregates a single tuple). In the absence of trivial compromises, one has to manipulate more than one aggregation vector to compromise any tuple. This is called *non-trivial compromises*.

Definition 4 (Compromisability)**.**

1. In a given data cube $< C_{core}, S_{all} >$, let C_{aggr} be any set of aggregation vectors, C_{aggr} **compromises** C_{core} if there exists at least one unit row vector in the reduced row echelon form (RREF) of $M_{C_{core},C_{aggr}}$.
2. Suppose C_{aggr} compromises C_{core}. We say C_{aggr} **trivially compromises** C_{core} if there exists at least one unit row vector in $M_{C_{core},C_{aggr}}$, and C_{aggr} **non-trivially compromises** C_{core}, otherwise. We say that the ith tuple is compromised, if the RREF of $M_{C_{core},C_{aggr}}$ contains a unit row vector whose ith element is 1.

Table 7 gives an example of compromisability. The matrix in Table 7 is the RREF of the aggregation matrix M_1 in Table 6. The first row of M_1 is a unit row vector.

Table 7

The reduced row echelon form of M_1 in Table 6

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	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	−1
6	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
7	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Hence in Table 6, the set of 1-* aggregation cuboids S_1 compromises the core cuboid C_{core}. Because M_1 contains one unit row vector e_1, the first tuple in C_{core} is compromised. Moreover, this is a non-trivial compromise because the original aggregation matrix M_1 does not contain any unit row vector.

3.3. Formalization rationale

It is a common approach in the literature to model dimensions using integer intervals. The actual values of different data types in the domain of a dimension is related to a set of integers by a one-to-one mapping. For example, dimensions month and employee in Table 3 are mapped to integer intervals $[1, 4]$ in Table 5. Such an abstraction ignores the specific values in each dimension and focuses on the structure of the data cube. Although dimensions may have continuous domains and infinitely many values, such as commissions, any given instance of data cube must contain only finite number of values. As stated in Definition 1, we map a value to an integer only if it appears in one or more tuples in the core cuboid. Hence it is sufficient to use an arbitrarily large but fixed integer interval for each dimension, as in Definition 1. Notice that some inference problems depend on the specific values in each tuple, such as those discussed in [30,31,33]. We do not address those problems in this paper.

The core cuboid, aggregation cuboids, and data cube in Definition 1 are similar to those in [26]. For example, the real-world data cube in Table 3 is modeled in Table 5. However, unlike in [26], we model data cubes using sets of vectors rather than using a relational operator. In doing so we are able to conveniently refer to any specific tuple or aggregation without complex relational queries. This choice simplifies our notation. We define the core cuboid and the aggregation cuboid separately, while they are not explicitly distinguished in [26]. The reason for our choice is that only tuples in core cuboid are assumed to contain sensitive values, but aggregations are not. This may not be valid for some special applications, where some users are prohibited from accessing certain aggregated values as well. Our ongoing work address this issue.

The missing tuples formalized in Definition 1 reflect the external knowledge of users (that is, the knowledge obtained through channels other than queries). One example of external knowledge is the unpopulated cells (that is, combinations of dimension values) in a sparse data cube. Users usually know which cells of the data cube are populated and which are not. This is so if the dimension values of each tuple is made public since they are not sensitive. One may argue that those values should be kept secret, because then inferences become impossible. However, even if all the dimension values are hidden, users may still infer the positions of unpopulated cells through queries containing COUNT. Preventing inferences through COUNT queries has been shown as intractable [29]. Hence we make the safer assumption that the positions of unpopulated cells are public.

Our definition of missing tuples captures a broader concept of external knowledge than unpopulated cells in a data cube. In practice users may learn values of sensitive

Table 8
Two variations of the example in Table 3

Quarter	Month/Employee	Alice	Bob	Jim	Mary	Sub total
4	October	?	?	?	0	7100
	November		?	?		4100
	December	?			?	4100
*	Bonus	?			?	6000
	Sub total	7000	4300	3000	7000	
4	October	?	?	?	1000	8100
	November		?	?		4100
	December	?			?	4100
*	Bonus	?			?	6000
	Sub total	7000	4300	3000	8000	

attributes through many channels. We use missing tuples to characterize such known values, regardless of the specific channels in which they are learned. From the viewpoint of both malicious users and inference control, values become irrelevant once they are learned through external knowledge, and hence are removed from the core cuboid. The specific values being learned are also irrelevant. For example, Table 8 shows two variations of the data cube shown in Table 3. The upper cross tabulation assumes that users know employee Mary to have a zero (but valid) commission for October. The lower cross tabulation assumes that users know Mary to have a commission of 1000. For those two varied data cubes, no change is necessary for the model given in Table 5.

Our definition of aggregation matrix and compromisability is similar to those used by Chin [13]. In [13], a set of sum queries over sensitive values is modeled as a linear system of equations. It then follows that determining compromisability of the sensitive values is equivalent to determining the existence of unit row vectors in the RREF of the coefficient matrix of the linear system. In our study, we directly define the compromisability based on the RREF of the aggregation matrix without referring to this well known equivalence. We distinguish between trivial and non-trivial compromises to facilitate our study, because they exhibit different cardinality-based characterizations as we shall show in the following sections. In the literature trivial and non-trivial compromises are referred to as *small query set attack* (or *single query attack*) [15], and *linear system attack* (or *multiple query attack*) [21], respectively.

4. Cardinality-based sufficient conditions for inference-free data cubes

In this section we prove cardinality-based sufficient conditions for safe data cubes. Those conditions relate the cardinality of core cuboids to the compromisability of

data cubes. We discuss trivial compromises in Section 4.1 and non-trivial compromises in Section 4.2. As stated in Section 3, the difference between the two cases is whether or not compromises can be achieved with a single aggregation.

4.1. Trivial compromisability

We have two results for trivial compromises as stated in Theorem 1. The first says that full core cuboids cannot be trivially compromised. This holds because in a full core cuboid all aggregation sets are larger than one. First consider the 1-* aggregation vectors. The aggregation set of any 1-* aggregation vector is equal to the size of the dimension having *-value, which is larger than one. Moreover, the aggregation set of any aggregation vector with more than one *-elements has a cardinality no less than some 1-* aggregation vectors. Hence any aggregation set contains more than one tuple. For example, in Table 4, the aggregation set of all the 1-* aggregation vectors contain at least two tuples, and the only 2-* aggregation vector contains all the nine tuples in the core cuboid. The second claim of Theorem 1 states that any core cuboid containing fewer tuples than the given upper bound is always trivially compromised. Intuitively, this is so because not enough tuples exist in the core cuboid in order for all the aggregation sets to contain more than one tuple. Notice if the core cuboid contains no tuple at all, then no aggregation set contains one tuple. However, this extreme case does not conform to Definition 1, because we require all dimension values to be present in at least one tuple.

Theorem 1. *In a given k dimensional data cube $< C_{core}, S_{all} >$ with dimensions $D_1, D_2, \ldots, D_k$, we have*:

1. *C_{full} cannot be trivially compromised by any aggregation cuboid $C \in S_{all}$.*
2. *C_{core} is trivially compromised by S_1 if $|C_{core}| < 2^{k-1} \cdot max(d_1, d_2, \ldots, d_k)$.*

Proof.

1. By Definition 4 we need to show that for any $t \in C$, we have $|Q\,set(t)| > 1$. Without loss of generality, let t be the j-* aggregation vector $(*, *, \ldots, *, x_{j+1}, x_{j+2}, \ldots, x_k)$. By Definition 2, we have $Q\,set(t) = \{t' : t' \in C_{full}, t'[j+1] = x_{j+1}, t'[j+2] = x_{j+2}, \ldots, t'[k] = x_k\}$. Because $C_{full} = \Pi_{i=1}^{k}[1, d_i]$ we have $|Q\,set(t)| = \prod_{i=1}^{j} d_i$. Because $d_i > 1$ for all $1 \leqslant i \leqslant k$, we have $|Q\,set(t)| > 1$.
2. Suppose C_{core} is not trivially compromised. We show that $|C_{core}| \geqslant 2^{k-1} \cdot max(d_1, d_2, \ldots, d_k)$. Without loss of generality, assume $d_k \geqslant D_i$ for all $1 \leqslant i \leqslant k$. By Definition 1, there are totally d_k slices of C_{core} on the kth dimension. Without loss of generality it suffices to show that $|P_k(C_{core}, 1)| \geqslant 2^{k-1}$. We do so by mathematical induction on $i \leqslant k$ as given below.
   The Inductive Hypothesis: For $i = 1, 2, \ldots, k$, there exists $C_i \subseteq P_k(C_{core}, 1)$ satisfying $|C_i| = 2^{i-1}$, and for any $t_1, t_2 \in C_i$ we have $t_1[j] = t_2[j]$ for all $j \geqslant i$.

The Base Case: By Definition 1, there exists $t \in C_{core}$ satisfying $t[k] = 1$. Let C_1 be $\{t\}$. We have $C_1 \subseteq P_k(C_{core}, 1)$ and $|C_1| = 1$, validating the base case of our inductive hypothesis.

The Inductive Case: Suppose for all $1 \leqslant i < k$ there exists $C_i \subseteq P_k(C_{core}, 1)$ satisfying $|C_i| = 2^{i-1}$, and for any $t_1, t_2 \in C_i$, $t_1[j] = t_2[j]$ for all $j \geqslant i$. We show that there exists $C_{i+1} \subseteq P_k(C_{core}, 1)$ such that $|C_{i+1}| = 2^i$, and for any $t_1, t_2 \in C_{i+1}$, $t_1[j] = t_2[j]$ for all $j \geqslant i + 1$.

For any $t_1 \in C_i$, let $t'_1[i] = *$ and $t'_1[j] = t[j]$ for all $j \neq i$. Because $t_1 \in Q\,set(t'_1)$ we have $|Q\,set(t'_1)| \geqslant 1$. Since C_{core} is not trivially compromised by S_1, according to Definition 4, we have $|Q\,set(t'_1)| > 1$. Hence, there exists $t''_1 \in Q\,set(t'_1) \subseteq C_{core}$ such that $t''_1[i] \neq t_1[i]$ and $t''_1[j] = t_1[j]$ for all $j \neq i$; which implies $t''_1 \notin C_i$.

Similarly for any $t_2 \in C_i$ satisfying $t_1 \neq t_2$, there exists $t''_2 \in C_{core}$ such that $t''_2[i] \neq t_2[i]$ and $t''_2[j] = t_2[j]$ for all $j \neq i$. Now we show that $t''_2 \neq t''_1$. Because $t_1[j] = t_2[j]$ for all $j \geqslant i$, there must be $l < i$ such that $t_1[l] \neq t_2[l]$. Because $t''_1[j] = t_1[j]$ and $t''_2[j] = t_2[j]$ for all $j < i$, we have that $t''_1[l] \neq t''_2[l]$. That is, $t''_1 \neq t''_2$.

Hence there exists $C'_i \subset C_{core}$ satisfying $|C'_i| = |C_i|$, and for any $t \in C_i$, there exists one and only one $t' \in C'_i$ such that $t[i] \neq t'[i]$ and $t[j] = t'[j]$ for all $j \neq i$. Let C_{i+1} be $C_i \cup C'_i$. Then $|C_{i+1}| = 2^i$.

This proves the inductive case of our induction, from which the claim $|P_k(C_{core}, 1)| \geqslant 2^{k-1}$ follows. □

For data cubes with cardinalities between the two limits stated in Theorem 1, the trivial compromisability cannot be determined solely by the cardinality of core cuboids. Two data cubes whose core cuboids have the same cardinality but different missing tuples can have different trivial compromisability. For example, the core cuboid C_{core} in Table 4 is not trivially compromised. Without changing the cardinality of C_{core}, we delete the tuple $(2, 2)$ and add a new tuple $(1, 4)$ to obtain a new core cuboid C'_{core}. C'_{core} is trivially compromised although $|C'_{core}| = |C_{core}|$, because in C'_{core} the aggregation sets of $(2, *)$ and $(*, 2)$ contain exactly one tuple.

The trivial compromisability of data cubes can also be determined by computing the cardinalities of all the rows in the aggregation matrix. With an $m \times n$ aggregation matrix this can be done in $O(mn)$. For example, for the aggregation matrix M_1 given in Table 6, counting the 1-elements in each row shows that C_{core} is not trivially compromised by S_1. The complexity can be further reduced considering that $M_1[i, j] = 1$ only if $M_{C_{full},S_1}[i, j] = 1$. For example, to calculate the aggregation set $Q\,set(t)$ for the aggregation vector $t = (1, *)$, we only need to consider the four elements $M_1[1, 1]$, $M_1[1, 2]$, $M_1[1, 3]$ and $M_1[1, 4]$, as we know that $M_1[1, j] = 0$ for all $j > 4$.

4.2. Non-trivial compromisability

Our results for non-trivial compromises require the observations stated in Lemma 1. Intuitively, the first claim of Lemma 1 holds, because aggregation vectors in any single aggregation cuboid always have disjoint aggregation sets (that is, any tuple is aggregated by at most one aggregation vector), and hence they do not help each other in non-trivial compromises. The second claim of Lemma 1 holds because aggregation vectors having more than one *-value can be derived from some 1-* aggregation vectors. For example, in Table 4 the aggregation set of the 2-* aggregation vector $(*, *)$ is equal to the aggregation set of one 1-* aggregation cuboid. Because of the second claim, it is sufficient to consider S_1 instead of S_{all} to determine the compromisability of data cubes. The last condition in Lemma 1 says that it is impossible to determine the non-trivial compromisability of a data cube by merely observing its k dimensions. This implies that any large enough data cube is vulnerable to non-trivial compromises. Here a data cube is large enough if $d_i \geqslant 4$ for all $1 \leqslant i \leqslant k$. For example, when $k = 2$ and $d_1 = d_2 = 2$, no data cube will be non-trivially compromisable.

Lemma 1. 1. *In any given data cube* $< C_{core}, S_{all} >$, C_{core} *can not be non-trivially compromised by any single cuboid* $C \in S_{all}$.
2. *In any given data cube* $< C_{core}, S_{all} >$, *if* C_{core} *cannot be compromised by* S_1, *then it cannot be compromised by* S_{all}.
3. *For any integers* k *and* $d_1, d_2, \ldots, d_k$ *satisfying* $d_i \geqslant 4$ *for* $1 \leqslant i \leqslant k$, *there exists a* k*-dimensional data cube* $< C_{core}, S_{all} >$ *with the* k *dimensions* $D_1, D_2, \ldots, D_k$, *such that* C_{core} *is non-trivially compromised by* S_1.

Proof.

1. Let $C \in S_{all}$ be any aggregation cuboid. For any $t \in C_{core}$ there exists one and only one $t_{aggr} \in C$ such that $t \in Q\,set(t_{aggr})$. Hence, in the aggregation matrix M each non-zero column is a unit column vector, implying that M could be transformed into its RREF by permuting its columns. Furthermore, each row of M must contain at least two 1's because no trivial compromise is assumed. Hence no unit row vector is in the RREF of M. That is, C_{core} cannot be non-trivially compromised by C.
2. Without loss of generality, let t_{aggr} be any j-* ($j > 1$) aggregation vector satisfying that $t_{aggr}[i] = *$ for any $1 \leqslant i \leqslant j$. Let C be the set of 1-* aggregation vectors defined as: $\{t\colon\ t[1] = *, t[i] \neq * \forall i \in [2, j], t[i] = t_{aggr}[i] \forall i \in [j + 1, k]\}$. We have that $Q\,set(t_{aggr}) = Q\,set(C)$. Hence in the aggregation matrix $M_{C_{core}, S_{all}}$, the row corresponding to t_{aggr} can be represented as the linear combination of the rows corresponding to C. The rest of the proof follows from linear algebra.

3. First we justify the case $k = 2$, then we extend the result to $k > 2$.

For the proof of $k = 2$, without loss of generality, we use mathematical induction on d_1, for an arbitrary, but fixed value of $d_2 \geqslant 4$.

The Inductive Hypothesis: For any $d_1, d_2 \geqslant 4$, we can build a two dimensional data cube $< C_{core}, S_{all} >$ with dimensions D_1, D_2, such that C_{core} is non-trivially compromised by S_1.

The Base Case: When $d_1 = d_2 = 4$, the data cube shown in Table 4 validates the base case of our inductive hypothesis.

The Inductive Case: Assuming that there exists non-trivially compromisable two dimensional data cube $< C_{core}, S_{all} >$ with dimensions $[1, d_1]$ and $[1, d_2]$, we show how to obtain a non-trivially compromisable two dimensional data cube with dimensions $[1, d_1 + 1]$ and $[1, d_2]$.

Without loss of generality suppose that the tuple (1, 1) is non-trivially compromised by S_1. Then, there exists a row vector a such that $a \cdot M_1 = e_1$.

First define a set of tuples C as:

- for any $t \in C$, $t[1] = d_1 + 1$
- for any $i \in [1, d_2]$, $(d_1 + 1, i) \in C$ if and only if $(d_1, i) \in P_1(C_{core}, d_1)$

We define $C'_{core} = C_{core} \cup C$, and $< C'_{core}, S'_{all} >$ as a new data cube with dimensions $[1, d_1 + 1]$ and $[1, d_2]$. We have $P_1(C'_{core}, d_1 + 1) = C$. Let M'_1 be the 1-* aggregation matrix of the new data cube. We have $M'_1 = (M_1 \mid M_c)$. The non-zero columns in M_c correspond to the tuples in C. According to the definition of C we further have $M_1 = (M''_1 \mid M_c)$, where the non-zero columns of M_c correspond to the tuples in $P_1(C_{core}, d_1)$. Thus, $M'_1 = (M''_1 \mid M_c \mid M_c)$. Because $a \cdot M_1 = (a \cdot M''_1 \mid a \cdot M_c) = e_1$, we have that $a \cdot M'_1 = (a \cdot M''_1 \mid a \cdot M_c \mid a \cdot M_c) = (e_1|0)$. Hence, the tuple [1, 1] in C'_{core} is non-trivially compromised, validating the inductive case of our inductive hypothesis.

We briefly describe how to extend this result for $k = 2$ to $k > 2$. We do so by regarding part of the k dimensional data cube as a special two dimensional data cube. Specifically, given k dimensional data cube $< C_{core}, S_{all} >$, let $C'_{core} = \{t\colon\ t \in C_{core}, t[j] = 1 \forall 3 \leqslant j \leqslant k\}$ and C be the collection of all 1-* aggregation vectors satisfying $\forall t \in C \forall j \in [3, k], t[j] = 1$. We have $Q\,set(C) = C'_{core}$. Hence M_1 can be represented as:

$$\left(\frac{M'_1|0}{M''_1}\right),$$

where M'_1 is a $|C|$ by $|C'_{core}|$ sub-matrix and 0 is the $|C|$ by $|C_{core} \setminus C'_{core}|$ zero matrix. M'_1 can be treated as the 1-* aggregation matrix of a special two dimensional data cube. We can build C'_{core} in such a way that this two dimensional data cube is non-trivially compromised. Hence the RREF of M'_1 contains at least one unit row vector, which implies the RREF of M_1 does so, too. □

We have two results for non-trivial compromises as stated in Theorem 2. The first says that full core cuboids cannot be non-trivially compromised. The first claim of Theorem 2 together with the first claim of Theorem 1 proves that any data cube with a full core cuboid is non-compromisable. In the proof of the first claim in Theorem 2, we construct a set of 2^k column vectors to contain any given column vector in the aggregation matrix. This set of 2^k vectors satisfies the property that any of its 2^k members can be represented as the linear combination of the other $2^k - 1$ members. The elementary row transformation used to obtain RREF of a matrix does not change the linear dependency of the column vectors. Hence the linear dependency among the 2^k column vectors also holds in the RREF of the aggregation matrix. Consequently the 2^k tuples corresponding to those columns cannot be non-trivially compromised.

The second and third claims of Theorem 2 give a tight lower bound on the cardinality of any core cuboid with missing tuples, such that it remains free of non-trivial compromises. The lower bound $2d_l + 2d_m - 9$ is a function of the two least dimension cardinalities. Intuitively, the justification of the lower bound is based on the following fact. The least number of missing tuples necessary for any non-trivial compromise increases monotonically with the number of aggregation cuboids involved in the compromise. This number reaches its lower bound when exactly two aggregation cuboids are used for non-trivial compromises. This is exactly the case given by the second claim of Theorem 2. The second claim shows that it is impossible to derive non-trivial compromisability criteria solely based on the cardinality of core cuboids, when the cardinality is not greater than the given lower bound. Thus the lower bound given by the third claim is the best possible.

Theorem 2 (Non-trivial Compromisability)**.**

1. *In any given data cube* $< C_{core}, S_{all} >$, C_{full} *cannot be non-trivially compromised by* S_{all}.
2. *For any integers* k *and* $d_1, d_2, \ldots, d_k$ *satisfying* $d_i \geqslant 4$ *for all* $1 \leqslant i \leqslant k$, *let* d_l *and* d_m *be the least two among* d_i*s. Then there exists a* k*-dimensional data cube* $< C_{core}, S_{all} >$ *with* k *dimensions* $D_1, D_2, \ldots, D_k$, *such that* $|C_{full} \setminus C_{core}| = 2d_m + 2d_n - 9$ *and* C_{core} *is non-trivially compromised by* S_1.
3. *Given a* k *dimensional data cube* $< C_{core}, S_{all} >$ *with dimensions* $D_1, D_2, \ldots, D_k$, *suppose* D_l *and* D_m *(*$1 \leqslant l, m \leqslant k$*) are the two with the least cardinalities. If* $|C_{full} \setminus C_{core}| < 2|D_l| + 2|D_m| - 9$ *holds, then* C_{core} *cannot be non-trivially compromised.*

Proof.

1. Due to the second claim of Lemma 1 we only need to shown that C_{full} cannot be non-trivially compromised by S_1. Without loss of generality, we show that $t_0 = (1, 1, \ldots, 1)$ cannot be non-trivially compromised by S_1. In order to do so, we define $C'_{full} = \{t\colon \forall i \in [1, k], t[i] = 1 \vee t[i] = 2\}$. We then have $C'_{full} \subseteq C_{full}$ and $|C'_{full}| = 2^k$. Let M'_1 be a matrix comprising of the 2^k columns of M_1

that corresponds to C'_{full}. In the rest of the proof we formally show that each of those 2^k column vectors can be represented as the linear combination of the rest $2^k - 1$ column vectors. It then follows from linear algebra that the RREF of M_1 does not contain e_1. Hence t_0 cannot be non-trivially compromised by S_1. First we define the *sign assignment vector* as an 2^k dimensional column vector t_{sign} as follows:

- $t_{sign}[1] = 1$
- $t_{sign}[2^i + j] = -t_{sign}[j]$ for all $0 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant 2^i$

Claim: $M'_1 \cdot t_{sign} = 0$, where 0 is the 2^k dimensional zero column vector.
Justification:
Let t_{aggr} be the ith 1-* aggregation vector and suppose $t_{aggr}[l] = *$ for some $1 \leqslant l \leqslant k$.
Let r be the ith row of M'_1.
If $t_{aggr}[j] \leqslant 2$ for all $j \neq l$.
  Then $|Q\,set(t_{aggr}) \cap C'_{full}| = 2$, and suppose $Q\,set(t_{aggr}) = \{t_1, t_2\}$
    where $t_1 = C'_{full}[j_1]$ and $t_2 = C'_{full}[j_2]$, $t_1[l] = 1$, $t_2[l] = 2$
    and $t_1[j] = t_2[j] = t_{aggr}[j]$ for all $j \neq l$
  Hence, j_1, j_2 satisfy
    $r[j_1] = r[j_2] = 1$ and $r[j] = 0$ for any $j \neq j_1 \wedge j \neq j_2$.
    The j_1th and j_2th columns of M'_1 correspond to t_1 and t_2 respectively.
  Since C'_{full} is in dictionary order, we have $j_2 = j_1 + 2^{l-1}$.
  Hence, we have $r \cdot t_{sign} = 0$.
Otherwise, $|Q\,set(t_{aggr}) \cap C'_{full}| = 0$.
  Hence, $r = 0$ and $0 \cdot t_{sign} = 0$.

Hence, as stated earlier, the justification of our claim concludes the main proof.

2. Without loss of generality assume $m = 1$ and $n = 2$. Analogous to the proof for the third condition of Lemma 1, it suffices to consider only the case $k = 2$. For an arbitrary but fixed value of d_2, we show by induction on d_1 that the data cube as constructed in the proof for the third condition of Lemma 1 satisfies $|C_{full} \setminus C_{core}| = 2d_1 + 2d_2 - 9$.
The Inductive Hypothesis: C_{core} as constructed in the proof of Lemma 1 satisfies:

- $|C_{full} \setminus C_{core}| = 2d_1 + 2d_2 - 9$.
- $|P_1(C_{full}, d_1) \setminus P_1(C_{core}, d_1)| = 2$.

The Base Case: In Table 4, the core cuboid C_{core} satisfies $|C_{full} \setminus C_{core}| = 2d_1 + 2d_2 - 9$. We also have $|P_1(C_{full}, 4) - P_1(C_c, 4)| = 2$. This validates the base case of our inductive hypothesis.
The Inductive Case: Suppose we have two-dimensional data cube $< C_{core}, S_{all} >$ with dimensions D_1 and D_2 satisfying $|C_{full} - C_{core}| = 2d_1 + 2d_2 - 9$ and $|P_1(C_{full}, d_1) - P_1(C_c, d_1)| = 2$. Use $< C'_{core}, S'_{all} >$ and C'_{full} for the data

cube and full core cuboid with dimensions $[1, d_1 + 1]$ and D_2, respectively. By the definition of C in the proof of Lemma 1 $|C| = |P_1(C_{core}, d_1)|$, and as a consequence $|C'_{full} \setminus C'_{core}| = |C_{full} \setminus C_{core}| + 2 = 2(d_1 + 1) + 2d_2 - 9$. Since $P_1(C'_{core}, d_1 + 1) = C$, we have $|P_1(C'_{full}, d_1 + 1) - P_1(C'_{core}, d_1 + 1)| = 2$. This validates the inductive case of our inductive argument and consequently concludes our proof.

Lower Bound: Similarly we assume $m = 1$ and $n = 2$. We show that if C_{core} is non-trivially compromised then we have $|C_{full} \setminus C_{core}| \geqslant 2d_1 + 2d_2 - 9$. First we make following assumptions.

(a) Tuple $t = (1, 1, \ldots, 1) \in C_{core}$ is non-trivially compromised by S_1.
(b) No tuple in C_{core} is trivially compromised by S_1.
(c) There exists a minimal subset S of S_1, such that for any $C \in S$, t cannot be non-trivially compromised by $S \setminus C$.
(d) For any $t' \in C_{full} \setminus C_{core}$, t cannot be non-trivially compromised by S_1 in data cube $< C_{core} \cup \{t'\}, S_{all} >$. That is, $|C_{full} \setminus C_{core}|$ reaches its lower bound.

Assumption 2 holds by Definition 4. Assumption 3 holds as by the first claim of Lemma 1, S must contain at least two 1-* aggregation cuboids. Assumption 4 holds because by Theorem 2, $|C_{full} \setminus C_{core}|$ has a lower bound if C_{core} is non-trivially compromisable.

Claim: Let $C \in S$ and $t[i] = *$ for any $t \in C$. We have $|P_i(C_{full}, 1) \setminus P_i(C_{core}, 1)| \geqslant 1$, and $|P_i(C_{full}, j) \setminus P_i(C_{core}, j)| \geqslant 2$ for any $2 \leqslant j \leqslant d_i$].

Justification: The proof is by contradiction. Without loss of generality, we only justify the claim for $i = k$ and $j = 2$. That is, given $C \in S$ satisfying $t[k] = *$ for any $t \in C$ we prove that $|P_i(C_{full}, 1) \setminus P_i(C_{core}, 1)| \geqslant 1$ and $|P_i(C_{full}, 2) \setminus P_i(C_{core}, 2)| \geqslant 2$.

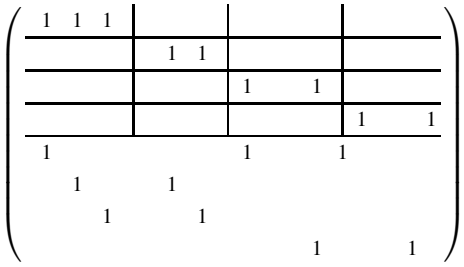
First we transform $M_{C_{core},S}$ into a singly bordered block diagonal form (SBBDF) [38] with row permutations, denoted by $M_{m \times n}$. The ith diagonal block of M corresponds to $P_k(C_{core}, i)$ and $\{t\colon\ t \in S \setminus C \wedge t[k] = i\}$, and the border of M corresponds to C. For example, Table 9 illustrates the SBBDF of $M_{S_1,C_{core}}$ in Table 6. We call the columns of M containing the ith diagonal block as the ith *slice of* M, also we use notation $M[-, i]$ for the ith column of M and $M[i, -]$ for the ith row.

Due to Assumption 1, there exists a m-dimensional row vector a satisfying $a \cdot M = e_1$. Use r_i for $M[i, -]$ then we get $e_1 = \sum_{i=1}^{m} a[i] \cdot r_i$. Suppose each diagonal block of M has size $m' \times n'$. Use r_i^j, for $1 \leqslant j \leqslant d_k$ to represent the sub-row vector of r_i intersected by the jth slice of M. We have $|r_i^j| = n'$. We also use e'_1 and $0'$ to represent the n' dimensional unit row vector and n' dimensional zero row vector, respectively. Then the following are true:

i. $e'_1 = \sum_{i=1}^{m'} a[i] r_i^1 + \sum_{i=m-m'+1}^{m} a[i] r_i^1$
ii. $0' = \sum_{i=m'+1}^{2m'} a[i] r_i^2 + \sum_{i=m-m'+1}^{m} a[i] r_i^2$.

Table 9
An example of SBBDF

SBBDF of $M_{S_1,C_{core}}$ Shown in Table 6 With 0-Elements Omitted



First we suppose $|P_k(C_{full}, 2) \setminus P_k(C_{core}, 2)| = 0$, that is, the second slice of M contains no zero column. We then derive a contradiction to our assumptions.

Because $|P_k(C_{full}, 2) \setminus P_k(C_{core}, 2)| = 0$, the first slice of M contains no less zero columns than the second slice of M. Intuitively if the latter can be transformed into a zero row vector by some elementary row transformation, then applying the same transformation on the former leads to a zero vector, too. This is formally represented as:

iii. $0' = \sum_{i=1}^{m'} a[m'+i]r_i^1 + \sum_{i=m-m'+1}^{m} a[i]r_i^1$.

Subtracting (iii) from (i) gives $e_1' = \sum_{i=1}^{m'}(a[i] - a[m'+i])r_i^1$. That implies C_{core} is non-trivially compromised by $S \setminus \{C_k\}$, contradicting Assumption 3. Thus $|P_k(C_{full}, 2) \setminus P_k(C_{core}, 2)| > 0$.

Next assume $|P_k(C_{full}, 2) \setminus P_k(C_{core}, 2)| = 1$ and derive a contradiction to our assumptions.

Row vector r_i^3 satisfies the following condition:

iv. $0' = \sum_{i=2m'+1}^{3m'} a[i]r_i^3 + \sum_{i=m-m'+1}^{m} a[i]r_i^3$.

Let $t' \in P_k(C_{full}, 2) \setminus P_k(C_{core}, 2)$. Notice that (i), (ii) still hold. Suppose t' corresponds to $M[-, y]$, which is a zero column. Now assume that we add t' to $P_k(C_{core}, 2)$. Consequently we have that $M[-, y] \neq 0$. Due to Assumption 4, the left side of (ii) must now become e_1'. That is, $a \cdot M[-, y] = 1$. There will also be an extra 1-element $M[x, y]$ in the border of M.

Now let t'' be the tuple corresponding to $M[-, y + n']$ in the third slice of M. Suppose $t'' \in P_k(C_{core}, 3)$ and consequently $M[-, y + n'] \neq 0$. We have that $M[-, y + n'] = M[-, y]$ and consequently $a \cdot M[-, y + n'] = 1$.

Now removing t' from $P_k(C_{core}, 2)$. Now we show by contradiction that $t'' \in P_k(C_{core}, 3)$ cannot be true. Intuitively, because t' is the only missing tuple in the second slice of M, the third slice of M contains no less zero vectors than the second slice of M does, except t''. Because $a \cdot M[-, y + n'] = 1$, the elements of the vector a, which transforms the second slice of M to a zero

vector as shown by (ii), can also transform the third slice of M to a unit vector. This is formally represented in (v):

v. $e'' = \sum_{i=2m'+1}^{3m'} a[i-m']r_i^3 + \sum_{i=m-m'+1}^{m} a[i]r_i^3$.

Subtracting (iv) from (v) we get $e'' = \sum_{i=2m'+1}^{3m'}(a[i-m'] - a[i])r_i^3$; implying C_{core} is compromised by $S \setminus \{C_i\}$. Hence, Assumption 3 is false. Consequently, $t'' \notin C_{core}$.

A similar proof exists for the ith slice of C_c for any $4 \leqslant i \leqslant d_k$. However, $M[x,-] \neq 0$ because otherwise we can let a_x be zero and then decrease $|C_{full} \setminus C_{core}|$, contradicting Assumption 4. Hence $M[x,-]$ is a unit vector with the 1-element in the first slice of M. However, this further contradicts Assumption 2. That is, no trivial compromise is assumed. Hence we have that $|P_k(C_{full},2) \setminus P_k(C_{core},2)| = 1$ is false.

Now consider $|P_k(C_{full},1) \setminus P_k(C_{core},1)|$. Suppose all the assumptions hold and $|P_k(C_{full},1) \setminus P_k(C_{core},1)| = 0$. Let $t_1, t_2 \in P_k(C_{full},2) \setminus P_k(C_{core},2)$. Now define $C'_{core} = C_{core} \setminus \{t\} \cup \{t_1\}$, and use M' for $M_{C'_{core},S}$. From $a \cdot M = e_1$ and Assumption 4 we get that $a \cdot M' = e_i$, and $M[-,i]$ corresponds to t_1. This implies that t_1 is non-trivially compromised in $< C'_{core}, S_{all} >$, with $|P_k(C_{full},1) \setminus P_k(C'_{core},1)| = 1$, which contradicts what we have already proved. Hence, we get $|P_k(C_{full},1) \setminus P_k(C_{core},1)| \geqslant 1$. This concludes the justification of our claim.

The justified claim implies that the number of missing tuples in C_{core} increases monotonically with the following:

- The number of aggregation cuboids in S.
- d_i, if there exists $C \in S$ satisfying $t[i] = *$ for any $t \in C$.

Hence $|C_{full} \setminus C_{core}|$ reaches its lower bound when $|S| = 2$, which is equal to $2D_1 + 2D_2 - 9$, as shown in the first part of the current proof - concluding the proof of Theorem 2. □

The results stated in Corollary 1 follow from Theorem 2 and Lemma 1 but have value of their own. The first claim of Corollary 1 says that if the ith 1-* aggregation cuboid is essential for any non-trivially compromises, then every slice of the core cuboid on the ith dimension must contain at least one missing tuples. As an example, in the core cuboid shown in Table 4, every slice on the two dimensions contains either one or two missing tuples. The second claim of Corollary 1 says that any core cuboid that have full slices on $k-1$ of the k dimensions must be safe from non-trivial compromises.

Corollary 1 (Non-trivial Compromisability With Full Slices)**.** *In any data cube* $< C_{core}, S_{all} >$ *with dimensions* $D_1, D_2, \ldots, D_k$:

Table 10
Example of a data cube with full slices on every dimension

	1	2	3	4	*
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,*)
2	(2,1)		(2,3)	(2,4)	(2,*)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,*)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,*)
*	(*,1)	(*,2)	(*,3)	(*,4)	(*,*)

1. *Let* $C \in S \subseteq S_1$ *and suppose* $t[i] = *$ *for any* $t \in C$*, where* $i \in [1, k]$ *is fixed. If* S *non-trivially compromises* C_{core} *but* $S \setminus \{C\}$ *does not, then* $|P_i(C_{full}, j) \setminus P_i(C_{core}, j)| \geqslant 1$ *for any* $1 \leqslant j \leqslant d_i$.
2. *If there exists* $S \subset [1, k]$ *satisfying* $|S| = k - 1$ *and for any* $i \in S$*, there exists* $j \in D_i$ *such that* $|P_i(C_{full}, j) \setminus P_i(C_{core}, j)| = 0$*, then* C_{core} *cannot be non-trivially compromised by* S_{all}.

Proof. The first claim follows directly from the proof of Theorem 2. The second claim then follows from the first claim of Corollary 1 taken together with the first claim of Lemma 1. □

As an example of Corollary 1, we have stated in Section 1 that the second quarter data shown in Table 1 is non-compromisable. This is so because the core cuboid, as shown in Table 10, contains full slices on any of the two dimensions. Hence it cannot be non-trivially compromised. Moreover, its trivial non-compromisability is straightforward.

5. Three-tier inference control model

Traditional view of inference control has two tiers, that is, the data set and the answerable queries. The data set is usually modeled as a set of tuples, similar to the core cuboid defined in Section 3. A query selects a subset of tuples in the data set, called the *query set* of the query. The query then aggregates (for example, sums) the values of those tuples in the query set. Inference becomes a concern when the values being aggregated by the query are sensitive. With this two tier view, a typical restriction-based inference control mechanism checks a given set of queries for unwanted inferences and answers only those queries that do not compromise the sensitive values.

Inference control based on the two tier view has some inherent drawbacks. First of all, allowing ad-hoc queries unnecessarily complicates inference control. Because any subset of a given data set may potentially be the query set of a query, a total of 2^n queries with different query sets are possible on a data set containing n vectors.

Second of all, inferences can be obtained with a single query as well as by manipulating multiple queries, as shown by the third and fourth quarter data in Table 1, respectively. Hence totally 2^{2^n} different sets of queries can be formed on the data set. Such a large number of possibilities partially contributes to the high complexity of most existing inference control mechanisms. In practice most ad-hoc queries are either meaningless to users or redundant because their results can be derived from other previously answered queries. For example, for SUM queries, at most n queries with different query sets can be formed on a data set of size n before the result of any new query can be derived from the old ones, because the rank of an aggregation matrix with n columns cannot exceed n.

Inference control mechanisms developed under the two tier view usually have high *on-line* computational complexity. Here on-line computations refer to the computations conducted after queries have been posed to the system, and conversely *off-line* computations occur before queries are posed. In the two tier view of inference control, it is difficult for restriction-based inference control mechanisms to predict how incoming queries will aggregate data. Hence the major part of computational effort required by inference control mechanisms cannot be started until queries are received. Consequently, the time used by inference control adds to the system response time. This is unacceptable considering the high complexity of most inference control mechanisms.

Finally, rich dimension hierarchies embedded in most multi-dimensional data sets are ignored by the two tier view of inference control. That prevents inference control mechanisms from benefiting from those hierarchies. In Table 1, the dimension hierarchy composing of month, quarter and year naturally divides the data set into four blocks, shown as the four cross-tabulations. In OLAP systems, most meaningful queries are formed on the basis of those partitions. As an example, a query that sums Alice's commission in January and Bob's commission in August conveys little useful information to users. Without taking dimension hierarchies into account, inference control mechanisms have to take the whole data set as their input, even when queries involve only a block of the data set.

To address the listed issues, we propose a three-tier inference control model consisting of three tiers with three relations in between, as shown in Fig. 1. The *data tier* D is a set of *tuples*. Both *aggregation tier* A and *query tier* Q are sets of *queries*. We do not consider the details of tuples and queries here, but instead we consider them as elements of sets. The relation R_{QD} is defined as the composition of R_{QA} and R_{AD}. We assume that a suitable definition of *compromisability* has been given. In addition we enforce three properties on the model as the follows.

1. **Three Tiers:**
   (a) Data Tier D.
   (b) Aggregation Tier A.
   (c) Query Tier Q.

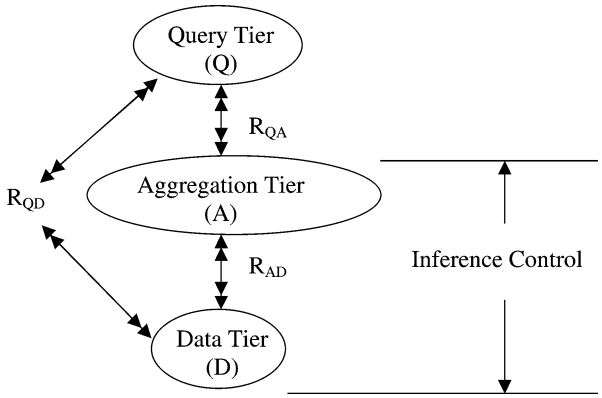


Fig. 1. Three-tier model for controlling inferences.

2. **Relations Between Tiers:**
   (a) $R_{AD} \subseteq A \times D$.
   (b) $R_{QA} \subseteq Q \times A$.
   (c) $R_{QD} = R_{AD} \circ R_{QA}$.
3. **Properties:**
   (a) $|A|$ is polynomial in $|D|$.
   (b) D and A can be partitioned into $D_1, D_2, \ldots, D_m$ and $A_1, A_2, \ldots, A_m$, satisfying that $(a, d) \in R_{AD}$ only if $d \in D_i$ and $a \in A_i$ for some $1 \leqslant i \leqslant m$.
   (c) A does not compromise D.

Proposition 1 explains how the three-tier model controls inferences. Intuitively, the aggregation set A provides a set of intermediate aggregations that are guaranteed to be safe from compromises. Queries in Q are then answered by deriving results from these intermediate aggregations. The query results will not compromise any tuple in D because they do not convey any information beyond those contained in the aggregation tier A.

Proposition 1. *The three-tier model guarantees that Q does not compromise D.*

Proof. For any given set of queries $Q' \subseteq Q$, we have that $R_{QD}(Q') = R_{AD}(R_{QA}(Q'))$ because $R_{QD} = R_{AD} \circ R_{QA}$. Hence Q' does not compromises D if the aggregations given by $R_{QA}(Q')$ do not compromises D. We have that $R_{QA}(Q') \subseteq A$ holds by definition. The third property of the model then guarantees that $R_{QA}(Q')$ does not compromise D. □

In contrast to traditional two tier view of inference control, the three-tier model improves the performance of inference control mechanisms in several ways. Firstly,

the size of the input to inference control mechanisms is dramatically reduced. Different from the two tier view, the three-tier view makes aggregation tier A the input of inference control mechanisms. The third property of the three-tier model requires an aggregation tier A to have a size comparable to that of the data tier D. Choosing such an aggregation tier A is possible, because as we have explained, the number of non-redundant aggregations is bounded by the size of the data tier D.

Secondly, the three-tier model facilitates using a divide-and-conquer approach to further reduce the size of inputs to inference control mechanisms by *localizing* the inference control problem. Due to the second property of the model, for any $1 \leqslant i \leqslant m$ we have that $R_{AD}(A_i) \subseteq D_i$ and $R_{AD}(D_i) \subseteq A_i$. Hence for any given set of queries $Q' \subseteq Q$ satisfying $R_{QA}(Q') \subseteq A_i$ for some $1 \leqslant i \leqslant m$, the compromisability depends on A_i and D_i only. Intuitively, if a set of queries can be answered with some blocks of the aggregation tier A, then the inference control problem of the queries can be localized to these blocks of A and their R_{AD}-related blocks of the data tier D. In practice, partitioning D and A to satisfy the second property is possible because most multi-dimensional data sets contain dimension hierarchies.

Finally, the three-tier model shifts the major computational effort required by inference control to off-line processing, thereby reduces on-line performance cost. Composing aggregation tier A by defining R_{AD} to satisfy the three properties of the model is the most computationally expensive task. This task can be processed off-line, before answering any queries. The on-line processing consists of decomposing any given set of queries $Q' \subseteq Q$ by computing $R_{QA}(Q')$. Because most OLAP systems utilize pre-computed aggregations for query-answering, this query decomposition mechanism is in place or can be implemented easily. Hence by decoupling off-line and on-line processing of inference control and pushing computational complexity to the off-line part, the three-tier model can eliminate or reduce the delay of query-answering caused by inference control.

Defining R_{QD} as the composition of R_{AD} and R_{QA} may reduce the total number of answerable queries. This restriction reflects the unavoidable trade-off between availability and security. However, the design of aggregation tier A should enable it to convey as much useful information as possible, while not endangering the sensitive information stored in the data tier D. The usefulness of queries usually depends on application settings. For example, in OLAP applications data cube style aggregations, as modeled in Section 3, are the most popular queries users may pose to the system.

6. Cardinality-based inference control for data cubes

In this section we describe an inference control algorithm that integrates the cardinality-based compromisability criteria developed in Section 4. We then show its correctness and computational complexity. Finally we discuss implementation issues and improvements to the algorithm.

6.1. Inference control algorithm

Our inference control algorithm is based on the three-tier inference control model discussed in Section 5. According to the model, inference control resides between the data tier and aggregation tier. Hence we shall focus on those two tiers. We briefly address the query tier in Section 6.3.

Given a data cube $< C_{core}, S_{all} >$, the core cuboid C_{core} constitutes the data tier. A collection of aggregation vectors S_A, to be defined shortly, constitutes the aggregation tier. A tuple $t \in C_{core}$ is R_{AD}-related to an aggregation vector $t_{aggr} \in S_A$ if and only if $t \in Q\,set(t_{aggr})$.

The definition of S_A is based on the following partition on C_{core}. For $i = 1, 2, \ldots, k$, we divide the ith dimension $[1, d_i]$ into integer intervals $[1, d_i^1], [d_i^1 + 1, d_i^2], \ldots, [d_i^{m_i-1} + 1, d_i]$, where m_i is fixed for each i. Then we partition C_{core} into $\prod_{l=1}^{k} m_i$ blocks using those intervals. The partition of C_{core} satisfies the property that any two tuples $t, t' \in C_{core}$ are in the same block if and only if their ith elements are in the same block of the ith dimension for all $i = 1, 2, \ldots, k$. In the partition of C_{core}, we regard each block, denoted as C^s_{core}, as the core cuboid of a *sub-data cube* $< C^s_{core}, S^s_{all} >$ (from now on we use symbols with superscripts for sub-data cubes and their components). S_A is the collection of 1-* aggregation vectors with non-empty aggregation sets in all the safe sub-data cubes. Notice that when integrating the aggregation vectors defined in sub-data cubes into S_A, the aggregation set of each aggregation vector remains the same as it is originally defined in the sub-data cube. For example, the data cube in Table 1 is partitioned into four sub-data cubes, each represented by a cross-tabulation. The subtotals shown in the table correspond to the 1-* aggregation vectors defined in sub-data cubes, and therefore constitute the members of S_A.

The inference control algorithm *Ctrl_Inf_Cube* shown in Fig. 2 constructs the aggregation set tier S_A by partitioning the core cuboid C_{core}. The main routine of *Ctrl_Inf_Cube* accepts as input a given data cube $< C_{core}, S_{all} >$ and its k dimensions $D_1, D_2, \ldots, D_k$. For $i = 1, 2, \ldots, k$, a set of $m_i - 1$ integers between 1 and d_i partition the ith dimension $[1, D_i]$ into m_i blocks, and consequently partition C_{core} into $\prod_{i=1}^{k} m_i$ blocks. Each block of C_{core} is then normalized to be a core cuboid C^s_{core} as stated in Definition 1, by converting its k dimensions to integer intervals starting from 1. The core cuboid C^s_{core} is then passed to the subroutine *Ctrl_Inf_Sub*. The subroutine *Ctrl_Inf_Sub* applies the cardinality-based sufficient conditions given in Section 4 to the sub-data cube $< C^s_{core}, S^s_{all} >$, in order to determine its compromisability. The 1-* aggregation vectors with non-empty aggregation sets in those non-compromisable sub-data cubes are returned to the main routine, and collectively constitute the aggregation tier.

As an example, suppose we feed a core cuboid similar to Table 1 into the algorithm *Ctrl_Inf_Cube*, with each quarter as a block. The subroutine *Ctrl_Inf_Sub* conducts five tests consecutively for each such block. Because the first block corresponds to a full core cuboid, the second test succeeds. Hence the subroutine returns all the 1-*

Algorithm *Ctrl_Inf_Cube*
Input: data cube $< C_{core}, S_{all} >$ with dimensions $D_1, D_2, \ldots, D_k$, and integers $1 < d_i^1 < d_i^2 \ldots < d_i^{m_i - 1} < d_i$ for $1 \leqslant i \leqslant k$, where m_i is fixed for each i and let $d_i^0 = 0$, $d_i^{m_i} = d_i$
Output: a set of aggregation vectors S_A
Method:

1. Let $S_A = \phi$;
2. **For** each k dimensional vector v in vector space $\prod_{i=1}^{k}[1, m_i]$
   Let $C_{tmp} = \{t\colon\ t \in C_{core}, \forall i \in [1,k]\ \ t[i] \in [d_i^{v[i]-1} + 1, d_i^{v[i]}]\}$;
   Let $C_{core}^s = \{t : \exists\, t' \in C_{tmp}, \forall i \in [1,k]\ \ t[i] = t'[i] - d_i^{v[i]-1}\}$;
   Let S_A^s = *Ctrl_Inf_Sub(* C_{core}^s *)*;
   Let $S_A = S_A \cup S_A^s$;
3. **Return** S_A.

Subroutine *Ctrl_Inf_Sub*
Input: k dimensional core cuboid C_{core}^s
Output: a set of 1-* aggregation vectors if S_1^s does not compromise C_{core}^s, and ϕ otherwise
Method:

1. **If** $|C_{core}^s| < 2^{k-1} \cdot max(d_1^s, d_2^s, \ldots, d_k^s)$
   Return ϕ;
2. **If** $|C_{core}^s| = |C_{full}^s|$
   Return $\cup_{C \in S_1^s} C$;
3. **If** C_{core}^s is trivially compromised by S_1^s
   Return ϕ;
4. **If** $|C_{full}^s - C_{core}^s| < 2d_l^s + 2d_m^s - 9$, where d_l^s, d_m^s are the two smallest among d_i^ss
   Return $\cup_{C \in S_1^s} C$;
5. **If** there exists $D \subset [1,k]$ satisfying $|D| = k - 1$ and for any $i \in D$, there exists $j \in [1, d_i^s]$ such that $|P_i(C_{full}^s, j) \setminus P_i(C_{core}^s, j)| = 0$
   Return $\cup_{C \in S_1^s} C$;
6. **Return** ϕ.

Fig. 2. The algorithm of inference control in data cube.

aggregation vectors as its output. The second block has full slices on both dimensions, and hence the fifth test for full slices succeeds with all the 1-* aggregation vectors returned. The third block is determined by the third test as trivially compromised, so nothing is returned for this block. Finally the fourth block fails all the five tests and nothing is returned.

6.2. Correctness and time complexity

Now we prove the correctness of algorithm *Ctrl_Inf_Cube*. More specifically, we show that the aggregation tier constructed by the algorithm satisfies the first and second properties of the model discussed in Section 5. The last property holds trivially. The first property, namely, $|S_A|$ being a polynomial in $|C_{core}|$ is justified in Proposition 2.

Proposition 2. *S_A constructed in the algorithm* Ctrl_Inf_Cube *satisfies*: $|S_A| = O(|C_{core}|)$.

Proof. Let $n = |C_{core}|$. The 1-* aggregation matrix M_1^s of any sub-data cube $< C_{core}^s, S_{all}^s >$ has $n^s = |C_{core}^s|$ non-zero columns. Hence M_1^s has $(n^s \cdot k)$ many 1-elements, implying at most $(n^s \cdot k)$ non-zero rows in M_1^s. Hence the set of 1-* aggregation vectors with non-empty aggregation sets has a cardinality no bigger than $(n^s \cdot k)$, that is, $|S_A^s| \leqslant n^s \cdot k$. Consequently, we have $|S_A| \leqslant k \cdot \sum n^s = O(n)$, assuming k is constant compared to n. □

The second property of the model, namely, the ability to partition S_A and C_{core} is satisfied by the partitions constructed in the algorithm *Ctrl_Inf_Cube*. Let t_{aggr}^s be any aggregation vector in sub-data cube $< C_{core}^s, S_{all}^s >$. We have $Q\,set(t_{aggr}^s) \subseteq C_{core}^s$. Because as stated earlier, in the data cube $< C_{core}, S_{all} >$, $Q(t_{aggr}^s)$ remains the same as it is defined in $< C_{core}^s, S_{all}^s >$. Hence it is also a subset of C_{core}^s. Consequently, a tuple $t \in C_{core}$ is in $Q\,set(t_{aggr}^s)$ only if $t \in C_{core}^s$. That is, t and t_{aggr}^s are R_{AD} related only if they are in the same sub-data cube $< C_{core}^s, S_{all}^s$.

Proposition 3. *The computational complexity of the algorithm* Ctrl_Inf_Cube *is* $O(|C_{core}|)$.

Proof. Let $n = |C_{core}|$. The main routine *Ctrl_Inf_Cube* partitions C_{core} by evaluating the k elements of each tuple and assigning the tuple to the appropriate block of the partition. This operation has a runtime of $O(nk) = O(n)$. The subroutine *Ctrl_Inf_Sub* is called for each of the $\prod_{i=1}^{k} m_i$ blocks of the partition. For any block with cardinality n^s, we show the complexity of the subroutine to be $O(n^s)$.

First consider the matrix M_1' that contains all non-zero rows and zero columns in M_1^s. From the proof of Proposition 2, M_1' has n^s columns, $O(n^s)$ rows and $(n^s \cdot k)$ many 1-elements. Hence M_1' can be populated in $O(n^s)$ time (we only need to populate the 1-elements).

Once M_1' is established, we show that the cardinality-based tests in the subroutine *Ctrl_Inf_Sub* can be done in $O(n^s)$ time. The first, second and fourth tests can be done in constant time because they only need the cardinalities $|C_{core}^s| = n^s$ and $|C_{full}^s| = \prod_{i=1}^{k} d_i^s$. The third test for trivial compromisability can be done by counting the 1-elements of M_1' in each row. The complexity is $O(n^s)$ because the total number of 1-* elements is $(n^s \cdot k)$. The last test for full slices requires counting the number

of columns of M_1' in each slice along all k dimensions. Hence the complexity is $k \cdot n^s = O(n^s)$. □

The computational complexity of the algorithm, as stated in Proposition 3, is $O(n)$, where n is the cardinality of the core cuboid C_{core}. On the other hand, determining compromisability by transforming an m by n aggregation matrix to its RREF has a complexity of $O(m^2 n)$, and the maximum subset of non-compromisable aggregation vectors cannot be found in polynomial time [13]. Moreover, the complexity of our algorithm is handled by off-line processing, according to the three-tier inference control model.

6.3. Implementation issues

6.3.1. Integrating inference control into OLAP

Using pre-computed aggregations to reduce the response time in answering queries is a common approach in OLAP [19,27]. OLAP queries are split into sub-queries whose results are computed from cached aggregations. Our algorithm also uses pre-computed aggregations, only for a different purpose: inference control. It is a natural choice to integrate the inference control algorithm with caching mechanisms that already exist in OLAP applications. However, inference control may require modifications to such mechanisms. For example, after splitting a query into sub-queries, one may find that some sub-queries do not correspond to any aggregation in the aggregation tier. A caching mechanism may choose to answer them by computing them from the raw data. However, answering such sub-queries may lead to inferences because the disclosed information goes beyond the aggregation tier. In such a case the query should either be denied or modified in order for them to be answerable from the aggregation tier. For example, in Table 1, suppose that a query asks for the total commission of each employee in the first two quarters. This query can be split into two sub-queries corresponding to the subtotals in the first two quarters. The query is safe to answer, because all the required subtotals are in the aggregation tier. Next, suppose another query asks for the total commission of each employee in the first four months. Splitting this query leads to unanswerable sub-queries, because only the data of April is selected by the query. Answering this query discloses all individual commissions in April if the subtotals in the first quarter are previously answered.

The partitions on the data tier and the aggregation tier has an important impact on the *usability* of OLAP systems with inference control enforced. Partitioning should be based on dimension hierarchies, so that most useful queries correspond to whole blocks in the partitions. The choice of dimension granularity of the partitioning is also important. Most practical data sets have deep dimension hierarchies composing of many different granularities for each dimension, with the coarser ones at the top of the hierarchy and finer ones at the bottom. Choosing coarser dimensions as the basis for partitioning leads to fewer and larger blocks. Larger blocks cause less answerable queries. In the above example the query for commissions in the first four months

cannot be answered because the granularity used in the query (month) is finer than the dimension used to partition the data (quarter). This can be avoided if blocks are formed by months. However, such partitioning does not provide inference control at all. Because of such subtleties, it may be attractive to use a query-driven or dynamic partition. However, varying partitions at run-time causes more online performance overhead and hence is less feasible.

6.3.2. Re-ordering tuples in unordered dimensions

In practice, many data cubes have unordered dimensions. That is, the order of values in the domain of those dimensions have no apparent semantics associated with it. For example, in Table 1 the dimension employee has no natural ordering. In the core cuboid of a data cube tuples can usually be re-ordered such that their orders in ordered dimensions are not affected. For example, in Table 1 assuming the dimension employee is unordered, tuples can be horizontally re-ordered along employee dimension.

Cardinality-based compromisability of data cubes depends on the *density* of each block of the core cuboid. As shown in Section 4, full blocks or dense blocks with cardinalities above the upper bound given in Theorem 2 cannot be non-trivially compromised. Without losing any useful information, the tuples in a core cuboid can be re-ordered such that partitioning the core cuboid leads to more full blocks and dense blocks. One consequence of this re-ordering is that aggregation tier will contain more safe aggregations leading to better *usability* of the system.

Techniques already exist in increasing the number of dense blocks in data cubes by re-ordering tuples along un-ordered dimensions. For example, the row shuffling algorithm presented in [4] re-orders tuples in the core cuboid, so that the tuples containing similar values are moved closer to each other. We can implement the row shuffling algorithm as a step prior to the algorithm *Ctrl_Inf_Cube*, by applying it to the full core cuboid of data cubes. In [4] the similarity between two tuples is defined as their p-norm distance. To apply the row shuffling algorithm, this similarity definition needs to be revised, such that two tuples are similar if they are both missing, or none of them are. The algorithm yields outputs desired by inference control because tuples in the core cuboid are clustered to form more dense blocks that are non-compromisable. Consequently more answerable aggregations will be included in S_A. Other clustering techniques may also be used for this purpose as long as they do not lose the information contained in the ordering of tuples.

6.3.3. Update operations

Although update operations are less common in OLAP systems than they are in traditional databases, the data in data warehouses need to be modified over time. Typical update operations include inserting or deleting tuples, and modifying the values contained in tuples. Those updates need to be done efficiently to reduce their impact on availability of the system. Three-tier inference control facilitates pre-defined aggregations, which may also need to be updated as underlying data change.

Cardinality-based inference control is independent of the sensitive values. Hence modifying sensitive values usually has no effect on compromisability and can be ig-

Subroutine *Ctrl_Inf_Insert*
Input: tuple $t \in C_{core}$ to be inserted
Output: a set of aggregation vectors S_A
Method:

1. **If** t should be inserted into the sub-data cube C^s_{core} and $S^s_A = \phi$
   Let $S_A = S_A \setminus S^s_A$;
   Let S^s_A = *Ctrl_Inf_Sub_Insert(* C^s_{core}, t *)*;
   Let $S_A = S_A \cup S^s_A$;
3. **Let** $C_{core} = C_{core} \cup \{t\}$;
4. **Return** S_A.

Subroutine *Ctrl_Inf_Sub_Insert*
Input: tuple t and sub-data cube C^s_{core}
Output: a set of aggregation vectors if S^s_1 does not compromise $C^s_{core} \cup \{t\}$, and ϕ otherwise
Method:

1. **For** each $i \in [1, k]$
   If $t[i] \notin [1, d^s_i]$
   $d^s_i = d^s_i + 1$;
2. **If** $|C^s_{core}| + 1 < 2^{k-1} \cdot max(d^s_1, d^s_2, \ldots, d^s_k)$
   Return ϕ;
3. **If** $C^s_{core} \cup \{t\}$ is trivially compromised by S^s_1
   Return ϕ;
4. **If** $|C^s_{full} - C^s_{core}| < 2d^s_l + 2d^s_m - 8$, where d^s_l, d^s_m are the two smallest among d^s_is
   Return $\cup_{C \in S^s_1} C$;
5. **If** there exists $D \subset [1, k]$ satisfying $|D| = k - 1$ and for any $i \in D$, there exists $j \in [1, d^s_i]$ such that $|P_i(C^s_{full}, j) \cup P_i(C^s_{core}, j)| = 0$
   Return $\cup_{C \in S^s_1} C$;
6. **Return** ϕ.

Fig. 3. Insertion of a tuple.

nored by inference control mechanisms. For example, changing the commissions in Table 1 does not affect the compromisability of the data cube. On the other hand, modifying the non-sensitive values contained in a tuple may affect the compromisability, because the modified tuple may belong to a different block than the original tuple in the partition of the core cuboid. We treat the modification of non-sensitive values contained in a tuple as two separate operations, deletion of the original tuples and insertion of new tuples containing the modified values.

Figure 3 and Fig. 4 show the algorithms for the deletion and insertion of a tuple respectively. These two update operations are handled similarly, therefore we discuss

Subroutine *Ctrl_Inf_Delete*
Input: tuple $t \in C_{core}$ to be deleted
Output: a set of aggregation vectors S_A
Method:

1. Find the sub-data cube C^s_{core} containing t;
2. **If** $S^s_A \neq \phi$
   Let $S_A = S_A \setminus S^s_A$;
   Let S^s_A = *Ctrl_Inf_Sub_Delete(* C^s_{core}, t *)*;
   Let $S_A = S_A \cup S^s_A$;
3. **Let** $C_{core} = C_{core} \setminus \{t\}$;
4. **Return** S_A.

Subroutine *Ctrl_Inf_Sub_Delete*
Input: tuple t and sub-data cube C^s_{core}
Output: a set of aggregation vectors if S^s_1 does not compromise $C^s_{core} \setminus \{t\}$, and ϕ otherwise
Method:

1. **For** each $i \in [1, k]$
   If $|P_i(C^s_{core}, t[i])| = 1$
   $d^s_i = d^s_i - 1$;
2. **If** $|C^s_{core}| - 1 < 2^{k-1} \cdot max(d^s_1, d^s_2, \ldots, d^s_k)$
   Return ϕ;
3. **If** $C^s_{core} \setminus \{t\}$ is trivially compromised by S^s_1
   Return ϕ;
4. **If** $|C^s_{full} - C^s_{core}| < 2d^s_l + 2d^s_m - 8$, where d^s_l, d^s_m are the two smallest among d^s_is
   Return $\cup_{C \in S^s_1} C$;
5. **If** there exists $D \subset [1, k]$ satisfying $|D| = k - 1$ and for any $i \in D$, there exists $j \in [1, d^s_i]$ such that $|P_i(C^s_{full}, j) \setminus P_i(C^s_{core}, j)| = 0$
   Return $\cup_{C \in S^s_1} C$;
6. **Return** ϕ.

Fig. 4. Deletion of a tuple.

deletion only. The subroutine *Ctrl_Inf_Delete* in Fig. 4 updates S_A upon the deletion of a tuple. It first finds the sub-data cube that contains the tuple to be deleted. If the sub-data cube is already compromised before the deletion, then it must remain so after the deletion. Hence in such a case the subroutine *Ctrl_Inf_Delete* returns immediately. If the sub-data cube is not compromised before the deletion, the subroutine *Ctrl_Inf_Sub_Delete* is called to determine the compromisability of the sub-data cube after the deletion of the tuple. The subroutine *Ctrl_Inf_Sub_Delete* reduces a dimension cardinality by one if the corresponding value is contained in

the deleted tuple only. The cardinality-based compromisability criteria are then applied to the sub-data cube similarly as in *Ctrl_Inf_Sub*. There is no need to check if the core cuboid is full after deleting a tuple from it. The complexity of the subroutine *Ctrl_Inf_Sub_Delete* is bound by $O(|C^s_{core}|)$. The complexity can be reduced if we keep the cardinality results computed in the subroutine *Ctrl _Inf_Sub* and *Ctrl_Inf_Sub_Delete*, such as $|C^s_{core}|$ and $|P_i(C^s_{core}, j)|$. Although more space is needed to do so, the subroutine *Ctrl_Inf_Sub_Delete* runs in constant time if those results do not need to be re-computed.

6.3.4. Aggregation operators other than sum

Although sum queries compose an important portion of OLAP queries, other aggregation operators such as count, average, max and min are also used in practice. In some applications counts also need to be protected from compromises. Count queries can be treated as sum queries on binary values, when considering the value of each tuple as one, and the value of each missing tuple as zero. It has been shown in [29] that compromisability of queries on binary values has a much higher complexity than the case of real numbers. Hence efficiently controlling inference of counting queries is still an open problem. Because we do not restrict counting queries, inference control of the queries that contain averages becomes equivalent to that of sums. Other aggregation operators exhibiting similar algebraic property with sum may also be handled by our algorithm. Holistic aggregation operators such as median invalidates the partitioning approach adopted by the three-tier model, because they can not be calculated with partial results obtained in each block of the partition. Our on-going work addresses these issues.

7. Conclusions

We have derived sufficient conditions for sum-only data cubes safe from inferences. We have shown that compromisability of multi-dimensional aggregations can be reduced to those of one dimensional aggregations. Using the results, we have shown that core cuboids with no values known from external knowledge are free from inferences, and that there is a tight bound on the number of such know values for the data cube to remain safe from inferences. To apply our results for inference control of data cube queries, we have shown an inference control algorithm based on a three-tier model. Our ongoing work addresses aggregation operations other than sums.

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